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PEXIDER FUNCTION-COEFFICIENTS QUADRATIC FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. The Hyers-Ulam stability of a Pexider function-coefficients quadratic functional equation in Banach spaces and the Hyers-Ulam stability of a Pexider quadratic functional equation in Banach modules are investigated by using the direct method.

1. Introduction

In mathematical and scientific research, it is common to encounter objects that only approximately satisfy certain properties. A natural question arises: can we find a special object that precisely meets these conditions? One powerful approach to addressing this problem is through the framework of Hyers-Ulam stability.

The core idea of Hyers-Ulam stability can be summarized as follows: Given a class of mappings, if every mapping in this class that approximately satisfies a given equation is guaranteed to be “close" to an exact solution (or a stable approximate solution), then the equation is said to exhibit Hyers-Ulam stability. This concept ensures that approximate solutions can be rigorously controlled and related to true solutions.

The study of stability in functional equations traces its origin to a fundamental question posed by Ulam [58] in 1940 concerning the stability of metric group homomorphisms. In 1941, Hyers [17] provided the first rigorous solution to Ulam's problem in the context of Banach spaces, specifically for the Cauchy functional equation. His approach, now known as the *direct method*, established a foundational framework for analyzing such stability problems. The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchy equation. More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [1, 9, 18, 21, 24–26, 32, 53].

The functional equation

$$f(x+y) = g(x) + h(y)$$

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is called a Pexider additive functional equation. In 1993, Chmieliński and Tabor [6] investigated the stability of the Pexider equation. This paper seems to be the first one concerning the stability problem of the Pexider equation. We introduce a theorem presented by Jun, Shin and Kim [23].

THEOREM 1.1. [23] Let G and E be an Abelian group and a Banach space, respectively. Let $\varphi : G^2 \to [0,\infty)$ be a function satisfying

$$\Phi(x) = \sum_{i=1}^{\infty} 2^{-i}(\varphi(2^{i-1}x, 0) + \varphi(0, 2^{i-1}x) + \varphi(2^{i-1}x, 2^{i-1}x)) < \infty$$

and

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n x) = 0$$

for all $x \in G$. If mappings $f, g, h : G \to E$ satisfy the inequality

$$\|f(x+y) - g(x) - h(y)\| \le \varphi(x,y)$$

for all $x, y \in G$, then there exists a unique additive mapping $A : G \to E$ such that

$$\begin{aligned}
&\|f(x) - A(x)\| \le \|g(0)\| + \|h(0)\| + \Phi(x),\\
&\|g(x) - A(x)\| \le \|g(0)\| + 2\|h(0)\| + \varphi(x,0) + \Phi(x),\\
&\|h(x) - A(x)\| \le 2\|g(0)\| + \|h(0)\| + \varphi(0,x) + \Phi(x)
\end{aligned}$$

for all $x \in G$.

The study in [23] is a valuable addition to the stability theory of functional equations. It successfully generalizes prior works and provides a robust framework for analyzing the stability of the Pexider equation. The rigorous proofs and detailed corollaries make it a useful reference for researchers in this field. Future work could explore relaxing the assumptions on $\varphi(x,y)$ or applying these results to specific problems in analysis and applied mathematics.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [57] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. Eshaghi Gordji and Khodaei [14] investigated the generalized Hyers-Ulam-Rassias stability of quadratic functional equations involving fixed coefficients. Stability results for more general quadratic functional equations have also been investigated in [60]. Under the assumption that $(G,*)$ is a group, X is a real or complex Hausdorff topological vector space and f, g, h, k are mappings from G into X, Jun et al. [22] proved the Hyers-Ulam stability of the following Pexiderized quadratic equation

$$f(x * y) + g(x * y^{-1}) = 2h(x) + 2k(y).$$

The study in [22] is a significant addition to the Hyers-Ulam stability literature, offering elegant generalizations of quadratic equation stability in algebraic and topological settings. The results deepen our understanding of how approximate solutions to functional equations can be systematically approximated by exact solutions, with potential applications in areas such as operator theory, mathematical physics, and numerical analysis.

The stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 21, 27, 29, 33–36, 47, 48, 50, 53, 55, 59, 60]). Some of these studies offer several notable strengths. Firstly, they provide a comprehensive analysis of Hyers-Ulam stability for nonlinear integral and integro-differential equations within the generalized framework of some metric spaces, extending classical results to more flexible settings. The use of fixed-point methods ensures rigorous and systematic proofs, enhancing the reliability of the findings. Practical examples are included to illustrate the theoretical results, bridging abstract mathematics with real-world applications. The work also highlights the advantages of some metric spaces, such as broader applicability and adaptability, making it valuable for researchers in functional analysis and differential equations [12, 56].

Mohiuddine [40] and Mohiuddine and Sevli [41] made a significant contribution to the generalization of the Hyers-Ulam stability theory, particularly, for Pexider quadratic functional equations. They extended previous results by considering more general approximate remainders $\psi(x, y)$ in both group and topological vector space settings. Direct and fixed-point methods have been widely used to establish stability results for various functional equations [3–5, 11, 28, 31, 42]. A notable strength is the separate treatment of even and odd functions, providing comprehensive results for each case.

Various stability problems for Pexider and Pexiderized functional equations have also been investigated in different spaces and settings [2, 16, 30, 37–39, 43–45].

In recent years, the Hyers-Ulam stability of Pexider functional equations has attracted significant attention from the academic community. A series of remarkable research achievements have emerged in this field [52, 54]. These recent research results have deepened our understanding of the Hyers-Ulam stability of Pexider functional equations, and they also laid a solid foundation for further exploration in related fields.

Stability problems for homomorphisms, derivations, and related functional equations in Banach algebraic structures have also been investigated in [13, 15, 19, 20, 27, 46, 49, 51, 55].

Definition 1.2. Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Then a quadratic mapping $H : P \to Q$ is called a quadratic B-homomorphism if

$$H(wx) = w^2 H(x)$$

holds for all $w \in B$ and all $x \in P$.

In this paper, we introduce the following Pexider function-coefficients quadratic functional equation

$$f(ax + by) + f(ax - by) = \alpha(x)h(x) + \beta(y)g(y) \tag{1}$$

and prove the Hyers-Ulam stability of the functional equation (1) in Banach spaces and the Hyers-Ulam stability of the functional equation (1) in Banach modules for the case that $\alpha(x)$ and $\beta(y)$ are fixed.

2. Hyers-Ulam stability of the Pexider function-coefficients quadratic functional equation in Banach spaces

In this section, we prove the Hyers-Ulam stability of the Pexider function-coefficients quadratic functional equation (1) in Banach spaces by using the direct method.

THEOREM 2.1. *Suppose that X is a real normed space and Y is a real Banach space and $\varphi : X^2 \to [0,\infty)$ is a function such that there exists a positive real number $L < 1$ with*

$$\varphi(2x, 2y) \le 4L\varphi(x,y)$$

for all $x, y \in X$. Assume that a, b are nonzero real numbers and $f, h, g : X \to Y$ are even mappings with $f(0) = h(0) = g(0) = 0$ such that

$$\|f(ax+by) + f(ax-by) - \alpha(x)h(x) - \beta(y)g(y)\| \le \varphi(x,y), \quad x, y \in X, \tag{2}$$

where $\alpha(x)$ and $\beta(y)$ are real valued functions on X. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{1}{4(1-L)}\left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\} \tag{3}$$

for all $x \in X$. Furthermore,

$$\begin{aligned} 2H(ax) &= \lim_{n\to\infty} \frac{1}{4^n}\alpha(2^n x)h(2^n x), \\ 2H(by) &= \lim_{n\to\infty} \frac{1}{4^n}\beta(2^n y)g(2^n y) \end{aligned}$$

for all $x, y \in X$.

Proof. Letting $y = 0$ in (2), we have

$$\|2f(ax) - \alpha(x)h(x)\| \le \varphi(x, 0) \tag{4}$$

for all $x \in X$. Letting $x = 0$ in (2), we have

$$\|2f(by) - \beta(y)g(y)\| \le \varphi(0, y) \tag{5}$$

for all $y \in X$. Thus

$$\|f(ax+by) + f(ax-by) - 2f(ax) - 2f(by)\| \le \varphi(x,y) + \varphi(x,0) + \varphi(0,y),$$

and

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi\left(\frac{x}{a}, \frac{y}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{y}{b}\right) \tag{6}$$

for all $x, y \in X$. Letting $y = x$ in (6), we have

$$\|f(2x) - 4f(x)\| \le \varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)$$

for all $x \in X$. Thus

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\}$$

for all $x \in X$. So

$$\begin{aligned}\left\|\frac{1}{4^n}f(2^nx)-\frac{1}{4^{n+1}}f(2^{n+1}x)\right\| &\leq \frac{1}{4^{n+1}}\left\{\varphi\left(\frac{2^nx}{a},\frac{2^nx}{b}\right)+\varphi\left(\frac{2^nx}{a},0\right)+\varphi\left(0,\frac{2^nx}{b}\right)\right\}\\ &\leq \frac{4^nL^n}{4^{n+1}}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\}\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx)\right\| &\leq \frac{L^m+\cdots+L^{n-1}}{4}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\}\\ &\leq \frac{L^m}{4(1-L)}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\}\end{aligned}$$

for all $x \in X$ and $n > m$. So the sequence $\left\{\frac{1}{4^n}f(2^nx)\right\}$ is a Cauchy sequence for each $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{4^n}f(2^nx)\right\}$ converges. Letting $H(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$, by using Hyers' direct method, we have

$$\|f(x)-H(x)\| \leq \frac{1}{4(1-L)}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\}.$$

It follows from (6) and the definition of H that

$$H(x+y)+H(x-y)=2H(x)+2H(y)$$

for all $x, y \in X$.

Let $G : X \to Y$ be another quadratic mapping satisfying (3). Then

$$\begin{aligned}\|H(x)-G(x)\| &\leq \lim_{n\to\infty}\left(\frac{1}{4^n}\|f(2^nx)-H(2^nx)\|+\frac{1}{4^n}\|f(2^nx)-G(2^nx)\|\right)\\ &\leq \lim_{n\to\infty}\frac{2}{4^{n+1}(1-L)}\left\{\varphi\left(\frac{2^nx}{a},\frac{2^nx}{b}\right)+\varphi\left(\frac{2^nx}{a},0\right)+\varphi\left(0,\frac{2^nx}{b}\right)\right\}\\ &\leq \lim_{n\to\infty}\frac{2\cdot 4^nL^n}{4^{n+1}(1-L)}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\}\\ &= 0\end{aligned}$$

for all $x \in X$. So $H(x) = G(x)$, which proves the uniqueness.

It follows from (4) and the definition of H that

$$\left\|\frac{1}{4^n}2f(2^nax)-\frac{1}{4^n}\alpha(2^nx)h(2^nx)\right\| \leq \frac{1}{4^n}\varphi(2^nx,0) \leq \frac{4^nL^n}{4^n}\varphi(x,0),$$

which tends to zero as $n \to \infty$. So

$$2H(ax) = \lim_{n\to\infty}\frac{1}{4^n}\alpha(2^nx)h(2^nx)$$

for all $x \in X$.

It follows from (5) and the definition of H that

$$\left\|\frac{1}{4^n}2f(2^nby)-\frac{1}{4^n}\beta(2^ny)g(2^ny)\right\| \leq \frac{1}{4^n}\varphi(0,2^ny) \leq \frac{4^nL^n}{4^n}\varphi(0,y),$$

which tends to zero as $n \to \infty$. So

$$2H(by) = \lim_{n\to\infty}\frac{1}{4^n}\beta(2^ny)g(2^ny)$$

for all $y \in X$. This completes the proof. □

Remark 2.2. If $h = g = f$ in Theorem 2.1, then

$$2H(ax) = \lim_{n\to\infty} \alpha(2^n x)H(x), \quad 2H(by) = \lim_{n\to\infty} \beta(2^n y)H(y)$$

for all $x, y \in X$. Furthermore, if a and b are nonzero rational numbers, then

$$2a^2 H(x) = 2H(ax) = \lim_{n\to\infty} \alpha(2^n x)H(x), \quad 2b^2 H(y) = 2H(by) = \lim_{n\to\infty} \beta(2^n y)H(y)$$

for all $x, y \in X$.

Corollary 2.3. *Let $0 < r < 2$ and θ be nonnegative real numbers and $f, h, g : X \to Y$ be even mappings satisfying $h(0) = g(0) = 0$ and*

$$(7) \qquad \|f(ax + by) + f(ax - by) - \alpha(x)h(x) - \beta(y)g(y)\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$, where $\alpha(x)$ and $\beta(y)$ are real valued functions. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \tfrac{2\theta}{4-2^r} \tfrac{|a|^r + |b|^r}{|ab|^r} \|x\|^r$$

for all $x \in X$. Furthermore,

$$2H(ax) = \lim_{n\to\infty} \frac{1}{4^n} \alpha(2^n x)h(2^n x), \quad 2H(by) = \lim_{n\to\infty} \frac{1}{4^n} \beta(2^n y)g(2^n y)$$

for all $x, y \in X$.

Proof. Letting $x = y = 0$ in (7), we have $f(0) = 0$. Let $L = 2^{r-2}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 2.1. Then the result follows from Theorem 2.1. □

Theorem 2.4. *Suppose that X is a real normed space and Y is a real Banach space and $\varphi : X^2 \to [0, \infty)$ is a function such that there exists a positive real number $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4} \varphi(x, y)$$

for all $x, y \in X$. Assume that a, b are nonzero real numbers and $f, h, g : X \to Y$ are even mappings satisfying $f(0) = h(0) = g(0) = 0$ and (2), where $\alpha(x)$ and $\beta(y)$ are real valued functions on X. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \le \frac{L}{4(1-L)} \left\{ \varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) \right\}$$

for all $x \in X$. Furthermore,

$$2H(ax) = \lim_{n\to\infty} 4^n \alpha\left(\frac{x}{2^n}\right) h\left(\frac{x}{2^n}\right), \quad 2H(by) = \lim_{n\to\infty} 4^n \beta\left(\frac{y}{2^n}\right) g\left(\frac{y}{2^n}\right)$$

for all $x, y \in X$.

Proof. By the same reasoning as in the proof of Theorem 2.1, we have Eq. (6) and it follows from (6) that

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \le \varphi\left(\frac{x}{2a}, \frac{x}{2b}\right) + \varphi\left(\frac{x}{2a}, 0\right) + \varphi\left(0, \frac{x}{2b}\right)$$

for all $x \in X$. So

$$\begin{aligned}\left\|4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| &\leq 4^n \left\{\varphi\left(\frac{x}{2^{n+1}a}, \frac{x}{2^{n+1}b}\right) + \varphi\left(\frac{x}{2^{n+1}a}, 0\right) + \varphi\left(0, \frac{x}{2^{n+1}b}\right)\right\} \\ &\leq \frac{4^n L^{n+1}}{4^{n+1}} \left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\}\end{aligned}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right)\right\| &\leq \frac{L^{m+1} + \cdots + L^n}{4} \left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\} \\ &\leq \frac{L^{m+1}}{4(1-L)} \left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\}\end{aligned}$$

for all $x \in X$ and $n > m$. So the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for each $x \in X$. Since Y is complete, the sequence $\{4^n f\left(\frac{x}{2^n}\right)\}$ converges. Letting $H(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$, by using Hyers' direct method, we have

$$\|f(x) - H(x)\| \leq \frac{L}{4(1-L)} \left\{\varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right)\right\}$$

for all $x \in X$. It follows from (6) and the definition of H that

$$H(x+y) + H(x-y) = 2H(x) + 2H(y)$$

for all $x, y \in X$.

It follows from (4) and the definition of H that

$$\left\|4^n \cdot 2f\left(\frac{ax}{2^n}\right) - 4^n \alpha\left(\frac{x}{2^n}\right) h\left(\frac{x}{2^n}\right)\right\| \leq 4^n \varphi\left(\frac{x}{2^n}, 0\right) \leq \frac{4^n L^n}{4^n} \varphi(x, 0),$$

which tends to zero as $n \to \infty$. So

$$2H(ax) = \lim_{n\to\infty} 4^n \alpha\left(\frac{x}{2^n}\right) h\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

It follows from (5) and the definition of H that

$$\left\|4^n \cdot 2f\left(\frac{by}{2^n}\right) - 4^n \beta\left(\frac{y}{2^n}\right) g\left(\frac{y}{2^n}\right)\right\| \leq 4^n \varphi\left(0, \frac{y}{2^n}\right) \leq \frac{4^n L^n}{4^n} \varphi(0, y),$$

which tends to zero as $n \to \infty$. So

$$2H(by) = \lim_{n\to\infty} 4^n \beta\left(\frac{y}{2^n}\right) g\left(\frac{y}{2^n}\right)$$

for all $y \in X$. The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof.

$\square$

REMARK 2.5. If $h = g = f$ in Theorem 2.4, then

$$2H(ax) = \lim_{n\to\infty} \alpha\left(\frac{x}{2^n}\right) H(x), \quad 2H(by) = \lim_{n\to\infty} \beta\left(\frac{y}{2^n}\right) H(y)$$

for all $x, y \in X$. Furthermore, if a and b are nonzero rational numbers, then

$$\begin{aligned}2a^2 H(x) &= 2H(ax) = \lim_{n\to\infty} \alpha\left(\frac{x}{2^n}\right) H(x), \\ 2b^2 H(y) &= 2H(by) = \lim_{n\to\infty} \beta\left(\frac{y}{2^n}\right) H(y)\end{aligned}$$

for all $x, y \in X$.

COROLLARY 2.6. *Let $r > 2$ and θ be nonnegative real numbers and $f, h, g : X \to Y$ be even mappings satisfying $h(0) = g(0) = 0$ and (7), where $\alpha(x)$ and $\beta(y)$ are real valued functions. Then there exists a unique quadratic mapping $H : X \to Y$ such that*

$$\|f(x) - H(x)\| \le \tfrac{2\theta}{2^r - 4} \tfrac{|a|^r + |b|^r}{|ab|^r} \|x\|^r$$

for all $x \in X$. Furthermore,

$$2H(ax) = \lim_{n\to\infty} 4^n \alpha\left(\frac{x}{2^n}\right) h\left(\frac{x}{2^n}\right), \quad 2H(by) = \lim_{n\to\infty} 4^n \beta\left(\frac{y}{2^n}\right) g\left(\frac{y}{2^n}\right)$$

for all $x, y \in X$.

Proof. Letting $x = y = 0$ in (7), we have $f(0) = 0$. Let $L = 2^{2-r}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 2.4. Then the result follows from Theorem 2.4. □

3. Hyers-Ulam stability of a Pexider quadratic functional equation in Banach modules

In this section, we prove the Hyers-Ulam stability of a Pexider quadratic functional equation (1) with fixed coefficients ($\alpha(x) = p$ and $\beta(y) = q$), extended by an element $w \in B$, in Banach modules by using the direct method.

LEMMA 3.1. *Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Assume that an even mapping $H : P \to Q$ satisfies $H(0) = 0$ and*

$$H(wax + by) + H(wax - by) = w^2 pH(x) + qH(y) \tag{8}$$

for fixed invertible elements $a, b \in B$, fixed elements $p, q \in B$, for all $x, y \in P$ and all $w \in B$. Then the mapping $H : P \to Q$ is a quadratic B-homomorphism.

Proof. Letting $w = e$ and $y = 0$ in (8), we have

$$2H(ax) = pH(x) \tag{9}$$

for all $x \in P$. Letting $w = e$ and $x = 0$ in (8), we have

$$2H(by) = qH(y) \tag{10}$$

for all $y \in P$. It follows from (8), (9) and (10) that

$$H(ax + by) + H(ax - by) = 2H(ax) + 2H(by) \tag{11}$$

for all $x, y \in P$. Replacing x by $a^{-1}x$ and y by $b^{-1}y$ in (11), we obtain

$$H(x + y) + H(x - y) = 2H(x) + 2H(y)$$

for all $x, y \in P$.

Let $y = 0$ in (8). It follows from (9) that

$$2H(wax) = w^2 pH(x) = 2w^2 H(ax) \tag{12}$$

for all $x \in P$. Replacing x by $a^{-1}x$ in (12), we have $2H(wx) = 2w^2 H(x)$ for all $x \in P$. Thus $H : P \to Q$ is a quadratic B-homomorphism. □

THEOREM 3.2. *Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Suppose that $\varphi : P^2 \to [0,\infty)$ is a function such that there exists a positive real number $L < 1$ with*

$$\varphi(2x, 2y) \leq 4L\varphi(x, y)$$

for all $x, y \in P$. Assume that a, b are fixed invertible elements in B, and p, q are fixed elements in B and $f : P \to Q$ is an even mapping with $f(0) = 0$ such that

$$\|f(wax + by) + f(wax - by) - w^2pf(x) - qf(y)\| \leq \varphi(x, y) \tag{13}$$

for all $x, y \in P$ and all $w \in B$. Then there exists a unique quadratic B-homomorphism $H : P \to Q$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{4(1-L)} \left\{\varphi\left(a^{-1}x, b^{-1}x\right) + \varphi\left(a^{-1}x, 0\right) + \varphi\left(0, b^{-1}x\right)\right\} \tag{14}$$

for all $x \in P$.

Proof. Let $w = e$ in (13). By the same reasoning as in the proof of Theorem 2.1, we obtain a unique quadratic mapping $H : P \to Q$ satisfying (14) and

$$H(x + y) + H(x - y) = 2H(x) + 2H(y)$$

for all $x, y \in P$. Moreover, the quadratic mapping $H : P \to Q$ is given by

$$H(x) := \lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right).$$

It follows from (13) that

$$\begin{aligned}
&\|\frac{1}{4^n}f(2^n(wax + by)) + \frac{1}{4^n}f(2^n(wax - by)) - \frac{1}{4^n}w^2pf(2^nx) - \frac{1}{4^n}qf(2^ny)\| \\
&\leq \frac{1}{4^n}\varphi(2^nx, 2^ny) \\
&\leq \frac{4^nL^n}{4^n}\varphi(x, y),
\end{aligned}$$

which tends to zero as $n \to +\infty$. Thus

$$H(wax + by) + H(wax - by) - w^2pH(x) - qH(y) = 0$$

for all $x, y \in P$. By Lemma 3.1, the quadratic mapping $H : P \to Q$ is a quadratic B-homomorphism. □

COROLLARY 3.3. *Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Assume that a, b are fixed invertible elements in B, and p, q are fixed elements in B. Let $0 < r < 2$ and θ be nonnegative real numbers and $f : P \to Q$ be an even mapping satisfying $f(0) = 0$ and*

$$\|f(wax + by) + f(wax - by) - w^2pf(x) - qf(y)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{15}$$

for all $x, y \in P$ and all $w \in B$. Then there exists a unique quadratic B-homomorphism $H : P \to Q$ such that

$$\|f(x) - H(x)\| \leq \tfrac{2\theta}{4-2^r}\left(\|a^{-1}\|^r + \|b^{-1}\|^r\right)\|x\|^r$$

for all $x \in P$.

Proof. Let $L = 2^{r-2}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in P$ in Theorem 3.2. Then the result follows from Theorem 3.2. □

Theorem 3.4. *Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Assume that a, b are fixed invertible elements in B, and p, q are fixed elements in B. Suppose that $\varphi : P^2 \to [0, \infty)$ is a function such that there exists a positive real number $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4}\varphi(x, y)$$

for all $x, y \in P$. Assume that $f : P \to Q$ is an even mapping satisfying $f(0) = 0$ and (13). Then there exists a unique quadratic B-homomorphism $H : P \to Q$ such that

$$\|f(x) - H(x)\| \leq \frac{L}{4(1-L)}\left\{\varphi\left(a^{-1}x, b^{-1}x\right) + \varphi\left(a^{-1}x, 0\right) + \varphi\left(0, b^{-1}x\right)\right\}$$

for all $x \in P$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. □

Corollary 3.5. *Let B be a unital Banach algebra with unit e and P and Q be left Banach B-modules. Assume that a, b are fixed invertible elements in B, and p, q are fixed elements in B. Let $r > 2$ and θ be nonnegative real numbers and $f : P \to Q$ be an even mapping satisfying $f(0) = 0$ and (15) for all $x, y \in P$ and all $w \in B$. Then there exists a unique quadratic B-homomorphism $H : P \to Q$ such that*

$$\|f(x) - H(x)\| \leq \tfrac{2\theta}{2^r-4}\left(\|a^{-1}\|^r + \|b^{-1}\|^r\right)\|x\|^r$$

for all $x \in P$.

Proof. Let $L = 2^{2-r}$ and $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in P$ in Theorem 3.4. Then the result follows from Theorem 3.4. □

4. Conclusion

In this paper, we systematically investigated the Hyers-Ulam stability of the Pexider function-coefficients quadratic functional equation (1) in Banach spaces by using the direct method. This technique enables us to establish rigorous stability results for a class of generalized functional equations, which extend and enrich the existing literature on functional analysis and stability theory. The study begins by reviewing the historical background of Hyers-Ulam stability, tracing its origin from Ulam's problem in 1940 to Hyers' pioneering work on the Cauchy functional equation. We then built upon recent advancements in Pexider type equations, particularly those involving function-coefficients and different mappingsf, g, h. By introducing functional equations with coefficients $a, b, \alpha(x), \beta(y)$, we generalized the classical Pexider equation to more complex settings, where the interactions between coefficients and mappings were carefully analyzed using the direct method. This research advances the understanding of stability in functional equations with function-coefficients and Pexider type structures, offering both theoretical insights and practical tools for researchers in functional analysis and related fields. Future directions may include extending these results to non-Archimedean spaces and fuzzy normed spaces.

Availability of data and materials

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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