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CROSS-SHARING PROBLEMS OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract. This paper deals with the determination of meromorphic functions for which the differential polynomials or difference-differential polynomials generated by meromorphic functions f and g share a small function with weighted sharing. The results extend and generalize those obtained by Guo and Liu [7]

1. Introduction

Let $f(z)$, $g(z)$ be two non-constant meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations and basic results from Nevanlinna's theory; for details, we refer the reader to [9, 19]. For a meromorphic function $f(z)$, we use $S(f)$ to denote the family of all meromorphic functions $a(z)$ ($\not\equiv 0, \infty$) satisfying $T(r, a) = S(r, f)$, where $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Functions belonging to the class $S(f)$ are called small functions with respect to $f(z)$.
Let $a(z) \in S(f) \cap S(g)$ and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if the zero sets of $f(z)-a(z)$ and $g(z)-a(z)$ coincide without taking multiplicities into account. On the other hand, the zeros of $f(z) - a(z)$ and $g(z) - a(z)$ coincide with the same multiplicities, then $f(z)$ and $g(z)$ are said to share $a(z)$ CM (counting multiplicities).

Definition 1.1. The hyper-order $\rho_2(f)$ of a meromorphic function f is defined by

$$\rho_2(f) = \lim_{r\to\infty} \sup \frac{\log\log T(r, f)}{\log r}.$$

Hayman investigated many interesting results concerning the zero distribution of complex differential polynomials. One of his important studies [10], published in 1959, is presented below:

Theorem 1.2. [10] Let n $(\geq 2) \in \mathbb{Z}^+$ and $f(z)$ be a transcendental entire function. Then $f^n(z)f'(z) - a$ admits infinitely many zeros, where a ($\neq 0$) is a constant.

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Wang and Fang [17] generalized Theorem 1.2 by considering transcendental meromorphic functions and established the result given below.

Theorem 1.3. [17] Let $f(z)$ be a transcendental meromorphic function and $n, k \in \mathbb{Z}^+$ such that $n \geq k+1$. Then $(f^n)^{(k)} - a$ possesses an infinite number of zeros, where a $(\neq 0)$ is a constant.

A difference analogue of Theorem 1.2 was first introduced by Yang [13]. Several additional conclusions obtained in [15], stated as follows:

Theorem 1.4. [15] Let $f(z)$ be a transcendental meromorphic function and n $(\geq 4) \in \mathbb{Z}^+$. Then $f(z)^n f(z+c) - a(z)$ has possesses an infinite number of zeros, where $a(z)$ $(\not\equiv 0) \in S(f)$.

We proceed to consider the issue of uniqueness when$[f(z)^n]^{(k)}$ and $[g(z)^n]^{(k)}$ share a value, in this direction, Fang [5] found the outcome given below:

Theorem 1.5. [5] Let $f(z)$, $g(z)$ be transcendental entire functions and $k, n \in \mathbb{Z}^+$ satisfying $n > 2k+4$. Suppos $[f(z)^n]^{(k)}$ and $[g(z)^n]^{(k)}$ share the value 1 *CM*, then either $f(z) = tg(z)$ where $t^n = 1$ or $f(z) = c_1 e^{cz}$, $\quad g(z) = c_2 e^{-cz}$ where c, c_1 and $c_2 \in \mathbb{C}$ fulfilling $(-1)^k (c_1 c_2)^n (nc)^2 k = a^2$.

Numerous refinements of Theorem 1.5 have been introduced in the literature, particularly in [1–4]. The case of $k = 1$, where $f^{n-1} f'(z)$ and $g^{n-1} g'(z)$ share the value 1 *CM*, follows from Theorem 1.5.

The problems of cross-sharing of complex delay-differential polynomials of entire and meromorphic functions of differential types were investigated by [6, 17]. The following theorems were established by Liu [6], and are stated as follows:

Theorem 1.6. [6] Let $n \in \mathbb{Z}^+$ such that $n \geq 8$ and $f(z)$, $g(z)$ be transcendental entire functions. Suppose $f(z)^n g'(z)$ and $g(z)^n f'(z)$ share $a(\neq 0)$ *CM*, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ where c, c_1 and $c_2 \in \mathbb{C}$ fulfilling $(c_1 c_2)^{n+1} c^2 = a^2$ or $f(z) = tg(z)$ where $t^{n-1} = 1$.

To present the key finding of this work, we first take two polynomials $\Phi(z)$ and $\Psi(z)$ of degree p and q respectively, given by

$$\Phi(z) = b(z - \alpha_1)^{m_1} \cdots (z - \alpha_t)^{m_t} \text{ and } \Psi(z) = d(z - \beta_1)^{n_1} \cdots (z - \beta_t)^{n_s},$$

where $m_1 + \cdots + m_t = p, n_1 + \cdots + n_s = q, \alpha_1, \cdots, \alpha_t$ are distinct constants and $\beta_1, \cdots, \beta_s$ are distinct constants. Additionally, we assume that $q > p$ without losing generality.

Let us consider the zeros of $(f - \beta_1)^{n_1} \cdots (f - \beta_s)^{n_s}$ in the following way. Let u_1 meet the requirement that the multiplicity of any zero of $(f - \beta_{u_1+1})^{n_{u_1+1}}, \ldots, (f - \beta_s)^{n_s}$ is greater than or equal to $k+2$, and at least one of the multiplicities of zeros of $(f - \beta_1)^{n_1}, \ldots, (f - \beta_{u_1})^{n_{u_1}}$ is less than $k+2$. Similarly, for the zeros of $(g' - \alpha_1)^{m_1} \cdots (g' - \alpha_t)^{m_t}$, let v_1 satisfies the requirement that the multiplicity of any zero of $(g' - \alpha_{v_1+1})^{m_{v_1+1}}, \ldots, (g' - \alpha_t)^{m_t}$ is greater than or equal to $k+2$, and at least one of the multiplicities of zeros of $(g' - \alpha_1)^{m_1}, \ldots, (g' - \alpha_{v_1})^{m_{v_1}}$ is less than $k+2$. Similarly, for $(g - \beta_1)^{n_1} \cdots (g - \beta_s)^{n_s}$ and $(f' - \alpha_1)^{m_1} \cdots (f' - \alpha_t)^{m_t}$, we define u_2 and v_2, where $0 \leq u_1, u_2 \leq s$ and $0 \leq v_1, v_2 \leq t$.

In 2024, Guo et al. [7] improved Theorem 1.6 and proved the result presented below.

THEOREM 1.7. [7] Assume that $f(z)$ and $g(z)$ be transcendental meromorphic functions. Let $k \in \mathbb{Z}^+$ and u_1, u_2, v_1, v_2 be defined as above. If $[\Psi(f)\Phi(g^{'})]^{(k)}$ and $[\Psi(g)\Phi(f^{'})]^{(k)}$ share $\alpha(z)(\not\equiv 0) \in S(f) \cap S(g)$ CM with

$$q \geq 1 + 2p + \Big[2k + 2(k+2)\max\left((s + 2t - u_1 - 2v_2), (s + 2t - u_2 - 2v_1)\right) \\ +2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 4\max\left(\textstyle\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j\right) + 8\Big],$$

then either

$$[\Psi(f)\Phi(g')]^{(k)} \equiv [\Psi(g)\Phi(f')]^{(k)} \quad \text{or} \quad [\Psi(f)\Phi(g')]^{(k)} \cdot [\Psi(g)\Phi(f')]^{(k)} \equiv \alpha^2.$$

In the case $[|\Psi(f)\Phi(g')]^{(k)} \equiv [\Psi(g)\Phi(f')]^{(k)}$, if $f = gh$ where $h \in \mathbb{C}$, then we obtain either $h^q = h^p$ whenever $\Psi(z) = a_q z^q$ and $\Phi(z) = b_p z^p$, or $h^\mu = 1$, where μ denotes the gcd of all integers between 1 and q such that the associated coefficients a_j ($1 \leq j \leq q$) and b_i $(1 \leq i \leq p)$ of $\Psi(z)$ and $\Phi(z)$ are constants that are not zero.

The difference analogue of Theorem 1.6 can also be found in [6] as follows:

THEOREM 1.8. *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2 < 1$ and $n \geq 8$. Assume that α $(\neq 0)$ CM shared by $f(z)^n g(z + c)$ and $g(z)^n f(z + c)$, then either $f(z) = c_1 g(z)$ where $c_1^{n-1} = 1$, or $f(z)g(z) = c_2$ where $c_2^{n+1} = \alpha^2$.*

Now we figure out the multiplicity of zeros of $(g(z+c) - \alpha_1)^{m_1} \cdots (g(z+c) - \alpha_t)^{m_t}$ in the following way. Let v_3 be chosen such that the multiplicities of any zeros of $(g(z + c) - \alpha_{v_3+1})^{m_{v_3+1}}, \ldots, (g(z + c) - \alpha_t)^{m_t}$ are at least $k + 2$, whereas at least one of the zeros of $(g(z+c) - \alpha_1)^{m_1}, \ldots, (g(z+c) - \alpha_{v_3})^{m_{v_3}}$ has similarly less than $k+2$. Likewise, for $(f(z + c) - \alpha_1)^{m_1} \cdots (f(z + c) - \alpha_s)^{m_t}$, we define v_4, in the same way where $0 \leq v_3, v_4 \leq t$.

THEOREM 1.9. [7] Let $f(z)$ and $g(z)$ be transcendental meromorphic functions with $\rho_2 < 1$. Let $k \in \mathbb{Z}^+$ and u_1, u_2, v_3, v_4 be defined as above. If $[\Psi(f)\Phi(g(z + c))]^{(k)}$ and $[\Psi(g)\Phi(f(z + c))]^{(k)}$ share $\alpha(z)(\not\equiv 0) \in S(f) \cap S(g)$ CM with

$$q \geq 1 + p + \Big[2k + 2(k+2)\max\left((t + s - u_1 - v_4), (s + t - u_2 - v_3)\right) \\ +2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 2\max\left(\textstyle\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j\right) + 8\Big],$$

then either

$$[\Psi(f)\Phi(g(z+c))]^{(k)} \equiv [\Psi(g)\Phi(f(z+c))]^{(k)}$$
$$\text{or} \quad [\Psi(f)\Phi(g(z+c))]^{(k)} \cdot [\Psi(g)\Phi(f(z+c))]^{(k)} \equiv \alpha^2.$$

For the case $[\Psi(f)\Phi(g(z+c))]^{(k)} \equiv [\Psi(g)\Phi(f(z+c))]^{(k)}$, if $f = gh$ and $h \in \mathbb{C}$, then either $h^q = h^p$ when $\Psi(z) = a_q z^q$ and $\Phi(z) = b_p z^p$, or $h^\mu = 1$, where μ is the gcd of integers from 1 to q such that the associated coefficients a_j ($1 \leq j \leq q$) and b_i ($1 \leq i \leq p$) of $\Psi(z)$ and $\Phi(z)$ are constants that are not zero.

QUESTION 1.10. *What happens if the sharing of a small function $\alpha(z)$ CM is replaced by sharing $\alpha(z)$ with weight l in Theorem 1.7 and Theorem 1.9?*

First, we review the notion of weighted sharing that has been previously explored in the literature (see details in [11, 12]). This notion serves as the foundation for discussing the relaxed form of sharing. We now present a precise description of it below.

Definition 1.11. [11,12]. Let $l \in \mathbb{N} \cup \{0, \infty\}$ and $\alpha \in S(f)$. We use $E_l(\alpha, f)$ to represent the set of all zeros of $f - \alpha$, where a zero having multiplicity k is counted k times whenever $k \leq l$ and $l+1$ times whenever $k > l$. If $E_l(\alpha, f) = E_l(\alpha, g)$, then f, g are said to share the function α with weight l, denote by f, g share (α, l). Since the relation$E_l(\alpha, f) = E_l(\alpha, g)$ yields $E_s(\alpha, f) = E_s(\alpha, g)$ for any integer s satisfying $(0 \leq s < l)$, it can be concluded that whenever f and g share (α, l), they also share (α, s) for all $0 \leq s < l$. In addition, we remark that f and g share the function α IM or CM precisely when they share $(\alpha, 0)$ or (α, ∞), respectively.

In our present work, we consider the notion of weighted sharing and establish the theorem stated below that answers the Question 1.10.

Theorem 1.12. *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions. Let $k \in \mathbb{Z}^+$ and u_1, u_2, v_1, v_2 be stated as in Theorem 1.7. If $[\Psi(f)\Phi(g^{'})]^{(k)}$ and $[\Psi(g)\Phi(f^{'})]^{(k)}$ share $(\alpha(z), l)$ along with one of the subsequent conditions:*
(i) $l \geq 2$ and

$$q \geq 2p + 1 + \Big[2k + 2(k+2)\max\left((s+2t-u_1-2v_2), (s+2t-u_2-2v_1)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 4\max\left(\textstyle\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j\right) + 8\Big], \tag{1}$$

(ii) $l = 1$ and

$$q \geq 2p + 1 + \Big[3k + \tfrac{(k+1)(s+2t)}{2} + 2(k+2)\max\left((s+2t-u_1-2v_2), (s+2t-u_2-2v_1)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 4\max\left(\textstyle\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j\right) + 9\Big], \tag{2}$$

(ii) $l = 0$ and

$$q \geq 2p + 1 + \Big[8k + 3(k+1)(s+2t) + 2(k+2)\max\left((s+2t-u_1-2v_2), (s+2t-u_2-2v_1)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 4\max\left(\textstyle\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j\right) + 14\Big], \tag{3}$$

then the same conclusion as in Theorem 1.7 holds.

Theorem 1.13. *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions of hyper-order less than one. Let $k \in \mathbb{Z}^+$ and u_1, u_2, v_3, v_4 be defined as in Theorem 1.9. If $[\Psi(f)\Phi(g(z+c))]^{(k)}$ and $[\Psi(g)\Phi(f(z+c))]^{(k)}$ share $(\alpha(z), l)$ with one of the below conditions:*
(i) $l \geq 2$ and

$$q \geq p + 1 + \Big[2k + 2(k+2)\max\left((s+t-u_1-v_4), (t+s-u_2-v_3)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 2\max\left(\textstyle\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j\right) + 8\Big], \tag{4}$$

(ii) $l = 1$ and

$$q \geq p + 1 + \Big[3k + \tfrac{(k+1)(s+t)}{2} + 2(k+2)\max\left((t+s-u_1-v_4), (s+t-u_2-v_3)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 2\max\left(\textstyle\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j\right) + 9\Big], \tag{5}$$

(ii) $l = 0$ and

$$q \geq p + 1 + \Big[8k + 3(k+1)(s+t) + 2(k+2)\max\left((t+s-u_1-v_4), (s+t-u_2-v_3)\right) + 2\max\left(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i\right) + 2\max\left(\textstyle\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j\right) + 14\Big], \tag{6}$$

then we obtain the same conclusions as in Theorem 1.9.

2. Lemmas

In order to establish the main theorems, we provide a number of basic fundamental lemmas in this section. Assume F and G are two non-constant meromorphic functions, and Δ be a function precise as below.

$$\Delta = \left(\frac{F^{(2)}}{F^{(1)}} - \frac{2F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - \frac{2G^{(1)}}{G-1}\right). \tag{7}$$

Lemma 2.1. [18] Suppose that $f(z)$ be a non-constant meromorphic function and let $p(f)$ be a polynomial of degree n with coefficients in $S(f)$. Then

$$T(r, p(f)) = S(r, f) + nT(r, f).$$

Lemma 2.2. [14] Let $s,\ k \in \mathbb{Z}^+$ and $f(z)$ be non-constant meromorphic function. Then

$$(i)\ N_s\left(r, \frac{1}{f^{(k)}}\right) \le S(r, f) + N_{s+k}(r, \frac{1}{f}) + T(r, f^{(k)}) - T(r, f).$$

$$(ii)\ N_s\left(r, \frac{1}{f^{(k)}}\right) \le k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.3. [8] Consider $f(z)$ be a transcendental meromorphic function and $\rho_2(f) < 1$. Then $T(r, f(z+c)) = T(r, f) + S(r, f)$, $N(r, f(z+c)) = N(r, f) + S(r, f)$, and $N\big(r, \frac{1}{f(z+c)}\big) = N\big(r, \frac{1}{f}\big) + S(r, f)$.

Lemma 2.4. [19] Let $k \in \mathbb{Z}^+$ and $f(z)$ be a non-constant meromorphic function. Then

$$T(r, f^{(k)}) \le (k+1)T(r, f) + S(r, f).$$

Lemma 2.5. [7] Consider the transcendental meromorphic functions $f(z)$ and $g(z)$ for which

$$[\Psi(f)\Phi(g')]^{(k)} \equiv [\Psi(g)\Phi(f')]^{(k)}.$$

If inequality $q \ge 2p+1+2(s+2t+1)$, holds, then $\Psi(f)\Phi(g') \equiv \Psi(g)\Phi(f')$. Additionally, if f be expressed as $f = gh$ where $h \in \mathbb{C}$, then either $h^q = h^p$ whenever $\Psi(z) = a_q z^q$ and $\Phi(z) = b_p z^p$, or $h^\mu = 1$, where μ is the gcd of integers from 1 to q such that the associated coefficients a_j $(1 \le j \le q)$ and b_i $(1 \le i \le p)$ of $\Psi(z)$ and $\Phi(z)$ are non-zero constants.

Lemma 2.6. [7] Assume that $f(z)$ and $g(z)$ are transcendental meromorphic functions with $\rho_2 < 1$ that satisfy

$$[\Psi(f)\Phi(g(z+c))]^{(k)} \equiv [\Psi(g)\Phi(f(z+c))]^{(k)}.$$

If $q \ge 2p+1+2(s+2t+1)$, *then* $\Psi(f)\Phi(g(z+c)) \equiv \Psi(g)\Phi(f(z+c))$. Moreover, if $f = gh$ for some $h \in \mathbb{C}$, then either $h^q = h^p$ in the case $\Psi(z) = a_q z^q$, $\Phi(z) = b_p z^p$, or else $h^\mu = 1$, where μ is the gcd of integers from 1 to q such that the associated coefficients a_j $(1 \le j \le q)$ and b_i $(1 \le i \le p)$ of $\Psi(z)$ and $\Phi(z)$ are non-zero constants.

Lemma 2.7. [16] Assume that F and G be non-constant meromorphic functions share $(1, l)$ and $l \in \mathbb{N} \cup \{0, \infty\}$. Let Δ be defined in (7). If $\Delta \not\equiv 0$ then
(i) for $l = 0$,

$$T(r, F) \le 4\overline{N}(r, F) + N_2(r, \tfrac{1}{G}) + 3\overline{N}(r, G) + N_2(r, \tfrac{1}{F})$$
$$+2\overline{N}(r, \tfrac{1}{F}) + \overline{N}(r, \tfrac{1}{G}) + S(r, G) + S(r, F).$$

(ii) for $l = 1$,

$$T(r, F) \leq \tfrac{5}{2}\overline{N}(r, F) + 2\overline{N}(r, G) + N_2(r, \tfrac{1}{F}) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + S(r, G) + S(r, F).$$

(iii) for $l \geq 2$,

$$T(r, F) \leq 2\overline{N}(r, G) + 2\overline{N}(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, G) + S(r, F).$$

A similar inequality is valid for $T(r, G)$.

3. Proof of Main Theorems

Proof of Theorem 1.12: Let $F = \frac{[\Psi(f)\Phi(g')]^{(k)}}{\alpha}$ and $G = \frac{[\Psi(g)\Phi(f')]^{(k)}}{\alpha}$, since $[\Psi(f)\Phi(g')]^{(k)}$ and $[\Psi(g)\Phi(f')]^{(k)}$ share (α, l), we see that F and G share the value $(1, l)$. Also let $T(r) = T(r, f) + T(r, g)$ and $S(r) = S(r, f) + S(r, g)$.
Applying Lemmas 2.1 and 2.4, we have

$$\begin{aligned} T(r, \Psi(f)\Phi(g')) &\leq pT(r, g') + qT(r, f) + S(r) \\ &\leq qT(r, f) + 2pT(r, g) + S(r) \leq (q + 2p)T(r) + S(r). \end{aligned}$$

So we obtain $S(\Psi(f)\Phi(g')) = o(T(r))$. Similarly $S(\Psi(g)\Phi(f')) = o(T(r))$.
By Lemma 2.2, we get

$$N_2(r, \tfrac{1}{F}) = N_2(r, \tfrac{1}{[\Psi(f)\Phi(g')]^{(k)}}) \quad \leq T(r, F) - T(r, \Psi(f)\Phi(g')) + N_{2+k}(r, \frac{1}{\Psi(f)\Phi(g')}) + S(r).$$

Therefore, by a simple calculation, we obtain

$$\begin{aligned} qT(r, f) - 2pT(r, g) &\leq T(r, \Psi(f)\Phi(g')) + S(r) \\ &\leq T(r, F) - N_2(r, \tfrac{1}{F}) + N_{2+k}(r, \tfrac{1}{\Psi(f)\Phi(g')}) + S(r). \end{aligned} \tag{8}$$

Likewise, by (8) we have

$$qT(r, g) - 2pT(r, f) \leq N_{2+k}(r, \tfrac{1}{\Psi(g)\Phi(f')}) - N_2(r, \tfrac{1}{G}) + T(r, G) + S(r). \tag{9}$$

Also, we have

$$\begin{aligned} N_{2+k}(r, \tfrac{1}{\Psi(f)}) &= N_{2+k}(r, \tfrac{1}{d(f-\beta_1)^{n_1}.....(f-\beta_s)^{n_s}}) \leq n_1 N(r, \tfrac{1}{f-\beta_1}) + ... + n_{u_1} N(r, \tfrac{1}{f-\beta_{u_1}}) \\ &\quad + (k+2)\overline{N}(r, \tfrac{1}{f-\beta_{u_1+1}}) + ... + (k+2)\overline{N}(r, \tfrac{1}{f-\beta_s}) \\ &\leq \big[\textstyle\sum_{i=1}^{u_1} n_i + (s - u_1)(k+2)\big] T(r, f), \end{aligned}$$

and $N_{2+k}\big(r, \frac{1}{\Psi(g)}\big) \leq S(r, g) + \big[\sum_{i=1}^{u_2} n_i + (s - u_1)(k+2)\big] T(r, g)$. By the same arguments, we can also get

$$\begin{aligned} N_{2+k}(r, \tfrac{1}{\Phi(g')}) &= N_{2+k}(r, \tfrac{1}{b(g'-\alpha_1)^{m_1}.....(g'-\alpha_t)^{m_t}}) \leq m_1 N(r, \tfrac{1}{g'-\beta_1}) + ... + m_{v_1} N(r, \tfrac{1}{g'-\alpha_{u_1}}) \\ &\leq (k+2)\overline{N}(r, \tfrac{1}{g'-\alpha_{v_1+1}}) + ... + (k+2)\overline{N}(r, \tfrac{1}{g'-\alpha_s}) \\ &\leq \big[\textstyle\sum_{j=1}^{v_1} m_i + (t - v_1)(k+2)\big] T(r, g') \\ &\leq 2\big[\textstyle\sum_{j=1}^{v_1} m_j + (t - v_1)(k+2)\big] T(r, g) + S(r, g), \end{aligned}$$

and $N_{2+k}\big(r,\frac{1}{\Phi(f')}\big) \le S(r,f) + 2\big[\sum_{j=1}^{v_2} m_j + (t-v_2)(k+2)\big]T(r,f)$. Also, $\overline{N}(r,F) = \overline{N}(r,(r,\Psi(f)\Phi(g'))) \le S(r)+T(r)$, and $\overline{N}(r,G) = \overline{N}(r,(r,\Psi(g)\Phi(f'))) \le S(r)+T(r)$. Applying Lemma 2.2, we get

$$\begin{aligned} N_2(r,\tfrac{1}{F}) &\le S(r) + k\overline{N}(r,\Psi(f)\Phi(g')) + N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) \\ &\le S(r) + k\overline{N}(r,f) + k\overline{N}(r,g) + N_{2+k}(r,\tfrac{1}{\Psi(f)}) + N_{2+k}(r,\tfrac{1}{\Phi(g')}). \end{aligned} \tag{10}$$

Similarly,

$$N_2(r,\frac{1}{G}) \le k\overline{N}(r,f) + k\overline{N}(r,g) + N_{2+k}(r,\frac{1}{\Psi(g)}) + S(r) + N_{2+k}(r,\frac{1}{\Phi(f')}). \tag{11}$$

We now proceed to consider the cases below:
Case 1: $\Delta \not\equiv 0$. By (i) of Lemma 2.7 to F and G, we acquire the below subcases:
Subcase 1.1: $l \ge 2$, then

$$T(r,G)+T(r,F) \le 2N_2(r,\frac{1}{G})+2N_2(r,\frac{1}{F})+4\overline{N}(r,F)+4\overline{N}(r,G)+S(r,F)+S(r,G). \tag{12}$$

Using (8)–(12) and Lemma 2.2, we get

$$\begin{aligned} (q-2p)[T(r,f)+T(r,g)] &\le N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 4\overline{N}(r,F) + 4\overline{N}(r,G) \\ &\quad + N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) + N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f')}) + S(r) \\ &\le (2k+8)(T(r,f)+T(r,g)) + 2N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) + 2N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f')}) + S(r) \\ &\le (2k+8)T(r) + 2[\textstyle\sum_{i=1}^{u_1} n_i + 2(s-u_1)(k+2)]T(r,f) \\ &\quad + 2[\textstyle\sum_{i=1}^{u_2} n_i + 2(s-u_2)(k+2)]T(r,g) + \big[4\textstyle\sum_{j=1}^{v_1} m_j + 4(t-v_1)(k+2)\big]T(r,f) \\ &\quad + \big[4\textstyle\sum_{j=1}^{v_2} m_j + 4(t-v_2)(k+2)\big]T(r,g) + S(r) \\ &\le [2k+2(k+2)\max(s+2t-u_1-2v_2, s+2t-u_2-2v_1) + 2\max(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i) \\ &\quad + 4\max(\textstyle\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j) + 8]T(r) + S(r), \end{aligned}$$

This leads to a contradiction with (1).

Subcase 1.2: $l = 1$.

$$\begin{aligned} T(r,G)+T(r,F) &\le 2N_2(r,\tfrac{1}{F}) + 2N_2(r,\tfrac{1}{G}) + \tfrac{9}{2}\overline{N}(r,F) + \tfrac{9}{2}\overline{N}(r,G) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) \\ &\quad + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{G}) + S(r,F) + S(r,G). \end{aligned} \tag{13}$$

Using (8)–(11), (13) and Lemma 2.2, we get

$$\begin{aligned} (q-2p)[T(r,f)+T(r,g)] &\le N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + \tfrac{9}{2}\overline{N}(r,F) + \tfrac{9}{2}\overline{N}(r,G) \\ &\quad + N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) + N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f')}) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{G}) + S(r) \\ &\le (3k+9)T(r) + 2N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) + 2N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f')}) \\ &\quad + \tfrac{1}{2}N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g')}) + \tfrac{1}{2}N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f')}) + S(r) \\ &\le \{3k+9+\tfrac{(k+1)(s+2t)}{2}\}T(r) + 2[\textstyle\sum_{i=1}^{u_1} n_i + 2(k+2)(s-u_1)]T(r,f) \\ &\quad + 2[\textstyle\sum_{i=1}^{u_2} n_i + 2(k+2)(s-u_2)]T(r,g) + \big[4\textstyle\sum_{j=1}^{v_1} m_j + 4(k+2)(t-v_1)\big]T(r,f) \\ &\quad + \big[4\textstyle\sum_{j=1}^{v_2} m_j + 4(k+2)(t-v_2)\big]T(r,g) + S(r) \\ &\le [3k + \tfrac{(k+1)(s+2t)}{2} + 2(k+2)\max(2t+s-u_1-2v_2, s+2t-u_2-2v_1) \end{aligned}$$

$$+2\max(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i) + 4\max(\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j) + 9]T(r) + S(r),$$

This leads to a contradiction with (2).

Subcase 1.3: $l = 0$.

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2N_2(r, \tfrac{1}{F}) + 2N_2(r, \tfrac{1}{G}) + 7\overline{N}(r,F) + 7\overline{N}(r,G) + 3\overline{N}(r, \tfrac{1}{F}) \\ &+3\overline{N}(r, \tfrac{1}{G}) + S(r,F) + S(r,G). \end{aligned} \tag{14}$$

Applying (8)–(11), (14) and Lemma 2.2, we get

$$\begin{aligned} (q-2p)[T(r,f) + T(r,g)] &\leq N_2(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + 7\overline{N}(r,F) + 7\overline{N}(r,G) \\ &+N_{2+k}(r, \tfrac{1}{\Psi(f)\Phi(g')}) + N_{2+k}(r, \tfrac{1}{\Psi(g)\Phi(f')}) + 3\overline{N}(r, \tfrac{1}{F}) \\ &+3\overline{N}(r, \tfrac{1}{G}) + S(r,f) + S(r,g) \\ &\leq (8k+14)T(r) + 2N_{2+k}(r, \tfrac{1}{\Psi(f)\Phi(g')}) + 2N_{2+k}(r, \tfrac{1}{\Psi(g)\Phi(f')}) \\ &+3N_{1+k}(r, \tfrac{1}{\Psi(f)\Phi(g')}) + 3N_{1+k}(r, \tfrac{1}{\Psi(g)\Phi(f')}) + S(r) \\ &\leq \{8k + 14 + 3(k+1)(s+2t)\}T(r) + 2[\textstyle\sum_{i=1}^{u_1} n_i + 2(s-u_1)(k+2)]T(r,f) \\ &+2[\textstyle\sum_{i=1}^{u_2} n_i + 2(s-u_2)(k+2)]T(r,g) + \big[4\sum_{j=1}^{v_1} m_j + 4(t-v_1)(k+2)\big]T(r,f) \\ &+\big[4\textstyle\sum_{j=1}^{v_2} m_j + 4(k+2)(t-v_2)\big]T(r,g) + S(r) \\ &\leq [8k + 3(k+1)(s+2t) + 2(k+2)\max(s+2t-u_1-2v_2, s+2t-u_2-2v_1) \\ &+2\max(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i) + 4\max(\sum_{j=1}^{v_1} m_j, \sum_{j=1}^{v_2} m_j) + 14]T(r) + S(r), \end{aligned}$$

Therefore, (3) cannot hold.

Case 2: $\Delta \equiv 0$. Therefore, by integrating two times, we obtain

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

in which $A\ (\neq 0), B \in \mathbb{C}$.
Hence

$$G = \frac{(A-B-1) + (B+1)F}{FB + (A-B)}, \tag{15}$$

and

$$F = \frac{(A-B-1) + (B-A)G}{GB - (B+1)}. \tag{16}$$

The next three subcases are as follows:
Subcase 2.1: $B \neq 0, -1$. Then by (16) we obtain

$$\overline{N}\left(r, \frac{1}{G - \frac{B+1}{B}}\right) = \overline{N}(r,F).$$

From Nevanlinna's second fundamental theorem, for the function G, we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r, \tfrac{1}{G}) + \overline{N}\left(r, \tfrac{1}{G - \frac{B+1}{B}}\right) + S(r,G) \\ &\leq \overline{N}(r,G) + \overline{N}(r, \tfrac{1}{G}) + \overline{N}(r,F) + S(r,G). \end{aligned} \tag{17}$$

$A-B\neq 1$, then (15) implies that

$$\overline{N}\left(r, \frac{1}{F-\frac{-A+B+1}{B+1}}\right)=\overline{N}(r,\frac{1}{G}).$$

Applying, Nevanlinna's second fundamental theorem once again, we get the following

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}\left(r,\tfrac{1}{F-\frac{-A+B+1}{B+1}}\right)+S(r,F)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})+S(r,F)+S(r,G). \end{aligned} \tag{18}$$

Using (11), (17)–(18) and Lemma 2.2, we get

$$\begin{aligned} (q-2p)[T(r,f)+T(r,g)] &\leq 2\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})\\ &\quad+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g')})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f')})+S(r,f)+S(r,g)\\ &\leq 3T(r)+N_2(r,\tfrac{1}{G})+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g')})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f')})+S(r)\\ &\leq (k+3+(k+1)(s+2t))T(r)+[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)\\ &\quad+\big[\textstyle\sum_{j=1}^{v_2}m_j+2(t-v_2)(k+2)\big]T(r,f)+S(r).\\ \Rightarrow q &\leq 2p+(k+3+(k+1)(s+2t))T(r)+[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)\\ &\quad+\big[\textstyle\sum_{j=1}^{v_2}m_j+2(t-v_2)(k+2)\big]T(r,f)+S(r), \end{aligned}$$

which contradicts (1)–(3).

Hence, $A-B=1$. Then from (15)

$$\overline{N}\left(r, \frac{1}{F+\frac{1}{B}}\right)=\overline{N}(r,G).$$

From Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}\left(r,\tfrac{1}{F+\frac{1}{B}}\right)+S(r,F)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,G)+S(r,F). \end{aligned} \tag{19}$$

Using (17), (19) and Lemma 2.2, we get

$$\begin{aligned} (q-2p)[T(r,f)+T(r,g)] &\leq 2\overline{N}(r,F)+2\overline{N}(r,G)+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g')})\\ &\quad+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f')})+S(r,f)+S(r,g)\\ &\leq 4T(r)+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g')})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f')})+S(r)\\ &\leq (4+(k+1)(s+2t))T(r)+S(r).\\ \Rightarrow q &\leq 2p+(4+(k+1)(s+2t))T(r)+S(r), \end{aligned}$$

which contradicts (1)–(3).

Subcase 2.2: $B=-1$. Then

$$G=\frac{A}{A+1-F} \text{ and } F=\frac{(1+A)G-A}{G}.$$

If $A+1 \neq 0$, then

$$\overline{N}\left(r, \frac{1}{F-(A+1)}\right) = \overline{N}(r, G) \text{ and } \overline{N}\left(r, \frac{1}{G-\frac{A}{A+1}}\right) = \overline{N}(r, \frac{1}{F}).$$

Proceeding as in Subcase 2.1, we get a contradiction.
Hence $A=-1$, then $FG=1$ and we get

$$[\Psi(f)\Phi(g')]^{(k)} . [\Psi(g)\Phi(f')]^{(k)} \equiv \alpha^2.$$

Subcase 2.3: $B=0$. Then from (15) and (16) we get $G=\frac{F+A-1}{A}$ and $F=AG+1-A$. If $A-1 \neq 0$, then $\overline{N}\left(r, \frac{1}{A-1+F}\right) = \overline{N}\left(r, \frac{1}{G}\right)$ and $\overline{N}\left(r, \frac{1}{G-\frac{A-1}{A}}\right) = \overline{N}(r, \frac{1}{F})$. Proceeding similarly to Subcase 2.1, we get a contradiction. Therefore, $A-1=0$, then $F \equiv G$, i.e.,

$$[\Psi(f)\Phi(g')]^{(k)} \equiv [\Psi(g)\Phi(f')]^{(k)} .$$

Moreover for $[\Psi(f)\Phi(g')]^{(k)} \equiv [\Psi(g)\Phi(f')]^{(k)}$, by Lemma 2.5, we get the desired conclusion.

This establishes the proof of Theorem 1.12.

Proof of Theorem 1.13: Let $F = \frac{[\Psi(f)\Phi(g(z+c))]^{(k)}}{\alpha}$ and $G = \frac{[\Psi(g)\Phi(f(z+c))]^{(k)}}{\alpha}$, since $[\Psi(f)\Phi(g(z+c))]^{(k)}$ and $[\Psi(g)\Phi(f(z+c))]^{(k)}$ share (α, l), we see that F and G share the value $(1, l)$.
Also, let $T(r) = T(r, f) + T(r, g)$ and $S(r) = S(r, f) + S(r, g)$. Using Lemmas 2.1 and 2.3, we have

$$\begin{aligned} T(r, \Psi(f)\Phi(g(z+c))) &\leq S(r, f) + S(r, g) + qT(r, f) + pT(r, g(z+c)) \\ &\leq S(r, f) + S(r, g) + qT(r, f) + pT(r, g) \leq S(r) + (q+p)T(r). \end{aligned}$$

Therefore, $S(\Psi(f)\Phi(g(z+c))) = o(T(r))$. Similarly $S(\Psi(g)\Phi(f(z+c))) = o(T(r))$.
From Lemma 2.2, we have

$$\begin{aligned} N_2(r, \tfrac{1}{F}) &= N_2(r, \tfrac{1}{[\Psi(f)\Phi(g(z+c))]^{(k)}}) \\ &\leq T(r, F) - T(r, \Psi(f)\Phi(g(z+c))) + N_{2+k}(r, \tfrac{1}{\Psi(f)\Phi(g(z+c))}) + S(r). \end{aligned}$$

Now

$$\begin{aligned} qT(r, f) - pT(r, g) &\leq S(r, f) + S(r, g) + T(r, \Psi(f)\Phi(g(z+c))) \\ &\leq S(r) + T(r, F) - N_2(r, \tfrac{1}{F}) + N_{2+k}(r, \tfrac{1}{\Psi(f)\Phi(g(z+c))}). \end{aligned} \tag{20}$$

Similarly, as above,

$$qT(r, g) - pT(r, f) \leq S(r)T(r, G) - N_2(r, \tfrac{1}{G}) + N_{2+k}(r, \tfrac{1}{\Psi(g)\Phi(f(z+c))}). \tag{21}$$

Also, we estimate

$$\begin{aligned} N_{2+k}(r, \tfrac{1}{\Psi(f)}) = N_{2+k}(r, \tfrac{1}{d(f-\beta_1)^{n_1}.....(f-\beta_s)^{n_s}}) &\leq n_1 N(r, \tfrac{1}{f-\beta_1}) + ... + n_{u_1} N(r, \tfrac{1}{f-\beta_{u_1}}) \\ &\leq +(k+2)\overline{N}(r, \tfrac{1}{f-\beta_{u_1+1}}) + ... + (k+2)\overline{N}(r, \tfrac{1}{f-\beta_s}) \\ &\leq \left[\textstyle\sum_{i=1}^{u_1} n_i + (s-u_1)(k+2)\right]T(r, f), \end{aligned}$$

and

$$N_{2+k}(r, \tfrac{1}{\Psi(g)}) \leq S(r, g) + \left[\textstyle\sum_{i=1}^{u_2} n_i + (s-u_1)(k+2)\right]T(r, g).$$

In the same way, we obtain

$$\begin{aligned} N_{2+k}(r,\tfrac{1}{\Phi(g(z+c))}) &= N_{2+k}(r,\tfrac{1}{b(g(z+c)-\alpha_1)^{m_1}.....(g(z+c)-\alpha_t)^{m_t}}) \\ &\le m_1N(r,\tfrac{1}{g(z+c)-\beta_1}) + ... + m_{v_3}N(r,\tfrac{1}{g(z+c)-\alpha_{v_3}}) \\ &+(k+2)\overline{N}(r,\tfrac{1}{g(z+c)-\alpha_{v_3+1}}) + ... + (k+2)\overline{N}(r,\tfrac{1}{g(z+c)-\alpha_t}) \\ &\le \big[\textstyle\sum_{j=1}^{v_3} m_i + (k+2)(t-v_3)\big]T(r,g)] + S(r,g), \end{aligned}$$

and $N_{2+k}(r,\frac{1}{\Phi(f(z+c))}) \le \big[\sum_{j=1}^{v_4} m_j + (k+2)(t-v_4)\big]T(r,f) + S(r,f)$. Also, we get

$$\overline{N}(r,F) = \overline{N}(r,(r,\Psi(f)\Phi(g(z+c)))) \le S(r,g)+T(r,f)+T(r,g)+S(r,f) = T(r)+S(r),$$

and

$$\overline{N}(r,G) = \overline{N}(r,(r,\Psi(g)\Phi(f(z+c)))) \le T(r,g)+T(r,f)+S(r,f)+S(r,g) = T(r)+S(r).$$

Using Lemma 2.2 once more, we get

$$\begin{aligned} N_2(r,\tfrac{1}{F}) &\le S(r,f) + S(r,g) + k\overline{N}(r,\Psi(f)\Phi(g(z+c))) + N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))}) \\ &\le S(r) + k\overline{N}(r,f) + k\overline{N}(r,g) + N_{2+k}(r,\tfrac{1}{\Psi(f)}) + N_{2+k}(r,\tfrac{1}{\Phi(g(z+c))}). \end{aligned} \tag{22}$$

Similarly

$$N_2\left(r,\frac{1}{G}\right) \le k\overline{N}(r,f) + k\overline{N}(r,g) + N_{2+k}\left(r,\frac{1}{\Psi(g)}\right) + N_{2+k}\left(r,\frac{1}{\Phi(f(z+c))}\right) + S(r). \tag{23}$$

Now we examine the cases below:
Case 1: $\Delta \not\equiv 0$. By (i) of Lemma 2.7 to F and G, we obtain the subsequent subcases:
Subcase 1.1: $l \ge 2$, then by

$$T(r,F)+T(r,G) \le 2N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+4\overline{N}(r,F)+4\overline{N}(r,G)+S(r,F)+S(r,G). \tag{24}$$

Using (20)–(24) and Lemma 2.2, we get

$$\begin{aligned} (q-p)[T(r,f)+T(r,g)] &\le S(r) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 4\overline{N}(r,F) + 4\overline{N}(r,G) \\ &+N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))}) + N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))}) \\ &\le S(r) + (2k+8)(T(r,f)+T(r,g)) + 2N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))}) + 2N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))}) \\ &\le (2k+8)(T(r,f)+T(r,g)) + 2[\textstyle\sum_{i=1}^{u_1} n_i + (s-u_1)(k+2)]T(r,f) \\ &+2[\textstyle\sum_{i=1}^{u_2} n_i + (s-u_2)(k+2)]T(r,g) + 2\big[\textstyle\sum_{j=1}^{v_4} m_j + (t-v_4)(k+2)\big]T(r,f) \\ &+2\big[\textstyle\sum_{j=1}^{v_3} m_j + (t-v_3)(k+2)\big]T(r,g) + S(r,f) + S(r,g) \\ &\le 2\max(\textstyle\sum_{j=1}^{v_3} m_j, \sum_{j=1}^{v_4} m_j) + 8]T(r) + S(r) \\ &+[2k+2(k+2)\max(s+t-u_1-v_4, s+t-u_2-v_3) + 2\max(\textstyle\sum_{i=1}^{u_1} n_i, \sum_{i=1}^{u_2} n_i), \end{aligned}$$

this leads to a contradiction with (4).

Subcase 1.2: $l = 1$, then

$$\begin{aligned} T(r,F)+T(r,G) \le S(r,F)+S(r,G)+2N_2(r,\tfrac{1}{F})+2N_2(r,\tfrac{1}{G})+\tfrac{9}{2}\overline{N}(r,F)+\tfrac{9}{2}\overline{N}(r,G)+\tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) \\ +\tfrac{1}{2}\overline{N}(r,\tfrac{1}{G}). \end{aligned} \tag{25}$$

Using (8)–(11) and (25) and Lemma 2.2, we get

$$\begin{aligned}(q-p)[T(r,f)+T(r,g)] &\le N_2(r,\tfrac{1}{F})+N_2(r,\tfrac{1}{G})+\tfrac{9}{2}\overline{N}(r,F)+\tfrac{9}{2}\overline{N}(r,G)\\ &+N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+\tfrac{1}{2}\overline{N}(r,\tfrac{1}{F})+\tfrac{1}{2}\overline{N}(r,\tfrac{1}{G})+S(r)\\ &\le (3k+9)T(r)+2N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+2N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})\\ &+\tfrac{1}{2}N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+\tfrac{1}{2}N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r)\\ &\le \{3k+9+\tfrac{(k+1)(s+t)}{2}\}T(r)+2[\textstyle\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)]T(r,f)\\ &+2[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)+\big[\textstyle\sum_{j=1}^{v_4}m_j+(t-v_4)(k+2)\big]T(r,f)\\ &+2\big[\textstyle\sum_{j=1}^{v_3}m_j+(t-v_3)(k+2)\big]T(r,g)+S(r)\\ &\le [3k+\tfrac{(k+1)(s+t)}{2}+2(k+2)\max(s+t-u_1-v_4,s+t-u_2-v_3)\\ &+2\max(\textstyle\sum_{i=1}^{u_1}n_i,\sum_{i=1}^{u_2}n_i)+2\max(\sum_{j=1}^{v_3}m_j,\sum_{j=1}^{v_4}m_j)+9]T(r)+S(r),\end{aligned}$$

this leads to a contradiction with (5).

Subcase 1.3: $l=0$, then

$$\begin{aligned}T(r,F)+T(r,G) &\le 2N_2(r,\tfrac{1}{F})+2N_2(r,\tfrac{1}{G})+7\overline{N}(r,F)+7\overline{N}(r,G)+3\overline{N}(r,\tfrac{1}{F})\\ &+3\overline{N}(r,\tfrac{1}{G})+S(r,F)+S(r,G).\end{aligned}\tag{26}$$

Adding (8)–(11), (26) and Lemma 2.2, we get

$$\begin{aligned}(q-p)[T(r,f)+T(r,g)] &\le N_2(r,\tfrac{1}{F})+N_2(r,\tfrac{1}{G})+7\overline{N}(r,F)+7\overline{N}(r,G)\\ &+N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+3\overline{N}(r,\tfrac{1}{F})\\ &+3\overline{N}(r,\tfrac{1}{G})+S(r,f)+S(r,g)\\ &\le (8k+14)T(r)+2N_{2+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+2N_{2+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})\\ &+3N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+3N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r)\\ &\le \{8k+14+3(k+1)(s+t)\}T(r)+2[\textstyle\sum_{i=1}^{u_1}n_i+(s-u_1)(k+2)]T(r,f)\\ &+2[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)+2\big[\textstyle\sum_{j=1}^{v_3}m_j+(t-v_3)(k+2)\big]T(r,g)\\ &+2\big[\textstyle\sum_{j=1}^{v_4}m_j+(t-v_4)(k+2)\big]T(r,f)+S(r)\\ &\le [8k+3(k+1)(s+t)+2(k+2)\max(s+t-u_1-v_4,s+2t-u_2-v_3)\\ &+2\max(\textstyle\sum_{i=1}^{u_1}n_i,\sum_{i=1}^{u_2}n_i)+4\max(\sum_{j=1}^{v_3}m_j,\sum_{j=1}^{v_4}m_j)+14]T(r)+S(r),\end{aligned}$$

therefore, (6) cannot hold.

Case 2: $\Delta\equiv 0$. Therefore, by integrating two times, we obtain $\frac{1}{G-1}=\frac{A}{F-1}+B$, where $A\ (\neq 0), B\in\mathbb{C}$.
So

$$G=\frac{(B+1)F+(A-B-1)}{FB+(A-B)},\tag{27}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{GB-(B+1)}.\tag{28}$$

Now we analyze the three subcases below:

Subcase 2.1: For $B\neq 0,-1$. Then from (28) we have $\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)=\overline{N}(r,F)$.

From Nevanlinna's second fundamental theorem , we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}\left(r,\tfrac{1}{G-\frac{B+1}{B}}\right)+S(r,G)\\ &\leq \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,F)+S(r,G). \end{aligned}\tag{29}$$

If $A-B\neq 1$, then (27) implies

$$\overline{N}\left(r,\frac{1}{F-\frac{1+B-A}{B+1}}\right)=\overline{N}\big(r,\frac{1}{G}\big).$$

Applying Nevanlinna's second fundamental theorem once again, we get the following

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}\left(r,\tfrac{1}{F-\frac{-A+B+1}{B+1}}\right)+S(r,F)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})+S(r,F)+S(r,G). \end{aligned}\tag{30}$$

Using (11), (29)–(30) and Lemma 2.2, we get

$$\begin{aligned} (q-p)[T(r,f)+T(r,g)] &\leq 2\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})\\ &+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r,f)+S(r,g)\\ &\leq 3T(r)+N_2(r,\tfrac{1}{G})+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r)\\ &\leq (k+3+(k+1)(s+t))T(r)+[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)\\ &+\big[\textstyle\sum_{j=1}^{v_4}m_j+(t-v_4)(k+2)\big]T(r,f)+S(r).\\ \Rightarrow q &\leq p+(k+3+(k+1)(t+s))T(r)+[\textstyle\sum_{i=1}^{u_2}n_i+(s-u_2)(k+2)]T(r,g)\\ &+\big[\textstyle\sum_{j=1}^{v_4}m_j+(t-v_4)(k+2)\big]T(r,f)+S(r), \end{aligned}$$

which contradicts (4)–(6).

Therefore $A-B=1$. Then from (15)

$$\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)=\overline{N}(r,G).$$

By Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}\left(r,\tfrac{1}{F+\frac{1}{B}}\right)+S(r,F)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,G)+S(r,F). \end{aligned}\tag{31}$$

Using (17), (31) and Lemma 2.2, we get

$$\begin{aligned} (q-p)[T(r,f)+T(r,g)] &\leq 2\overline{N}(r,F)+2\overline{N}(r,G)+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})\\ &+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r,f)+S(r,g)\\ &\leq 4T(r)+N_{1+k}(r,\tfrac{1}{\Psi(f)\Phi(g(z+c))})+N_{1+k}(r,\tfrac{1}{\Psi(g)\Phi(f(z+c))})+S(r)\\ &\leq (4+(k+1)(s+t))T(r)+S(r)\\ \Rightarrow q &\leq p+(4+(k+1)(s+t))T(r)+S(r), \end{aligned}$$

which contradicts (4)–(6).

Subcase 2.2: $B=-1$. Then

$$G=\frac{A}{1-F+A} \text{ and } F=\frac{(1+A)G-A}{G}.$$

If $A+1\neq 0$,

$$\overline{N}\left(r,\frac{1}{F-(A+1)}\right)=\overline{N}(r,G) \text{ and } \overline{N}\left(r,\frac{1}{G-\frac{A}{A+1}}\right)=\overline{N}(r,\frac{1}{F}).$$

By a similar argument as in subcase 2.1, we have a contradiction.
Therefore, $A+1=0$, then $FG=1$ and we get

$$[\Psi(f)\Phi(g(z+c))]^{(k)}.[\Psi(g)\Phi(f(z+c))]^{(k)}\equiv\alpha^2.$$

Subcase 2.3: $B=0$. Then (15) and (16) give $G=\frac{F+A-1}{A}$ and $F=AG+1-A$.
If $A-1\neq 0$, $\overline{N}\left(r,\frac{1}{A-1+F}\right)=\overline{N}(r,\frac{1}{G})$ and $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right)=\overline{N}(r,\frac{1}{F})$. We obtain a contradiction by continuing as in subcase 2.1. Therefore, for $A-1=0$ we get $F\equiv G$ i.e.,

$$[\Psi(f)\Phi(g(z+c))]^{(k)}\equiv[\Psi(g)\Phi(f(z+c))]^{(k)}.$$

Moreover, for $[\Psi(f)\Phi(g(z+c))]^{(k)}\equiv[\Psi(g)\Phi(f(z+c))]^{(k)}$, by Lemma 2.6 we get the following. Hence, the proof of Theorem 1.13 is complete.

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