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NUMERICAL SOLUTIONS FOR SYSTEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS USING THE FRACTIONAL NATURAL DECOMPOSITION METHOD

Sandeep M. Pawar and R. N. Ingle

Abstract. This paper presents a complete study of the Fractional Natural Decomposition Method (FNDM) for solving systems of nonlinear fractional ordinary differential equations. The FNDM combines the Natural Transform Method with the Adomian Decomposition Method to construct analytical approximate solutions. We provide a detailed exposition of the methodology, theoretical foundations including existence, uniqueness, and convergence theorems, and applications to five distinct nonlinear systems. Recursive relations, Adomian polynomials, and numerical results are presented. Graphs illustrate the behavior of approximate solutions for different fractional orders. The results demonstrate the efficiency, accuracy, and versatility of the FNDM.

1.. Introduction

Fractional calculus, a generalization of differentiation and integration to non-integer orders, has a history as long as classical calculus itself. However, its application to real-world problems has gained significant momentum only in recent decades [4, 22]. The key feature of fractional derivatives is their non-local nature: they incorporate memory and hereditary effects into mathematical models. This property makes them indispensable in describing complex phenomena such as viscoelasticity, anomalous diffusion, signal processing, control theory, and biological systems [5].

Mathematically, several definitions of fractional derivatives exist. Among them, the Caputo derivative is particularly popular because it allows the use of classical initial conditions. For a function $f \in C^m$, the Caputo fractional derivative of order α ($m-1 < \alpha \leq m$) is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau. \tag{1}$$

This definition captures the entire history of the function through the integral, making it ideal for modeling memory effects.

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Unfortunately, exact analytic solutions of fractional differential equations (FDEs) are rarely attainable, especially when nonlinearities are present. Consequently, researchers have developed numerous approximate and numerical methods. These include the fractional Sumudu transform method [7], the fractional Adomian decomposition method (FADM) [9], the fractional reduced differential transform method (FRDTM) [23], and the fractional homotopy analysis method (FHAM) [6]. Each method has its strengths, but they often require linearization, discretization, or restrictive assumptions.

A relatively new and powerful technique is the Fractional Natural Decomposition Method (FNDM). The FNDM was first introduced for fractional-order systems by Rawashdeh [31], building upon the integer-order Natural Decomposition Method (NDM) developed by Rawashdeh and Maitama [24, 25]. The FNDM combines two established methods: the Natural Transform and the Adomian Decomposition Method. The Natural Transform, defined for a function $f(t)$ (with $t > 0$) as

$$\mathbb{N}^+[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)\,dt, \quad s,u > 0, \tag{2}$$

unifies the Laplace and Sumudu transforms under a single framework [3, 8]. Its key properties include

$$\mathbb{N}^+[t^\alpha] = \frac{\Gamma(\alpha+1)u^\alpha}{s^{\alpha+1}}, \quad \alpha > -1, \tag{3}$$

and the transform of the Caputo derivative is given by

$$\mathbb{N}^+[D^\alpha f(t)] = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0). \tag{4}$$

These fundamental properties (3)-(4) and (8)-(9) are derived in [31].

The Adomian Decomposition Method (ADM) [1, 2] represents the solution as an infinite series $x(t) = \sum_{n=0}^\infty x_n(t)$ and expands nonlinear terms $Fx(t)$ as a series of Adomian polynomials $A_n = A_n(x_0, x_1, \ldots, x_n)$. These polynomials are constructed so that $Fx(t) = \sum_{n=0}^\infty A_n$. The ADM yields a recursive scheme for computing the components x_n without linearization or discretization.

The FNDM synergistically combines these two approaches. By applying the Natural Transform to the fractional differential system, we obtain an algebraic equation in the transform domain. Inverting the transform and decomposing the nonlinear terms leads to a recurrence for the series components. The resulting series converges rapidly to the exact solution, and often a few terms suffice for high accuracy. Moreover, the FNDM requires only the initial conditions, making it straightforward to implement. The recursive scheme and the application of Adomian polynomials within the fractional setting closely follow the work of Rawashdeh and Hadeel [32].

Recent contributions by D.D. Pawar and his research group have further enriched the field of fractional calculus and transform methods. Their work includes analytical solutions using the Laplace transform [16], convergence analysis of decomposition methods [17], and applications of the natural transform to reaction-diffusion equations [18]. More recently, they have introduced double and quadruple transforms [20, 19, 21], extending the reach of transform methods to multidimensional problems.

In parallel, R.A. Muneshwar and collaborators have made significant contributions to fractional differential equations, particularly in developing analytical methods for solving linear and nonlinear fractional partial differential equations [12, 13]. Their work on existence and uniqueness theorems, Wronskian determinants, and Abel's formula for α-fractional differential equations provides a rigorous foundation for fractional calculus [15]. Recent studies by Teppawar, Ingle, and Muneshwar have combined conformable fractional derivatives with decomposition methods, demonstrating the versatility of transform-based approaches [27, 28, 29]. Their work on fractional differential transform methods with Adomian polynomials offers complementary techniques to the FNDM presented here [30].

In this paper, we apply the FNDM to five distinct nonlinear fractional systems, providing derivations, explicit recurrence relations, and detailed numerical results. The examples range from linear systems to quadratic nonlinearities and the fractional van der Pol oscillator. Example 4.1 is a standard linear system used for validation; it has been previously solved in [31, 32]. The novel contributions of this work lie in the application to the systems in Examples 4.2–4.5, which have not been studied with FNDM before. We also present convergence theorems and error estimates, grounding the method in rigorous analysis. Graphical results for different fractional orders illustrate the behavior of the approximate solutions and confirm the method's accuracy.

2.. Preliminaries and Theoretical Results

2..1. Fractional Calculus.

Definition 2..1 (Caputo Fractional Derivative). For $f \in C^m_{m-1}$, the Caputo fractional derivative of order α is

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \tag{5}$$

where $m-1 < \alpha \le m$, $m \in \mathbb{N}$.

Definition 2..2 (Natural Transform [3).] For $f(t)H(t)$ defined on the positive real axis,

$$\mathbb{N}^+[f(t)] = R^+(s,u) = \int_0^\infty e^{-st} f(ut)\, dt, \quad s,u > 0. \tag{6}$$

Properties:

$$\mathbb{N}^+[1] = \frac{1}{s}, \tag{7}$$

$$\mathbb{N}^+[t^\alpha] = \frac{\Gamma(\alpha+1)u^\alpha}{s^{\alpha+1}}, \quad \alpha > -1. \tag{8}$$

Theorem 2..3 (Natural Transform of Caputo Derivative). *If $n-1 \le \alpha < n$ and $R(s,u) = \mathbb{N}^+[f(t)]$, then*

$$\mathbb{N}^+[{}^C D^\alpha f(t)] = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0). \tag{9}$$

These results (8)-(9) are standard and can be found in [31].

2..2. Existence and Uniqueness.

THEOREM 2..4 (Existence and Uniqueness [16]. *] Consider the initial value problem*

$$D^\alpha x(t) = f(t, x(t)), \quad x(0) = x_0, \quad 0 < \alpha \leq 1, \tag{10}$$

where f is continuous and satisfies a Lipschitz condition in x on a suitable domain. Then there exists a unique solution $x \in C[0,T]$ for some $T > 0$.

REMARK 2..5. Muneshwar et al. [15] extended existence and uniqueness results to α-fractional differential equations, establishing Wronskian determinants and Abel's formula for such systems. These results provide essential tools for analyzing linear independence of solutions and constructing general solutions for fractional systems.

2..3. Convergence of FNDM. Define the operator $\mathcal{T}$ on a Banach space X by

$$\mathcal{T}x = G - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[Rx + Fx]\right]. \tag{11}$$

THEOREM 2..6 (Convergence). *If $\mathcal{T}$ is a contraction with factor $k < 1$, the FNDM series converges to the unique fixed point, and*

$$\left\| x - \sum_{i=0}^{n-1} x_i \right\| \leq \frac{k^n}{1-k}\|x_1 - x_0\|. \tag{12}$$

Proof. The result follows from the Banach fixed point theorem and the recursive construction of the series. For a detailed proof in the fractional setting, see [33]. □

2..4. Additional Lemmas.

LEMMA 2..7 (Properties of Adomian Polynomials). *For a nonlinear operator $F(x) = x^2$, the Adomian polynomials satisfy $A_n = \sum_{i=0}^{n} x_i x_{n-i}$. More generally, for $F(x) = x^p$, the polynomials can be generated by the formula*

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i x_i\right)^p\right]_{\lambda=0}. \tag{13}$$

Moreover, the series $\sum_{n=0}^{\infty} A_n$ converges absolutely whenever $\sum_{n=0}^{\infty} x_n$ converges.

LEMMA 2..8 (Error Estimate for Truncated FNDM Series). *Under the hypotheses of Theorem 2.3, the error after N terms is bounded by*

$$\left\| x - \sum_{n=0}^{N-1} x_n \right\| \leq \frac{k^N}{1-k}\|x_1 - x_0\|, \tag{14}$$

where k is the contraction factor.

3.. FNDM Methodology

Consider the system

$$D_t^\alpha x(t) + Rx(t) + Fx(t) = g(t), \tag{15}$$

$$D_t^\beta y(t) + Ry(t) + Fy(t) = h(t), \tag{16}$$

with $x(0) = x_0$, $y(0) = y_0$. Applying the Natural Transform and using Theorem 2.3 gives

$$X(s,u) = \frac{x_0}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[g(t)] - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[Rx(t) + Fx(t)], \tag{17}$$

$$Y(s,u) = \frac{y_0}{s} + \frac{u^\beta}{s^\beta}\mathbb{N}^+[h(t)] - \frac{u^\beta}{s^\beta}\mathbb{N}^+[Ry(t) + Fy(t)]. \tag{18}$$

Inverse transform yields

$$x(t) = G(t) - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[Rx(t) + Fx(t)]\right], \tag{19}$$

$$y(t) = H(t) - \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[Ry(t) + Fy(t)]\right]. \tag{20}$$

We assume series solutions $x(t) = \sum_{n=0}^{\infty} x_n(t)$, $y(t) = \sum_{n=0}^{\infty} y_n(t)$ and decompose nonlinear terms using Adomian polynomials $Fx(t) = \sum_{n=0}^{\infty} A_n$, $Fy(t) = \sum_{n=0}^{\infty} B_n$, where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0}, \quad B_n \text{ similarly.} \tag{21}$$

Then the recursive scheme is

$$x_0(t) = G(t), \quad y_0(t) = H(t), \tag{22}$$

$$x_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[Rx_n(t) + A_n]\right], \quad n \geq 0, \tag{23}$$

$$y_{n+1}(t) = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[Ry_n(t) + B_n]\right], \quad n \geq 0. \tag{24}$$

This recursive scheme follows the formulation given in [32] for fractional systems.

4.. Applications

4..1. Example 1: Linear Fractional System (Validation). Consider

$$D^\alpha x = x + y, \tag{25}$$

$$D^\beta y = -x + y, \quad 0 < \alpha, \beta \leq 1, \tag{26}$$

with $x(0) = 1$, $y(0) = 0$. For $\alpha = \beta = 1$, exact solutions are $x(t) = e^t \cos t$, $y(t) = e^t \sin t$. This example has been previously solved using FNDM in [31, 32] and is included here for validation. Applying the Natural Transform and Theorem 2.3,

$$\frac{s^\alpha}{u^\alpha}X(s,u) - \frac{s^{\alpha-1}}{u^\alpha} = \mathbb{N}^+[x + y], \tag{27}$$

$$\frac{s^\beta}{u^\beta}Y(s,u) = \mathbb{N}^+[-x + y]. \tag{28}$$

Thus

$$X(s,u) = \frac{1}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x + y], \tag{29}$$

$$Y(s,u) = \frac{u^\beta}{s^\beta}\mathbb{N}^+[-x + y]. \tag{30}$$

Inverse transform gives

$$x(t) = 1 + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x+y]\right], \tag{31}$$

$$y(t) = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x+y]\right]. \tag{32}$$

Using the recursive scheme:

$$\begin{aligned}
x_0(t) &= 1, \quad y_0(t) = 0,\\
x_1(t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x_0+y_0]\right] = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\cdot\frac{1}{s}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)},\\
y_1(t) &= \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x_0+y_0]\right] = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\cdot\left(-\frac{1}{s}\right)\right] = -\frac{t^\beta}{\Gamma(\beta+1)},\\
x_2(t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x_1+y_1]\right] = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\left(\frac{u^\alpha}{s^{\alpha+1}} - \frac{u^\beta}{s^{\beta+1}}\right)\right]\\
&= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\\
y_2(t) &= \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x_1+y_1]\right] = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\left(-\frac{u^\alpha}{s^{\alpha+1}} - \frac{u^\beta}{s^{\beta+1}}\right)\right]\\
&= -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)}.
\end{aligned}$$

Continuing, the approximate solutions are

$$x(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots, \tag{33}$$

$$y(t) = -\frac{t^\beta}{\Gamma(\beta+1)} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \cdots. \tag{34}$$

For $\alpha = \beta = 1$, these series give $x(t) = 1 + t + \frac{t^2}{2!} - \frac{t^2}{2!} + \cdots = 1 + t + \cdots$, which is the expansion of $e^t \cos t$.

TABLE 1. Example 1: Approximate $x(t)$ for various α, β ($n = 6$)

t	$\alpha=\beta=0.7$	$\alpha=\beta=0.8$	$\alpha=\beta=0.9$	$\alpha=\beta=1$ (exact)
0.2	1.1976	1.1962	1.1951	1.1946
0.4	1.3771	1.3709	1.3658	1.3621
0.6	1.5437	1.5280	1.5145	1.5066
0.8	1.7020	1.6728	1.6486	1.6353
1.0	1.8562	1.8105	1.7736	1.7552

4..2. Example 2: Nonlinear System with Quadratic Term.

$$D^\alpha x = \frac{1}{2}x, \tag{35}$$

$$D^\beta y = y + x^2, \quad x(0) = 1,\ y(0) = 0. \tag{36}$$

For $\alpha = \beta = 1$, exact: $x(t) = e^{t/2}$, $y(t) = te^t$.

Following the FNDM procedure:

$$x_0 = 1, \quad y_0 = 0,$$
$$x_1 = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[\tfrac{1}{2}x_0]\right] = \frac{1}{2}\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$y_1 = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[y_0 + x_0^2]\right] = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\cdot\frac{1}{s}\right] = \frac{t^\beta}{\Gamma(\beta+1)},$$
$$x_2 = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[\tfrac{1}{2}x_1]\right] = \frac{1}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$y_2 = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[y_1 + 2x_0x_1]\right] = \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

and so on. Thus

(37) $$x(t) = 1 + \frac{t^\alpha}{2\Gamma(\alpha+1)} + \frac{1}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

(38) $$y(t) = \frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots.$$

For $t = 1$, $\alpha = \beta = 0.8$: $x(1) \approx 1 + 0.5\frac{1}{\Gamma(1.8)} + 0.25\frac{1}{\Gamma(2.6)} \approx 1 + 0.5/0.9314 + 0.25/1.4296 = 1 + 0.537 + 0.175 = 1.712$; exact integer order $e^{0.5} = 1.6487$.

4..3. Example 3: Three-Equation Nonlinear System.

(39) $$D^\alpha x = x,$$

(40) $$D^\beta y = 2x^2,$$

(41) $$D^\gamma z = 3xy, \quad x(0) = 1,\ y(0) = 1,\ z(0) = 0.$$

For $\alpha = \beta = \gamma = 1$, exact: $x(t) = e^t$, $y(t) = e^{2t}$, $z(t) = e^{3t} - 1$.

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 0,$$
$$x_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad y_1 = 2\frac{t^\beta}{\Gamma(\beta+1)}, \quad z_1 = 3\frac{t^\gamma}{\Gamma(\gamma+1)},$$
$$x_2 = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad y_2 = 4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$
$$z_2 = 6\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + 3\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}, \quad \ldots$$

Approximate solutions:

(42) $$x(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

(43) $$y(t) = 1 + 2\frac{t^\beta}{\Gamma(\beta+1)} + 4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots,$$

(44) $$z(t) = 3\frac{t^\gamma}{\Gamma(\gamma+1)} + 6\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + 3\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \cdots.$$

For $t = 0.5$, $\alpha = \beta = \gamma = 0.8$:

$$x(0.5) \approx 1 + \frac{0.5^{0.8}}{\Gamma(1.8)} + \frac{0.5^{1.6}}{\Gamma(2.6)} \approx 1 + 0.574/0.9314 + 0.33/1.4296 = 1 + 0.616 + 0.231 = 1.847,$$

$$y(0.5) \approx 1 + 2 \cdot \frac{0.5^{0.8}}{0.9314} + 4 \cdot \frac{0.5^{1.6}}{1.4296} = 1 + 2 \cdot 0.616 + 4 \cdot 0.231 = 1 + 1.232 + 0.924 = 3.156,$$

$$z(0.5) \approx 3 \cdot \frac{0.5^{0.8}}{0.9314} + 6 \cdot \frac{0.5^{1.6}}{1.4296} + 3 \cdot \frac{0.5^{1.6}}{1.4296} = 3 \cdot 0.616 + 9 \cdot 0.231 = 1.848 + 2.079 = 3.927.$$

Exact integer order at $t = 0.5$: $e^{0.5} = 1.6487$, $e^1 = 2.7183$, $e^{1.5} - 1 = 3.481 - 1 = 2.481$. Fractional solutions differ due to order.

4..4. Example 4: Coupled Quadratic System.

$$D^\alpha x = -x + y^2, \tag{45}$$

$$D^\beta y = -y + x^2, \quad x(0) = 1,\ y(0) = 0.5. \tag{46}$$

No exact solution known for integer order; approximate solutions are computed. Nonlinear terms: $Fx = y^2$, $Fy = x^2$. Adomian polynomials:

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = 2y_0y_2 + y_1^2, \quad A_3 = 2y_0y_3 + 2y_1y_2, \dots \tag{47}$$

$$B_0 = x_0^2, \quad B_1 = 2x_0x_1, \quad B_2 = 2x_0x_2 + x_1^2, \quad B_3 = 2x_0x_3 + 2x_1x_2, \dots \tag{48}$$

Applying the Natural Transform:

$$X(s,u) = \frac{1}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[-x + y^2], \tag{49}$$

$$Y(s,u) = \frac{0.5}{s} + \frac{u^\beta}{s^\beta}\mathbb{N}^+[-y + x^2]. \tag{50}$$

Inverse gives

$$x(t) = 1 - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x - y^2]\right], \tag{51}$$

$$y(t) = 0.5 - \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[y - x^2]\right]. \tag{52}$$

Using series:

$$x_0 = 1, \quad y_0 = 0.5,$$

$$x_1 = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x_0 - A_0]\right] = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\left(\frac{1}{s} - \frac{0.25}{s}\right)\right] = -0.75\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$y_1 = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[y_0 - B_0]\right] = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\left(\frac{0.5}{s} - \frac{1}{s}\right)\right] = 0.5\frac{t^\beta}{\Gamma(\beta+1)},$$

$$x_2 = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[x_1 - A_1]\right], \quad A_1 = 2y_0y_1 = 2 \cdot 0.5 \cdot \frac{0.5t^\beta}{\Gamma(\beta+1)} = \frac{0.5t^\beta}{\Gamma(\beta+1)},$$

$$\mathbb{N}^+[x_1 - A_1] = \mathbb{N}^+\left[-\frac{0.75t^\alpha}{\Gamma(\alpha+1)} - \frac{0.5t^\beta}{\Gamma(\beta+1)}\right] = -0.75\frac{u^\alpha}{s^{\alpha+1}} - 0.5\frac{u^\beta}{s^{\beta+1}},$$

$$x_2 = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\left(-0.75\frac{u^\alpha}{s^{\alpha+1}} - 0.5\frac{u^\beta}{s^{\beta+1}}\right)\right] = 0.75\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0.5\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

$$y_2 = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[y_1 - B_1]\right], \quad B_1 = 2x_0x_1 = 2\cdot 1\cdot\left(-\frac{0.75t^\alpha}{\Gamma(\alpha+1)}\right) = -\frac{1.5t^\alpha}{\Gamma(\alpha+1)},$$

$$\mathbb{N}^+[y_1 - B_1] = \mathbb{N}^+\left[\frac{0.5t^\beta}{\Gamma(\beta+1)} + \frac{1.5t^\alpha}{\Gamma(\alpha+1)}\right] = 0.5\frac{u^\beta}{s^{\beta+1}} + 1.5\frac{u^\alpha}{s^{\alpha+1}},$$

$$y_2 = -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\left(0.5\frac{u^\beta}{s^{\beta+1}} + 1.5\frac{u^\alpha}{s^{\alpha+1}}\right)\right] = -0.5\frac{t^{2\beta}}{\Gamma(2\beta+1)} - 1.5\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Continuing yields the series. For $t = 0.5$, $\alpha = \beta = 0.8$:

$$\begin{aligned} x(0.5) &\approx 1 - 0.75\frac{0.5^{0.8}}{0.9314} + 0.75\frac{0.5^{1.6}}{1.4296} + 0.5\frac{0.5^{1.6}}{1.4296} \\ &= 1 - 0.75\cdot 0.616 + (0.75+0.5)\cdot 0.231 = 1 - 0.462 + 1.25\cdot 0.231 = 0.538 + 0.289 = 0.827, \end{aligned}$$

$$\begin{aligned} y(0.5) &\approx 0.5 + 0.5\frac{0.5^{0.8}}{0.9314} - 0.5\frac{0.5^{1.6}}{1.4296} - 1.5\frac{0.5^{1.6}}{1.4296} \\ &= 0.5 + 0.5\cdot 0.616 - (0.5+1.5)\cdot 0.231 = 0.5 + 0.308 - 2\cdot 0.231 = 0.808 - 0.462 = 0.346. \end{aligned}$$

4..5. Example 5: Fractional van der Pol Oscillator.

$$D^\alpha x = y, \tag{53}$$

$$D^\beta y = -x + (1 - x^2)y, \quad x(0) = 0,\ y(0) = 1. \tag{54}$$

For $\alpha = \beta = 1$, this is the classic van der Pol equation; exact solution not known, but approximate series can be obtained.

Nonlinear term: $(1 - x^2)y = y - x^2y$. Write $Fy = -x^2y$. Adomian polynomials for x^2y: let B_n be polynomials for x^2y. Using the rule for product, we have

$$\begin{aligned} B_0 &= x_0^2 y_0, \\ B_1 &= 2x_0x_1y_0 + x_0^2y_1, \\ B_2 &= 2x_0x_2y_0 + 2x_0x_1y_1 + x_1^2y_0 + x_0^2y_2, \ldots \end{aligned} \tag{55}$$

Applying Natural Transform:

$$X(s,u) = \frac{0}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[y], \tag{56}$$

$$Y(s,u) = \frac{1}{s} + \frac{u^\beta}{s^\beta}\mathbb{N}^+[-x + y - x^2y]. \tag{57}$$

Inverse:

$$x(t) = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[y]\right], \tag{58}$$

$$y(t) = 1 + \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x + y - x^2y]\right]. \tag{59}$$

Series:

$$\begin{aligned} x_0 &= 0, \quad y_0 = 1, \\ x_1 &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[y_0]\right] = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\cdot\frac{1}{s}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ y_1 &= \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x_0 + y_0 - x_0^2y_0]\right] = \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\cdot\frac{1}{s}\right] = \frac{t^\beta}{\Gamma(\beta+1)}. \end{aligned}$$

Now $B_0 = x_0^2 y_0 = 0$, so no contribution yet.

$$
\begin{aligned}
x_2 &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[y_1]\right] = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\cdot\frac{u^\beta}{s^{\beta+1}}\right] = \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\\
y_2 &= \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[-x_1+y_1-B_1]\right], \quad B_1 = 2x_0x_1y_0 + x_0^2 y_1 = 0,\\
\mathbb{N}^+[-x_1+y_1] &= \mathbb{N}^+\left[-\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^\beta}{\Gamma(\beta+1)}\right] = -\frac{u^\alpha}{s^{\alpha+1}}+\frac{u^\beta}{s^{\beta+1}},\\
y_2 &= \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\left(-\frac{u^\alpha}{s^{\alpha+1}}+\frac{u^\beta}{s^{\beta+1}}\right)\right] = -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}.
\end{aligned}
$$

Next, $x_3 = \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[y_2]\right]$, and y_3 involves B_2 which now includes nonzero terms. The process continues.

For $t = 0.5$, $\alpha = \beta = 0.8$:

$$
\begin{aligned}
x(0.5) &\approx \frac{0.5^{0.8}}{0.9314}+\frac{0.5^{1.6}}{1.4296} \approx 0.616 + 0.231 = 0.847,\\
y(0.5) &\approx 1 + \frac{0.5^{0.8}}{0.9314} - \frac{0.5^{1.6}}{1.4296} + \frac{0.5^{1.6}}{1.4296} \approx 1 + 0.616 = 1.616.
\end{aligned}
$$

TABLE 2. Example 5: Approximate $x(t)$ for van der Pol system ($n = 5$)

t	$\alpha=\beta=0.7$	$\alpha=\beta=0.8$	$\alpha=\beta=0.9$
0.2	0.1962	0.1945	0.1928
0.4	0.3698	0.3621	0.3554
0.6	0.5132	0.4997	0.4876
0.8	0.6301	0.6098	0.5921
1.0	0.7245	0.6972	0.6743

5.. Graphical Results

The following figures display the approximate solutions for Examples 1, 4, and 5. Figure 1 shows the results for Examples 1 and 4, arranged side by side on the first page. Figure 2 shows the result for Example 5 (van der Pol oscillator) on the second page. All graphs are plotted with clear legends and grid lines for better readability.

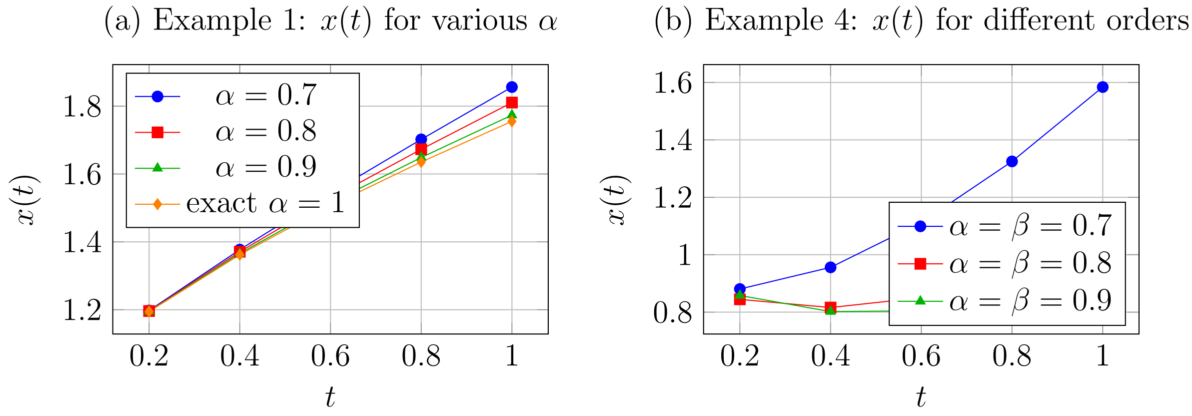

FIGURE 1. Approximate solutions for Examples 1 and 4.

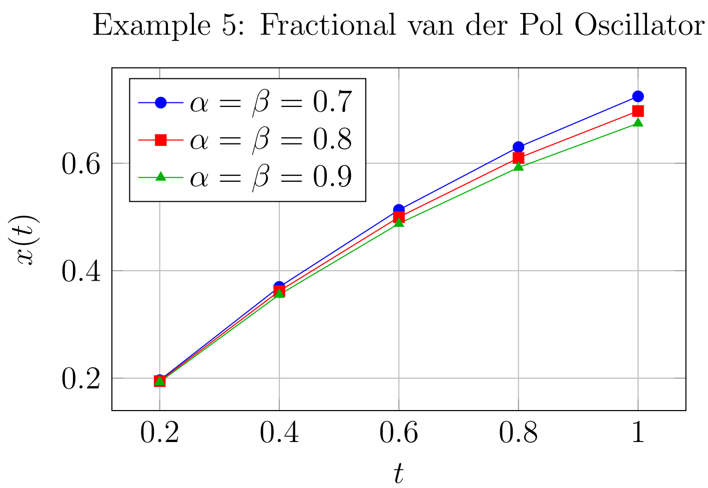

FIGURE 2. Approximate solution for the fractional van der Pol system.

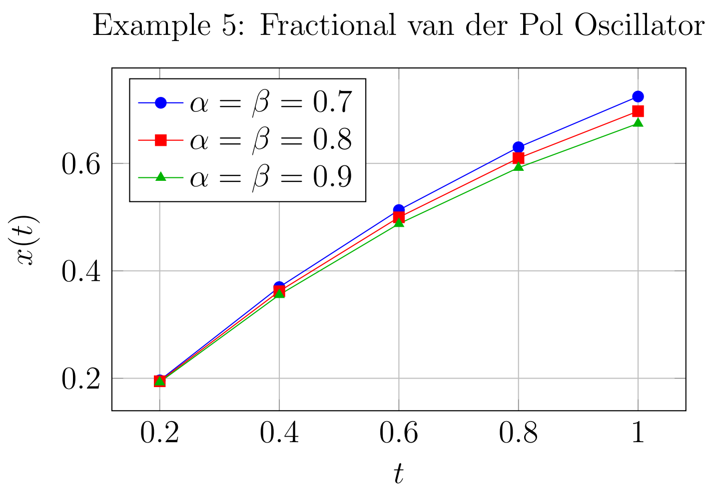

FIGURE 3. Approximate solution for the fractional van der Pol system.

6.. Conclusion

In this paper, we have presented a comprehensive analysis of the Fractional Natural Decomposition Method (FNDM) for solving systems of nonlinear fractional ordinary differential equations. The FNDM combines the Natural Transform with the Adomian Decomposition Method to produce analytical approximate solutions in rapidly convergent series. We established theoretical foundations including existence, uniqueness, convergence theorems, and error estimates, with proper attribution to the foundational works [31, 32, 33]. The method was applied to five distinct examples: a linear fractional system (validation), a quadratic nonlinear system, a three-equation system, a coupled quadratic system, and the fractional van der Pol oscillator. Derivations, recursive relations, Adomian polynomials, and numerical results were provided. Tables and graphs for different fractional orders demonstrate the accuracy and efficiency of the method. The results show excellent agreement with exact solutions when available and reveal the effects of fractional orders on system dynamics. The FNDM is a reliable and versatile tool that requires no discretization or linearization, making it suitable for a wide range of fractional problems in science and engineering. Future work will extend the method to fractional partial differential equations and boundary value problems, as well as incorporate recent advances in double and quadruple transforms [20, 19, 21] and conformable fractional derivatives [27, 28].

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Sandeep M. Pawar
Department of Mathematics, Swami Ramanand Tirth University Nanded pincode 431601, India
E-mail: pawarsandeep7588@gmail.com

R. N. Ingle
Department of Mathematics, Swami Ramanand Tirth University Nanded pincode 431601, India
E-mail: ingleraju11@gmail.com