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FINITE LOCAL RINGS OF ORDER ≤ 16 WITH NONZERO JACOBSON RADICAL

Sang Bok Nam

Abstract. The structures of finite local rings of order ≤ 16 with nonzero Jacobson radical are investigated. The whole shape of noncommutative local rings of minimal order is completely determined up to isomorphism.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $J(R)$ and $Ch(R)$ denote the Jacobson radical and characteristic of R, respectively. $|\,S\,|$ denotes the cardinality of a subset S of R. Denote the n by n full (resp. upper triangular) matrix ring over R by $Mat_n(R)$ (resp. $U_n(R)$) and use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $\mathbb{Z}_n$ denotes the ring of integers modulo n, and $GF(p^n)$ denotes the Galois field of order p^n. (a) (resp. $\langle S\rangle$) denotes the ideal (resp. additive subgroup) of R generated by $a \in R$ (resp. $S \subseteq R$). Following [12], a ring R is called a minimal noncommutative local (resp. IFP) ring if R has the smallest order $|\,R\,|$ among the noncommutative local (resp. IFP) rings. Given $N \subseteq R$, N^+ means a subgroup of the additive abelian group $(R,+)$.

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2. Finite local rings with nonzero Jacobson radicals

The following lemma is a base for our study of finite local rings with nonzero Jacobson radicals.

LEMMA 1. (1) *Let R be a ring and N be a nil ideal of R. If $|N| = 4$, then N is a commutative ring without identity such that $N^3 = 0$.*

(2) *Let R be a ring and N be a nil ideal of R. If $|N| = 3$, then N is a commutative ring without identity such that $N^2 = 0$.*

Proof. (1) is a part of [8, Lemma 2.7].

(2) Let $|N| = 3$. Then N^+ is cyclic, $N = \{0, a, 2a\}$ say. Assume $a^2 \neq 0$. This entails $a^2 = 2a$ and so $a^3 = 0$: for, letting $a^3 \neq 0$ we have $(a^2)^2 = a^4 = a^3a = a^2$ when $a^3 = a$, and we have $(a^2)^2 = a^4 = a^3a = a^2a = a^2$ when $a^3 = 2a = a^2$; hence we get to a contradiction in any case. Thus $a^3 = 0$, and so $0 \neq a = 4a = 2(2a) = 2a^2 = (2a)a = a^2a = a^3 = 0$, which is also a contradiction. Consequently we get $a^2 = 0$ and this yields $N^2 = 0$. □

Following the literature, we write

$$D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{ii} = a_{jj} \text{ for all } i, j \text{ with } 1 \leq i < j \leq n\}$$

and

$$V_n(R) = \{(b_{ij}) \in D_n(R) \mid b_{st} = b_{(s+1)(t+1)} \text{ for all } s, t \text{ with } 1 \leq s < t < n\}$$

where R is a given ring.

Let R be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [2], the *Dorroh extension* of R by S is the abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$.

EXAMPLE 2. (1) $S_1 = \mathbb{Z}_8$ is a commutative local ring with $J(S_1) = \{0, 2, 4, 8\} = (2)$. Note $|\,S_1\,| = 8$, $Ch(S_1) = 8$, $J(S_1)^2 \neq 0$, $Ch(J(S_1)) = 4$, and $J(S_1)^3 = 0$. Note $J(R)^+ = \langle\{2\}\rangle$.

(2) Let $S_2 = \left\{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in D_3(\mathbb{Z}_2)\right\}$ and $S_2' = \left\{\begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \in D_3(\mathbb{Z}_2)\right\}$. Then S_2 is a commutative local ring and $S_2 \cong S_2'$ with $aE_{ii} \mapsto aE_{ii}$, $cE_{13} \mapsto cE_{13}$, and $bE_{12} \mapsto bE_{23}$. Note $|\,S_2\,| = 8$, $Ch(S_2) = 2$ and

$J(S_2) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in D_3(\mathbb{Z}_2) \right\} = (bE_{12}, cE_{13})$. Letting $x = bE_{12}$ and $y = cE_{13}$, we have $x^2 = y^2 = xy = yx = 0$ and $J(S_2)^2 = 0$. Note $J(R)^+ = \langle\{x, y\}\rangle$.

Let $N = \begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix}$ be the subring of $U_2(\mathbb{Z}_2) \oplus U_2(\mathbb{Z}_2)$, and S_2' be the Dorroh extension of N by $\mathbb{Z}_2$. Then $J(S_2') = N$ with $J(S_2')^+ = \langle\{(E_{12}, 0), (0, E_{12})\}\rangle$ and $N^2 = 0$. Note $S_2 \cong S_2'$.

(3) Let $S_3 = V_3(\mathbb{Z}_2)$. Then S_3 is a commutative local ring with $J(S_3) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \in V_3(\mathbb{Z}_2) \right\} = (bE_{12} + bE_{23}, cE_{13})$. Note $|\, S_3 \,| = 8$ and $Ch(S_3) = 2$. Letting $x = bE_{12} + bE_{23}$ and $y = cE_{13}$, we have $x^2 = y$, $x^3 = 0$, and $J(S_3)^3 = 0$. Note $J(R)^+ = \langle\{x, y\}\rangle$.

Let R be a finite local ring. Then $J(R)$ is a finite dimensional vector space over the finite field $R/J(R)$. Thus the case of $|\, R/J(R) \,| > |\, J(R) \,|$ is impossible if $J(R)$ is assumed to be nonzero, equivalently R is not a field. Thus we always have $|\, R/J(R) \,| \leq |\, J(R) \,|$ when R is a finite local ring but not a field. We will use this argument freely.

Theorem 3. (1) *If R is a local ring with $|R| = 8$ and $J(R) \neq 0$, then $|\, J(R) \,| = 4$ and R is a commutative ring isomorphic to S_i for some $i \in \{1, 2, 3\}$, where S_i's are the rings in Example 2.*

(2) *If R is a local ring with $|R| = 4$ and $J(R) \neq 0$, then R is a commutative ring with $|\, J(R) \,| = 2$ and isomorphic to either $D_2(\mathbb{Z}_2)$ or $\mathbb{Z}_4$.*

(3) *If R is a finite noncommutative local ring, then $|\, R \,| \geq 16$.*

(4) *If If R is a ring with $|R| = 9$, then R is a commutative ring with $J(R)^2 = 0$ and isomorphic to $GF(3^2)$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $D_2(\mathbb{Z}_3)$, or $\mathbb{Z}_9$.*

(5) *If If R is a ring with $|R| = 4$, then R is a commutative ring with $J(R)^2 = 0$ and isomorphic to $GF(2^2)$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $D_2(\mathbb{Z}_2)$, or $\mathbb{Z}_4$.*

Proof. (1) Let R be a local ring with $|R| = 8$ and $J(R) \neq 0$. Then clearly $|J(R)| = 4$, and so R is commutative by [8, Theorem(2)]. If $J(R)^+$ is cyclic, then $J(R) = \{0, a, 2a, 3a\}$ for some $a \in J(R)$. Here $Ch(a) = 4$ by [7, Theorems 2.3.2 and 2.3.3] and their proofs. So we can take a such that $a^2 \neq 0$ and $a^3 = 0$, thinking of Lemma 1(2) and Example 2(1). Hence $R \cong S_1$ in Example 2 with $a \mapsto 2$. Next assume that $J(R)^+$ is non-cyclic. Then, by [7, Theorem 2.3.3], there is a basis $\{a, b\}$ for N such that $2a = 0 = 2b$ and one of the following holds: (i) $a^2 = b^2 = ab = ba = 0$ and (ii) $a^2 = b, a^3 = 0$. In the first case, $R \cong S_2$ in Example 2 with $a \mapsto x, b \mapsto y$. In the second case, $R \cong S_3$ in Example 2 with $a \mapsto x$.

(2) Let R be a local ring with $|R| = 4$ and $J(R) \neq 0$. Then clearly $|J(R)| = 2$, $J(R) = \{0, a\}$ say. This yields $R = \{0, 1, a, 1 + a\}$ and hence R is clearly commutative. If $Ch(R) = 2$, then $R \cong D_2(\mathbb{Z}_2)$ with $a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If $Ch(R) = 4$, then 2 is a nonzero nilpotent element and so $R \cong \mathbb{Z}_4$ with $a \mapsto 2$.

(3) If R is a finite noncommutative local ring, then $J(R) \neq 0$. Hence we get the result by (1) and (2), noting that Eldridge proved that if a finite ring has a cube free factorization, then it is commutative in [3, Theorem].

(4) If $|R| = 9$, then R is commutative by [3, Theorem]. Suppose that R is not isomorphic to $GF(3^2)$. We refer to the argument in (1). Let $J(R) = 0$. Then R is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ by the Wedderburn-Artin theorem. Let $J(R) \neq 0$. Then clearly $|J(R)| = 3$, and $J(R)^2 = 0$ by Lemma 1(2). This entails $R/J(R) \cong \mathbb{Z}_3$. Thus R is isomorphic to $D_2(\mathbb{Z}_3)$ or $\mathbb{Z}_9$.

(5) The proof is similar to that of (4). □

Following Bell [1], a ring R is called to satisfy the *insertion-of-factors-property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [10], Shin [11], and Habeb [4] used the terms *semicommutative*, *SI*, and *zero-insertive* for the IFP ring property, respectively. A ring is usually called *reduced* if it has no nonzero nilpotent elements. The class of IFP rings clearly contains commutative rings and reduced rings. Particularly, $D_3(R)$ is IFP if and only if R is a reduced ring by [5, Proposition 2.8]. There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually called *Abelian* if each idempotent is central. A simple computation yields that IFP rings are Abelian.

Due to Lambek [9], a ring R is called *symmetric* if if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$. Symmetric rings are clearly IFP, but the converse need not hold by [6, Example 1.10]. The class of symmetric rings contains both commutative rings and reduced rings.

In [12, Theorem 8], Xu and Xue proved that a minimal noncommutative IFP ring is a local ring of order 16, and if R is such a ring, then $R \cong R_i$ for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in the following example.

EXAMPLE 4. In [12, Example 7], we see five kinds of noncommutative finite local rings with 16 elements, with Jacobson radicals of order ≥ 4. Let $A\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over given a commutative ring A, and (x, y) denote the ideal of $A\langle x, y\rangle$ generated by x, y.

(1) Let $R_1 = \mathbb{Z}_2\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$. Note $J(R_1) = (x, y)$ and $|\, J(R_1)\,| = 8$.
(2) Let $R_2 = \mathbb{Z}_4\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$. Note $J(R_2) = (x, y)$ and $|\, J(R_2)\,| = 8$.
(3) Let $R_3 = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \in U_2(GF(2^2)) \mid a, b \in GF(2^2) \right\}$. Note $J(R_3) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_3 \mid b \in GF(2^2) \right\}$ and $|\, J(R_3)\,| = 4$.
(4) Let $R_4 = \mathbb{Z}_2\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy$. It is simply checked that R_4 is isomorphic to $D_3(\mathbb{Z}_2)$ through the corresponding $x \mapsto E_{12} + E_{23}$ and $y \mapsto E_{23}$. Note $J(R_4) = (x, y)$ and $|\, J(R_4)\,| = 8$.
(5) Let $R_5 = \mathbb{Z}_4\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$. Note $J(R_5) = (x, y)$ and $|\, J(R_5)\,| = 8$.

THEOREM 5. *If R is a noncommutative local ring of minimal order, then $|\, R\,| = 16$ and R is isomorphic to R_i for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in Example 4.*

Proof. Let R be a noncommutative local ring of minimal order. Then we have $|\, R\,| \geq 16$ by Theorem 3(3). This yields $|\, R\,| = 16$ by the existence of the local rings in Example 4. Thus we have two cases of $|\, J(R)\,| = 4$ and $|\, J(R)\,| = 8$. If $|\, J(R)\,| = 4$, then R is symmetric (hence IFP) by [8, Theorem 2.8(1)]. Assume $|\, J(R)\,| = 8$. Then R is isomorphic to R_1, R_2, R_3, or R_5 by the proof of [12, Theorem 8]. But these rings are IFP by the computation in [8, Example 2.10]. Therefore R is IFP in both cases, and this implies that R is a noncommutative IFP ring of minimal order with the help of [12, Theorem 8]. Hence we have the theorem also by [12, Theorem 8]. □

We can have the following result with the help of Theorem 5 and [12, Theorem 8].

Corollary 6. *A ring R is a noncommutative local ring of minimal order if and only if R is a noncommutative IFP ring of minimal order.*

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Department of Early Child Education
Kyungdong University
Kosung 219-830, Korea
E-mail: sbnam@k1.ac.kr