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SOME FIXED POINT RESULTS IN C^*-ALGEBRA VALUED BIPOLAR $b-$METRIC SPACES

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Abstract. In this paper, we prove the existence and uniqueness of fixed point and common fixed point under Banach and Kannan type contractions in the setting of C^*-algebra valued bipolar b-metric spaces. Further, few examples are given to justify the findings. Our results improve and generalize some results from the literature, at the end we give several open problems.

1. Introduction and preliminaries

Recently, Ma et al. [7] announced the notion of C^*-algebra-valued metric space and formulated some first fixed point theorems in the C^*-algebra-valued metric space. Several authors obtained results of fixed points in the C^*-algebra-valued metric spaces, see [3, 9, 11].

The concept of $b-$metric spaces was introduced by Bakhtin [1] in 1989. In 2015, Ma and Jiang [6] introduced a concept of C^*-algebra valued $b-$metric spaces which generalize an ordinary C^*-algebra valued space. In 2016, Mutlu and Gürdal [10] introduced bipolar metric spaces. Roy et al [12] introduced the concept of bipolar $b-$metric space and established fixed point theorems on bipolar $b-$metric space.

Throughout this paper, we denote with $\mathbb{A}$ an unital C^*-algebra. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$ if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x. Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq \theta$, where θ means the zero element in $\mathbb{A}$. The set of all positive elements will be denoted by A_+.

Definition 1.1. [2] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty sets. Suppose the mapping $d : \mathcal{U} \times \mathcal{V} \to \mathbb{A}_+$ satisfies:

(i) $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in \mathcal{U} \cap \mathcal{V}$,
(iii) $d(x, y) \preceq d(x, u) + d(v, u) + d(v, y)$ for all $x, v \in \mathcal{U}$ and $y, u \in \mathcal{V}$.

Then $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ is called a C^*-algebra-valued bipolar metric space.

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Lemma 1.2. *[11] Let $\mathbb{A}$ be an unital C^*-algebra with a unit identity element $I_{\mathbb{A}}$.*
1) *We have $a \preceq I_{\mathbb{A}}$ if and only if $\|a\| \leq 1$, for all $a \in \mathbb{A}_+$.*
2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I_{\mathbb{A}} - a$ is invertible and $\|a(I_{\mathbb{A}} - a)^{-1}\| < 1$.*
3) *If $a, b \in \mathbb{A}_+$ and $ab = ba$, then $ab \in \mathbb{A}_+$.*
4) *Let $a \in \mathbb{A}' = \{a \in \mathbb{A} : ab = ba, \text{ for all } b \in \mathbb{A}\}$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I_{\mathbb{A}} - a \in \mathbb{A}_+^{'}$ is an invertible operator, then $(I_{\mathbb{A}} - a)^{-1}b \succeq (I_{\mathbb{A}} - a)^{-1}c$.*

Definition 1.3. [5] Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty sets and $b \succeq I$.
Suppose the mapping $d : \mathcal{U} \times \mathcal{V} \to \mathbb{A}_+$ satisfies:
(i) $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in \mathcal{U} \cap \mathcal{V}$,
(iii) $d(x, y) \preceq b[d(x, u) + d(v, u) + d(v, y)]$ for all $x, v \in \mathcal{U}$ and $y, u \in \mathcal{V}$.
Then $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ is called a C^*-algebra-valued bipolar $b-$metric space.

Definition 1.4. [8] Let $(\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d)$ and $(\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d)$ be C^*-algebra-valued bipolar $b-$metric spaces and the mapping $T : \mathcal{U}_1 \cup \mathcal{V}_1 \to \mathcal{U}_2 \cup \mathcal{V}_2$.
If $T(\mathcal{U}_1) \subset \mathcal{U}_2$ and $T(\mathcal{V}_1) \subset \mathcal{V}_2$, then $T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d) \rightrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d)$ is called a covariant map.
If $T(\mathcal{U}_1) \subset \mathcal{V}_2$ and $T(\mathcal{V}_1) \subset \mathcal{U}_2$, then $T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d) \leftrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d)$ is called a contravariant map.

Proposition 1.5. [13] In a C^*-algebra valued bipolar $b-$metric space, every convergent Cauchy bisequence is biconvergent.

In this paper, we obtain results of existence and uniqueness fixed point and common fixed point in the setting of C^*-algebra-valued bipolar $b-$metric space. Our results improve and generalize some results from the literature, at the end we give several open problems.

2. Main result

Inspired by the notion of bipolar metric space in the setting of C^*-algebra introduced in 2023 [2], we discuss the existence of fixed point results for these contractive mappings in a C^*-algebra valued bipolar $b-$metric space.

Definition 2.1. Let $(\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d_1)$ and $(\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d_2)$ be two C^*-algebra-valued bipolar $b-$metric spaces. A covariant map $T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d_1) \rightrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d_2)$ such that

$$d_2(Tx, Ty) \preceq \alpha^* d_1(x, y)\alpha, \text{ for all } x \in \mathcal{U}_1, y \in \mathcal{V}_1$$

and a contravariant map $T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d_1) \leftrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d_2)$ such that

$$d_2(Tx, Ty) \preceq \alpha^* d_1(x, y)\alpha, \text{ for all } x \in \mathcal{U}_1, y \in \mathcal{V}_1,$$

for some $\alpha \in \mathbb{A}$ is called covariant Lipschitz continuous and contravariant Lipschitz continuous respectively. If $\|b\|\|\alpha\|^2 < 1$, then covariant map is said covariant contraction and contravariant map is said a contravariant contraction.

Example 2.2. Let $\mathbb{A} = M_2(\mathbb{C})$, the C^*-algebra of all 2×2 complex matrices. Let $\mathcal{U}_1 = \mathcal{V}_1 = \mathbb{R}, \mathcal{U}_2 = \mathcal{V}_2 = \mathbb{R}$. Define C^*-algebra-valued bipolar b-metrics

$$d_1(x, y) = |x - y| I_2, \qquad d_2(x, y) = |x - y| I_2, \qquad \text{where } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Then $\alpha^* = \alpha, \alpha^*\alpha = \frac{1}{4}I_2$.

Define the covariant map

$$T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d_1) \rightrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d_2)$$

by $Tx = \frac{1}{4}x$.

For every $x \in \mathcal{U}_1$ and $y \in \mathcal{V}_1$,

$$d_2(Tx, Ty) = \left| \frac{1}{4}x - \frac{1}{4}y \right| I_2 = \frac{1}{4}|x - y| I_2,$$

while

$$\alpha^* d_1(x, y)\alpha = \alpha \left(|x - y| I_2\right) \alpha = |x - y|\alpha^2 = \frac{1}{4}|x - y| I_2.$$

Hence $d_2(Tx, Ty) = \alpha^* d_1(x, y)\alpha$, which implies $d_2(Tx, Ty) \preceq \alpha^* d_1(x, y)\alpha$. If the coefficient of the bipolar b-metric is $b = 1$, then

$$\|b\| \, \|\alpha\|^2 = 1 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1.$$

Therefore, T is a covariant contraction.

Example 2.3. Let $\mathbb{A} = M_2(\mathbb{C})$, the C^*-algebra of all 2×2 complex matrices. Let

$$\mathcal{U}_1 = \mathbb{R}, \ \mathcal{V}_1 = \mathbb{R}, \ \mathcal{U}_2 = \mathbb{R}, \ \mathcal{V}_2 = \mathbb{R}.$$

Define the C^*-algebra-valued bipolar b-metrics

$$d_1(x, y) = |x - y| I_2, \qquad d_2(x, y) = |x - y| I_2, \qquad \text{where } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Then $\alpha^* = \alpha, \ \alpha^*\alpha = \frac{1}{4}I_2$.

Define the contravariant map

$$T : (\mathcal{U}_1, \mathcal{V}_1, \mathbb{A}, d_1) \leftrightarrows (\mathcal{U}_2, \mathcal{V}_2, \mathbb{A}, d_2)$$

by $Tx = \frac{1}{4}x$.

Since $T(\mathcal{U}_1) \subseteq \mathcal{V}_2$ and $T(\mathcal{V}_1) \subseteq \mathcal{U}_2$, the map T is contravariant.

Moreover, for every $x \in \mathcal{U}_1$ and $y \in \mathcal{V}_1$,

$$d_2(Tx, Ty) = \left| \frac{1}{4}x - \frac{1}{4}y \right| I_2 = \frac{1}{4}|x - y| I_2,$$

while

$$\alpha^* d_1(x, y)\alpha = \alpha \left(|x - y| I_2\right) \alpha = |x - y|\alpha^2 = \frac{1}{4}|x - y| I_2.$$

Hence, $d_2(Tx, Ty) = \alpha^* d_1(x, y)\alpha$, and therefore $d_2(Tx, Ty) \preceq alpha^* d_1(x, y)\alpha$.

If the coefficient of the bipolar b-metric satisfies $b = 1$, then

$$\|b\| \, \|\alpha\|^2 = 1 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1.$$

Therefore, T is a contravariant contraction.

Theorem 2.4. *Let $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ be a C^*-algebra-valued bipolar $b-$metric space and given a covariant contraction $T : (\mathcal{U}, \mathcal{V}, \mathbb{A}, d) \rightrightarrows (\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$. Then, T has a unique fixed point.*

Proof. Let $x_0 \in \mathcal{U}$ and $y_0 \in \mathcal{V}$, define a bisequence $(\{x_n\}, \{y_n\})$ on $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ by

$$x_{n+1} = Tx_n \text{and} y_{n+1} = Ty_n,$$

for all $n \in \mathbb{N}$. We have

$$d(x_n, y_n) = d(Tx_{n-1}, Ty_{n-1}) \preceq \alpha^* d(x_{n-1}, y_{n-1})\alpha \preceq \cdots \preceq (\alpha^*)^n d(x_0, y_0)\alpha^n$$

and

$$d(x_{n+1}, y_n) = d(Tx_n, Ty_{n-1}) \preceq \alpha^* d(x_n, y_{n-1})\alpha \preceq \cdots \preceq (\alpha^*)^n d(x_1, y_0)\alpha^n.$$

Let $n \leq m$, $n, m \in \mathbb{Z}$, we get

$$\begin{aligned}
&d(x_{n+m}, y_n) \preceq b[d(x_{n+m}, y_{n+1}) + d(x_n, y_{n+1}) + d(x_n, y_n)]\\
&= bd(x_{n+m}, y_{n+1}) + b(\alpha^*)^n[d(x_0, y_1) + d(x_0, y_0)]\alpha^n\\
&\preceq b^2[d(x_{n+m}, y_{n+2}) + d(x_{n+1}, y_{n+2}) + d(x_{n+1}, y_{n+1})] + b(\alpha^*)^n[d(x_0, y_1) + d(x_0, y_0)]\alpha^n\\
&\preceq b^2 d(x_{n+m}, y_{n+2}) + b^2(\alpha^*)^{n+1}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+1} + b(\alpha^*)^n\\
&[d(x_0, y_1) + d(x_0, y_0)]\alpha^n\\
&\vdots\\
&\preceq b^m d(x_{n+m}, y_{n+m}) + b^m(\alpha^*)^{n+m-1}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+m-1} +\\
&\cdots + b^2(\alpha^*)^{n+1}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+1} + b(\alpha^*)^n[d(x_0, y_1) + d(x_0, y_0)]\alpha^n\\
&\preceq b^m(\alpha^*)^{n+m}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+m} + b^{m-1}(\alpha^*)^{n+m-1}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+m-1} +\\
&\cdots + b^2(\alpha^*)^{n+1}[d(x_0, y_1) + d(x_0, y_0)]\alpha^{n+1} + b(\alpha^*)^n[d(x_0, y_1) + d(x_0, y_0)]\alpha^n\\
&\preceq b \sum_{k=n}^{n+m} b^{k-n}(\alpha^*)^k[d(x_0, y_1) + d(x_0, y_0)]\alpha^k\\
&\preceq \|b\| \sum_{k=n}^{n+m} \|b\|^{k-n}\|(\alpha^*)\|^k\|[d(x_0, y_1) + d(x_0, y_0)]\|^{\frac{1}{2}}\|[d(x_0, y_1) + d(x_0, y_0)]\|^{\frac{1}{2}}\|\alpha\|^k I_{\mathbb{A}}\\
&\preceq \|b\|^{1-n} \sum_{k=n}^{n+m} \|b\|^k\|(\alpha^k)\|^2\|[d(x_0, y_1) + d(x_0, y_0)]^{\frac{1}{2}}\|^2 I_{\mathbb{A}}\\
&= \|b\|^{1-n}\|[d(x_0, y_1) + d(x_0, y_0)]^{\frac{1}{2}}\|^2 \sum_{k=n}^{n+m} \|b\|^k\|(\alpha^2)\|^k I_{\mathbb{A}} \to \theta \text{ as } n, m \to +\infty.
\end{aligned}$$

and

$$\begin{aligned}
&d(x_n, y_{n+m}) \preceq b[d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+m})]\\
&= b(\alpha^*)^n[d(x_1, y_0) + d(x_0, y_0)]\alpha^n + bd(x_{n+1}, y_{n+m})\\
&\preceq b(\alpha^*)^n[d(x_1, y_0) + d(x_0, y_0)]\alpha^n + b^2[d(x_{n+1}, y_{n+1}) + d(x_{n+2}, y_{n+1}) + d(x_{n+2}, y_{n+m})]\\
&\preceq b^2(\alpha^*)^{n+1}[d(x_1, y_0) + d(x_0, y_0)]\alpha^{n+1} + b(\alpha^*)^n[d(x_1, y_0) + d(x_0, y_0)]\alpha^n\\
&+ b^2 d(x_{n+2}, y_{n+m})\\
&\vdots\\
&\preceq +b^m(\alpha^*)^{n+m-1}[d(x_1, y_0) + d(x_0, y_0)]\alpha^{n+m-1} +\\
&\cdots + b^2(\alpha^*)^{n+1}[d(x_1, y_0) + d(x_0, y_0)]\alpha^{n+1} + b(\alpha^*)^n[d(x_1, y_0) + d(x_0, y_0)]\alpha^n
\end{aligned}$$

$$+ b^m d(x_{n+m}, y_{n+m})$$
$$\preceq b^m(\alpha^*)^{n+m}[d(x_1,y_0)+d(x_0,y_0)]\alpha^{n+m} + b^{m-1}(\alpha^*)^{n+m-1}[d(x_1,y_0)+d(x_0,y_0)]\alpha^{n+m-1}+$$
$$\cdots + b^2(\alpha^*)^{n+1}[d(x_1,y_0)+d(x_0,y_0)]\alpha^{n+1} + b(\alpha^*)^n[d(x_1,y_0)+d(x_0,y_0)]\alpha^n$$
$$\preceq b\sum_{k=n}^{n+m} b^{k-n}(\alpha^*)^k[d(x_1,y_0)+d(x_0,y_0)]\alpha^k$$
$$\preceq \|b\|\sum_{k=n}^{n+m}\|b\|^{k-n}\|(\alpha^*)\|^k\|[d(x_1,y_0)+d(x_0,y_0)]\|^{\frac{1}{2}}\|[d(x_1,y_0)+d(x_0,y_0)]\|^{\frac{1}{2}}\|\alpha\|^k I_{\mathbb{A}}$$
$$\preceq \|b\|^{1-n}\sum_{k=n}^{n+m}\|b\|^k\|(\alpha^k)\|^2\|[d(x_1,y_0)+d(x_0,y_0)]^{\frac{1}{2}}\|^2 I_{\mathbb{A}}$$
$$= \|b\|^{1-n}\|[d(x_1,y_0)+d(x_0,y_0)]^{\frac{1}{2}}\|^2\sum_{k=n}^{n+m}\|b\|^k\|(\alpha^2)\|^k I_{\mathbb{A}} \to \theta \text{ as } n,m\to+\infty.$$

This implies that $(\{x_n\},\{y_n\})$ is a Cauchy bisequence in $(\mathcal{U},\mathcal{V},\mathbb{A},d)$. Hence, there exists $z\in\mathcal{U}\cup\mathcal{V}$ such that

$$\lim_{n\to+\infty} d(x_n,z)=\theta \text{ and } \lim_{n\to+\infty} d(y_n,z)=\theta.$$

Now, we shall show that z is fixed point of T. Letting $n\to+\infty$ and using the concept of continuity of the function of T. We have $d(z,Tz)=\theta$.
For uniqueness, let $y\in\mathcal{V}$ be another fixed point of T where

$$d(z,y)=d(Tz,Ty)\preceq\alpha^* d(z,y)\alpha$$

and

$$\|d(z,y)\|\le\|\alpha^* d(z,y)\alpha\|\le\|\alpha\|^2\|d(z,y)\|<\frac{1}{\|b\|}\|d(z,y)\|<\|d(z,y)\|,$$

so, $z=y$. □

EXAMPLE 2.5. Let $\mathcal{U}=]2,+\infty[$, $\mathcal{V}=]1,+\infty[$, $\mathbb{A}=\mathbb{R}^2$ and $d:\mathcal{U}\times\mathcal{V}\to\mathbb{A}_+$ a C^*-algebra-valued bipolar $b-$metric defined by $d(x,y)=((x+y)^2,0)$ with $b=(2,0)$. Define $T:(\mathcal{U},\mathcal{V},\mathbb{A},d)\rightrightarrows(\mathcal{U},\mathcal{V},\mathbb{A},d)$ by $Tx=\dfrac{x}{3}+1$, for all $x\in\mathcal{U}\cup\mathcal{V}$. Then we have

$$\begin{aligned} d(Tx,Ty) &= \left(\left(\frac{x}{3}-\frac{x}{3}\right)^2,0\right)\\ &= \left(\frac{1}{3},0\right)d(x,y)\left(\frac{1}{3},0\right)\\ &= \alpha^* d(x,y)\alpha \end{aligned}$$

where $\alpha=\left(\dfrac{1}{3},0\right)$, the mapping T satisfies hypothesis of Theorem 2.4 and T has a unique fixed point $x=\dfrac{3}{2}$.

THEOREM 2.6. *Let $(\mathcal{U},\mathcal{V},\mathbb{A},d)$ be a C^*-algebra-valued bipolar $b-$metric space and given a contravariant contraction $T:(\mathcal{U},\mathcal{V},\mathbb{A},d)\leftrightarrows(\mathcal{U},\mathcal{V},\mathbb{A},d)$, such that the mapping $T:\mathcal{U}\cup\mathcal{V}\to\mathcal{U}\cup\mathcal{V}$ satisfying*

$$d(Tx,Ty)\preceq\alpha^* d(x,y)\alpha,$$

with $\|b\|\|\alpha\|^2 < 1$. *Then* T *has a fixed point in* $\mathcal{U}$.

Proof. Assume $\mathbb{A} \neq \{\theta\}$, let $x_0 \in \mathcal{U}$ and $y_0 \in \mathcal{V}$, define a bisequence $(\{x_n\}, \{y_n\})$ on $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ by $y_n = Tx_n$ and $x_{n+1} = Ty_n$, for all $n \in \mathbb{N}$. We have

$$\begin{aligned} d(x_n, y_n) = d(Tx_{n-1}, Ty_n) &\preceq \alpha^* d(x_n, y_{n-1})\alpha \\ &= \alpha^* d(Tx_{n-1}, Ty_{n-1})\alpha \\ &\preceq (\alpha^*)^2 d(x_{n-1}, y_{n-1})\alpha^2 \\ &\vdots \\ &\preceq (\alpha^*)^{2n} d(x_0, y_0)\alpha^{2n}, \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, y_n) = d(Tx_n, Ty_n) &\preceq \alpha^* d(x_n, y_n)\alpha \\ &= \alpha^* d(Tx_{n-1}, Ty_{n-1})\alpha \\ &\preceq (\alpha^*)^{2n+1} d(x_0, y_0)\alpha^{2n+1}. \end{aligned}$$

Let $n \leq m$, $n, m \in \mathbb{N}$, we get

$$\begin{aligned} &d(x_{n+m}, y_n) \preceq b[d(x_{n+m}, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_n)] \\ &= bd(x_{n+m}, y_{n+1}) + b(\alpha^*)^{2n+2}[d(x_0, y_0)]\alpha^{2n+2} + b(\alpha^*)^{2n+1}[d(x_0, y_0)]\alpha^{2n+1} \\ &\preceq b^2[d(x_{n+m}, y_{n+2}) + d(x_{n+2}, y_{n+2}) + d(x_{n+2}, y_{n+1})] + b(\alpha^*)^{2n+2}[d(x_0, y_0)]\alpha^{2n+2} \\ &+ b(\alpha^*)^{2n+1}[d(x_0, y_0)]\alpha^{2n+1} \\ &\preceq b^2 d(x_{n+m}, y_{n+2}) + b^2(\alpha^*)^{2n+4}[d(x_0, y_0)+]\alpha^{2n+4} + b^2(\alpha^*)^{2n+3}[d(x_0, y_1)]\alpha^{2n+3} \\ &+ b(\alpha^*)^{2n+2}[d(x_0, y_0)]\alpha^{2n+2} + b(\alpha^*)^{2n+1}[d(x_0, y_0)]\alpha^{2n+1} \\ &\vdots \\ &\preceq b^m d(x_{n+m}, y_{n+m-1}) + b^m(\alpha^*)^{2n+2m-2}[d(x_0, y_0)]\alpha^{2n+2m-2} + \\ &\cdots + b^2(\alpha^*)^{2n+1}[d(x_0, y_0)]\alpha^{2n+1} \\ &\preceq b^m(\alpha^*)^{2n+2m-1}[d(x_0, y_0)]\alpha^{2n+2m-1} + b^{m-1}(\alpha^*)^{2n+2m-2}[d(x_0, y_0)]\alpha^{2n+2m-2} + \\ &\cdots + b^2(\alpha^*)^{2n+1}[d(x_0, y_0)]\alpha^{2n+1} \\ &\preceq b \sum_{k=2n+1}^{2n+2m-1} b^{k-n}(\alpha^*)^k[d(x_0, y_0)]\alpha^k \\ &\preceq \|b\| \sum_{k=2n+1}^{2n+2m-1} \|b\|^{k-n}\|(\alpha^*)\|^k\|[d(x_0, y_0)]\|^{\frac{1}{2}}\|[d(x_0, y_0)]\|^{\frac{1}{2}}\|\alpha\|^k I_{\mathbb{A}} \\ &\preceq \|b\|^{1-n} \sum_{k=2n+1}^{2n+2m-1} \|b\|^k\|(\alpha^k)\|^2\|[d(x_0, y_0)]^{\frac{1}{2}}\|^2 I_{\mathbb{A}} \\ &= \|b\|^{1-n}\|[d(x_0, y_0)]^{\frac{1}{2}}\|^2 \sum_{k=2n}^{2n+2m-1} \|b\|^k\|(\alpha^2)\|^k I_{\mathbb{A}} \to \theta \text{ as } n, m \to +\infty, \end{aligned}$$

and

$$d(x_n, y_{n+m}) = d(Ty_{n-1}, Tx_{n+m}) \preceq \alpha^* d(x_{n+m}, y_{n-1})\alpha.$$

This implies that $(\{x_n\},\{y_n\})$ is a Cauchy bisequence in $(\mathcal{U},\mathcal{V},\mathcal{A},d)$. Hence there exist $z\in\mathcal{U}\cup\mathcal{V}$ such that

$$\lim_{n\to+\infty} d(x_n,z)=\theta \text{ and } \lim_{n\to+\infty} d(y_n,z)=\theta.$$

Now, we shall show that z is fixed point of T.

Letting $n\to+\infty$ and using the concept of continuity of the function of T. We have $d(z,Tz)=\theta$.

For uniqueness, let $y\in\mathcal{V}$ be another fixed point of T where

$$d(z,y)=d(Tz,Ty)\preceq\alpha^* d(z,y)\alpha$$

and

$$\|d(z,y)\|\le\|\alpha^* d(z,y)\alpha\|=\|\alpha\|^2\|d(z,y)\|<\frac{1}{\|\beta\|}\|d(z,y)\|<\|d(z,y)\|,$$

now, we conclude that $z=y$. □

Example 2.7. Let $\mathcal{U}=[0,1]$, $\mathcal{V}=[1,2]$, $\mathbb{A}=\mathbb{M}_2(\mathbb{C})$ and $d:\mathcal{U}\times\mathcal{V}\to\mathbb{A}_+$ a C^*-algebra-valued bipolar $b-$metric defined by $d(x,y)=diag(|x-y|,\lambda|x-y|)$, $\lambda\ge 0$ with $b=2$.
Define $T:(\mathcal{U},\mathcal{V},\mathbb{A},d)\rightrightarrows(\mathcal{U},\mathcal{V},\mathbb{A},d)$ by $Tx=\dfrac{x+1}{2}$, for all $x\in\mathcal{U}\cup\mathcal{V}$.
Then we have

$$\begin{aligned} d(Tx,Ty)&=diag\left(\left|\left(\frac{x+1}{2}-\frac{y+1}{2}\right)\right|,\lambda\left(\left|\frac{x+1}{2}-\frac{y+1}{2}\right)\right|\right)\\ &=diag\frac{1}{2}(|x-y|,\lambda|x-y|)\\ &=diag\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)d(x,y)diag\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\end{aligned}$$

where $\alpha=diag\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$, the mapping T satisfies hypothesis of Theorem 2.6 and T has a unique fixed point.

Next, we present a generalization of Kannan's type contraction [4].

Theorem 2.8. *Let $(\mathcal{U},\mathcal{V},\mathbb{A},d)$ be a C^*-algebra-valued bipolar $b-$metric space and given a contravariant contraction $T:(\mathcal{U},\mathcal{V},\mathbb{A},d)\leftrightarrows(\mathcal{U},\mathcal{V},\mathbb{A},d)$ such that the mapping $T:\mathcal{U}\cup\mathcal{V}\to\mathcal{U}\cup\mathcal{V}$ satisfying*

$$d(Tx,Ty)\preceq\alpha(d(x,Tx)+d(y,Ty)),$$

with $\|b\|\|\alpha\|<\frac{1}{2}$. Then T has a unique fixed point.

Proof. Assume $\mathbb{A}\neq\{\theta\}$, let $x_0\in\mathcal{U}$ and $y_0\in\mathcal{V}$, define a bisequence $(\{x_n\},\{y_n\})$ on $(\mathcal{U},\mathcal{V},\mathbb{A},d)$ by $y_n=Tx_n$ and $x_{n+1}=Ty_n$, $\forall n\in\mathbb{N}$.
We have

$$\begin{aligned} d(x_n,y_n)&=d(Tx_{n-1},Ty_n)\\ &\preceq\alpha(d(x_n,Tx_n)+d(y_{n-1},Ty_{n-1}))\\ &=\alpha(d(x_n,y_n)+d(x_n,y_{n-1}))\\ &\Rightarrow(1-\alpha)d(x_n,y_n)\preceq\alpha d(x_n,y_{n-1}).\end{aligned}$$

By Lemma 1.2, we have $d(x_n, y_n) \preceq \alpha(1-\alpha)^{-1}d(x_n, y_{n-1})$ and

$$\begin{aligned} d(x_n, y_{n-1}) = d(Tx_{n-1}, Ty_{n-1}) &\preceq \alpha[d(x_{n-1}, Tx_{n-1}) + d(Ty_{n-1}, y_{n-1})] \\ &= \alpha[d(x_{n-1}, y_{n-1}) + d(x_n, y_{n-1})]. \end{aligned}$$

This implies that

$$(1-\alpha)d(x_n, y_{n-1}) \preceq \alpha d(x_{n-1}, y_{n-1}),$$

and

$$d(x_n, y_{n-1}) \preceq \alpha(1-\alpha)^{-1}d(x_{n-1}, y_{n-1})$$

then

$$d(x_n, y_n) \preceq (\alpha(1-\alpha)^{-1})^{2n}d(x_0, y_0)$$

and

$$d(x_n, y_{n-1}) \preceq (\alpha(1-\alpha)^{-1})^{2n-1}d(x_0, y_0).$$

Let $n \leq m$, $n, m \in \mathbb{N}$, we get

$$\begin{aligned} d(x_n, y_m) &\preceq b[d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_m)] \\ &\preceq b(\alpha(1-\alpha)^{-1})^{2n}d(x_0, y_0) + b(\alpha(1-\alpha)^{-1})^{2n+1}d(x_0, y_0) + bd(x_{n+1}, y_m) \\ &\vdots \\ &\preceq b(\alpha(1-\alpha)^{-1})^{2n}d(x_0, y_0) + b(\alpha(1-\alpha)^{-1})^{2n+1}d(x_0, y_0) + \\ &\quad ... + b(\alpha(1-\alpha)^{-1})^{2m}d(x_0, y_0) \\ &= b[(\alpha(1-\alpha)^{-1})^{2n} + (\alpha(1-\alpha)^{-1})^{2n+1} + ... + (\alpha(1-\alpha)^{-1})^{2m}]d(x_0, y_0) \\ &\preceq \|b\|\|\alpha\|[(1-\alpha)^{-1})^{2n} + (1-\alpha)^{-1})^{2n+1} + ... + (1-\alpha)^{-1})^{2m}]\|d(x_0, y_0)\|I_{\mathbb{A}} \\ &\preceq \|b\|\|\alpha\|[\|(1-\alpha)^{-1})\|^{2n} + \|(1-\alpha)^{-1})\|^{2n+1} + ... + \|(1-\alpha)^{-1})\|^{2m}]\|d(x_0, y_0)\|I_{\mathbb{A}} \\ &\to \theta \,\text{as}\, n, m \to +\infty, \text{since}\, \|b\|\|\alpha\| < \frac{1}{2}. \end{aligned}$$

This implies that $(\{x_n\}, \{y_n\})$ is a Cauchy bisequence in $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$. Hence there exist $z \in \mathcal{U} \cup \mathcal{V}$ such that

$$\lim_{n\to+\infty} d(x_n, z) = \theta \quad \text{and} \quad \lim_{n\to+\infty} d(y_n, z) = \theta.$$

Now, we shall show that z is fixed point of T

$$d(Tx_n, Tz) \preceq \alpha[d(z, Tz) + d(Tx_n, x_n)]$$

Letting $n \to +\infty$ and using the concept of continuity of the function of T. We have $d(z, Tz) = \theta$.

For uniqueness, let $y \in \mathcal{V}$ be another fixed point of T where

$$d(z, y) = d(Tz, Ty) \preceq \alpha[d(z, Tz) + d(y, Ty)] = \theta,$$

then $z = y$. □

Now, we prove our common fixed point result.

THEOREM 2.9. *Let $(\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ be a C^*-algebra-valued bipolar $b-$metric space and given $T, S : (\mathcal{U}, \mathcal{V}, \mathbb{A}, d) \leftrightharpoons (\mathcal{U}, \mathcal{V}, \mathbb{A}, d)$ be a contravariant contraction such that the mapping $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$ satisfying*

$$d(Tx, Sy) \preceq \alpha d(y, Sy) + \beta d(x, y) + \gamma[d(x, Tx) + d(y, Sy)], \text{ for all } x, y \in \mathcal{U} \times \mathcal{V}, x \neq y,$$

with $\alpha+\beta+2\gamma<1$. *Then* T *and* S *have a unique common fixed point.*

Proof. Let $x_0\in\mathcal{U}$ and $y_0\in\mathcal{V}$, define a bisequence $(\{x_n\},\{y_n\})$ on $(\mathcal{U},\mathcal{V},\mathbb{A},d)$ by

$$Ty_{2n}=x_{2n} \text{ and } x_{2n+1}=Sy_{2n+1}, Tx_{2n}=y_{2n+1} \text{ and } Sx_{2n+1}=y_{2n+2},$$

for all $n\in\mathbb{N}$. We have

$$\begin{aligned}
&d(y_{2n+1},x_{2n+1})=d(Tx_{2n},Sy_{2n+1})\\
&\preceq \alpha d(y_{2n+1},Sy_{2n+1})+\beta d(x_{2n},y_{2n+1})+\gamma[d(x_{2n},Tx_{2n})+d(y_{2n+1},Sy_{2n+1})]\\
&=\alpha d(y_{2n+1},x_{2n+1})+\beta d(x_{2n},y_{2n+1})+\gamma[d(x_{2n},y_{2n+1})+d(y_{2n+1},x_{2n+1})].
\end{aligned}$$

This implies that

$$d(y_{2n+1},x_{2n+1})\preceq\frac{\beta+\gamma}{1-\alpha-\gamma}d(y_{2n+1},x_{2n}). \tag{2.1}$$

Also,

$$\begin{aligned}
d(y_{2n+1},x_{2n})&=d(Tx_{2n},Sy_{2n})\\
&\preceq \alpha d(y_{2n},Sy_{2n})+\beta d(x_{2n},y_{2n})+\gamma[d(x_{2n},Tx_{2n})+d(y_{2n},Sy_{2n})]\\
&=\alpha d(y_{2n+1},x_{2n})+\beta d(x_{2n},y_{2n})+\gamma[d(x_{2n},y_{2n+1})+d(y_{2n},x_{2n})].
\end{aligned}$$

This implies that

$$d(y_{2n+1},x_{2n})\preceq\frac{\beta+\gamma}{1-\alpha-\gamma}d(y_{2n},x_{2n}). \tag{2.2}$$

Hence, from (2.1) and (2.2), we get for all $n\in\mathbb{N}$

$$d(x_{n+1},y_{n+1})\preceq\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+2}d(x_0,y_0),$$

$$d(x_n,y_{n+1})\preceq\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+1}d(x_0,y_0)$$

and

$$d(x_n,y_n)\preceq\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n}d(x_0,y_0).$$

For all $m,n\in\mathbb{N}$, we have two cases.
Case 1: If $m>n$.

$$\begin{aligned}
&d(y_n,x_m)\preceq b[d(y_n,x_n)+d(y_{n+1},x_n)+d(y_{n+1},x_m)]\\
&\preceq b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n}d(x_0,y_0)+b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+1}d(x_0,y_0)+bd(y_{n+1},x_m)\\
&\preceq b\left[\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n}+\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+1}\right]d(x_0,y_0)+b^2(d(y_{n+1},x_{n+1})\\
&+d(y_{n+2},x_{n+1})+d(y_{n+2},x_m))\\
&\preceq b\left[\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n}+\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+1}\right]d(x_0,y_0)+\\
&b^2\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+2}d(x_0,y_0)+b^2\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+3}d(x_0,y_0)+b^2d(y_{n+2},x_m)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\preceq b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n}\left[1+b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2}+b^2\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{4}+\cdots\right] \\
&d(x_0,y_0)+b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2n+1} \\
&\left[1+b\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{2}+b^2\left(\frac{\beta+\gamma}{1-\alpha-\gamma}\right)^{4}+\cdots\right]d(x_0,y_0) \\
&= b\Delta^{2n}\left(\frac{1}{1-b\Delta^2}\right)d(x_0,y_0)+b\Delta^{2n+1}\left(\frac{1}{1-b\Delta^2}\right)d(x_0,y_0)
\end{aligned}$$

where $\Delta=\dfrac{\beta+\gamma}{1-\alpha-\gamma}$. Since $\dfrac{\beta+\gamma}{1-\alpha-\gamma}<1$, we obtain $\lim\limits_{n,m\to+\infty}d(y_n,x_m)=\theta$.

Case 2: If $m<n$.

$$\begin{aligned}
d(y_n,x_m) &\preceq b[d(y_{m+1},x_m)+d(y_{m+1},x_{m+1})+d(y_n,x_{m+1})] \\
&\preceq b(\frac{\beta+\gamma}{1-\alpha-\gamma})^{2m+1}d(x_0,y_0)+b(\frac{\beta+\gamma}{1-\alpha-\gamma})^{2m+2}d(x_0,y_0)+bd(y_n,x_{m+1}) \\
&\preceq b((\frac{\beta+\gamma}{1-\alpha-\gamma})^{2m+1}+(\frac{\beta+\gamma}{1-\alpha-\gamma})^{2m+2})d(x_0,y_0)+b^2(d(y_{m+2},x_{m+1}) \\
&+d(y_{m+2},x_{m+2})+d(y_n,x_{m+2})) \\
&\vdots \\
&\preceq b\Delta^{2m+1}(\frac{1}{1-b\Delta^2})d(x_0,y_0)+b\Delta^{2m+2}(\frac{1}{1-b\Delta^2})d(x_0,y_0).
\end{aligned}$$

Since $\dfrac{\beta+\gamma}{1-\alpha-\gamma}<1$, $\lim\limits_{n,m\to+\infty}d(y_n,x_m)=\theta$. This implies that $(\{x_n\},\{y_n\})$ is a Cauchy bisequence in $(\mathcal{U},\mathcal{V},\mathbb{A},d)$. Hence there exists $z\in\mathcal{U}\cup\mathcal{V}$ such that

$$\lim_{n\to+\infty}d(x_n,z)=\theta \text{ and } \lim_{n\to+\infty}d(y_n,z)=\theta.$$

Also, $\{Ty_{2n}\}=\{x_{2n}\}\to z\in\mathcal{U}\cap\mathcal{V}$, then $\{y_n\}\to z$, the continuity of T implies that $Tz=z$. Similarly, we get $Sz=z$.

Let $w\in\mathcal{U}\cap\mathcal{V}$ such that $Tw=Sw=w$, we have

$$\begin{aligned}
d(w,z)=d(Tw,Sz) &\preceq \alpha d(w,Tw)+\beta d(z,w)+\gamma[d(w,Tw)+d(z,Sz)] \\
&= \alpha d(w,w)+\beta d(z,w)+\gamma[d(w,w)+d(z,z)] \\
&= \beta d(z,w).
\end{aligned}$$

Therefore, $d(z,w)\preceq\beta d(z,w)$, which is a contradiction, then $z=w$. □

Conclusion

In this paper, we obtain some results on fixed points and common fixed points under Banach and Kannan type covariant contractions and contravariant contractions in the setting of C^*-algebra valued bipolar b-metric spaces. Our results improve and

generalize some results from the literature. Using the techniques presented in this paper, the following problems remain open:
1. Prove the Reich result of fixed point for type covariant contractions and contravariant contractions in the setting of C^*-algebra valued bipolar b-metric spaces.
2. Prove the Hardy-Rogers result of fixed point for type covariant contractions and contravariant contractions in the setting of C^*-algebra valued bipolar b-metric spaces.
3. Prove the Ćirić result of fixed point for type covariant contractions and contravariant contractions in the setting of C^*-algebra valued bipolar b-metric spaces.
Of course, other questions such as Kirk theorem of fixed point, Suzuki fixed point theorem, Berinde fixed point theorem, Jungck common fixed point theorem etc.

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