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THE PROPERTIES OF JOIN AND MEET PRESERVING MAPS

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Abstract. We investigate the properties of join and meet preserving maps in complete residuated lattices. In particular, we give their examples.

1. Introduction

Pawlak [8,9] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1-3, 10,11,14]. Bělohlávek [1,2] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. Zhang [12,13] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy posets. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-3,5-8]. Kim [5] show that join (resp. meet, meet join, join meet) preserving maps and upper (resp. lower, meet join, join meet) approximation maps are equivalent in complete residuated lattices.

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In this paper, we investigate the properties of join and meet preserving maps in complete residuated lattice. In particular, we give their examples.

2. Preliminaries

DEFINITION 2.1. ([7]) A structure $(L, \vee, \wedge, \odot, \rightarrow, \bot, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:
(L1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;
(L2) $(L, \odot, \top)$ is a monoid;
(L3) adjointness properties,i.e.

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z.$$

A map $^* : L \rightarrow L$ defined by $a^* = a \rightarrow \bot$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top^*_x(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

DEFINITION 2.2. ([12,13]) Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:
(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,
(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,
(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.
If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

EXAMPLE 2.3. (1) We define a function $e_L : L \times L \rightarrow L$ as $e_L(x, y) = x \rightarrow y$. Then (L, e_L) is a fuzzy poset.
(2) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 2.10 (9).

DEFINITION 2.4. ([12,13]) Let (X, e_X) be a fuzzy poset and $A \in L^X$.
(1) A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies
(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x\in X}(A(x) \to e_X(x,y)) \leq e_X(x_0,y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1,x)$,

(M2) $\bigwedge_{x\in X}(A(x) \to e_X(y,x)) \leq e_X(y,x_1)$.

DEFINITION 2.5. ([12,13]) Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets.

(1) $\mathcal{H} : L^X \to L^Y$ is a join preserving map if $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$, where $\mathcal{H}^{\to}(\Phi)(B) = \bigvee_{\mathcal{H}(A)=B}\Phi(A)$.

(2) $\mathcal{J} : L^X \to L^Y$ is a meet preserving map if $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

THEOREM 2.6. ([5]) *Let X and Y be two sets. Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets. Then the following statements are equivalent:*

(1) $\mathcal{H} : L^X \to L^Y$ is a join preserving map iff $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ and $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I}\mathcal{H}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

(2) $\mathcal{J} : L^X \to L^Y$ is a meet preserving map iff $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$ and $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I}\mathcal{J}(A_i)$ for all $A, A_i \in L^X$, and $\alpha \in L$.

LEMMA 2.7. ([1,2,4]) *Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.*

(1) $\odot$ is isotone in both arguments.

(2) $\to$ is antitone in the first and isotone in the second argument.

(3) $x \to y = \top$ iff $x \leq y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(6) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(7) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i) \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(8) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(9) $x \to y \leq (y \to z) \to (x \to z)$.

(10) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(12) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

3. The properties of join and meet preserving maps

THEOREM 3.1. *Let (L^X, e_{L^X}) be a fuzzy poset. Let $\mathcal{H}, \mathcal{H}^{-1} : L^X \to L^X$ be join preserving maps such that $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{H}(\top_y)(x)$ for all $x, y \in X$. Let $\mathcal{J}, \mathcal{J}^{-1} : L^X \to L^X$ be meet preserving maps such that $\mathcal{J}^{-1}(\top^*_x)(y) = \mathcal{J}(\top^*_y)(x)$ and $\mathcal{H}(\top_x)(y) = \mathcal{J}^*(\top^*_x)(y)$ for all $x, y \in X$. For each $x, y \in X, \alpha \in L, A, B \in L^X$, we have the following properties.*

(1) $\mathcal{H}(A)(y) = \bigvee_x (A(x) \odot \mathcal{H}(\top_x)(y)$ *and* $\mathcal{H}^{-1}(A)(y) = \bigvee_x (A(x) \odot \mathcal{H}^{-1}(\top_x)(y) = \bigvee_x (A(x) \odot \mathcal{H}(\top_y)(x)$.

(2) $\mathcal{J}(A)(y) = \bigwedge_x (A^*(x) \to \mathcal{J}(\top^*_x)(y)) = \bigwedge_x (\mathcal{J}^*(\top^*_x)(y) \to A(x))$ *and* $\mathcal{J}^{-1}(A)(y) = \bigwedge_x (A^*(x) \to \mathcal{J}^{-1}(\top^*_x)(y)) = \bigwedge_x (\mathcal{J}^{-1*}(\top^*_x)(y) \to A(x))$.

(3) $\mathcal{J}(\top) = \mathcal{J}^{-1}(\top) = \top$ *and* $\mathcal{H}(\bot) = \mathcal{H}^{-1}(\bot) = \bot$.

(4) $\mathcal{J}(A) = (\mathcal{H}(A^*))^*$ *and* $\mathcal{H}(A) = (\mathcal{J}(A^*))^*$.

(5) $\mathcal{H}(\alpha \to A) \geq \alpha \to \mathcal{H}(A)$ *and* $\mathcal{J}(\alpha \odot A) \geq \alpha \odot \mathcal{J}(A)$.

(6) $\mathcal{J}(\top_x \to \alpha)(y) = H(\top_x)(y) \to \alpha = \mathcal{H}^*(\top_x \odot \alpha^*)(y)$.

(7) $\bigwedge_{\alpha \in L}((A(y) \to \alpha) \to \alpha) = A(y)$.

(8) $\mathcal{J}(A) = \bigwedge_{\alpha \in L}(\mathcal{H}(A \to \alpha) \to \alpha)$.

(9) $\mathcal{J}(A \to \alpha) = \mathcal{H}(A) \to \alpha$.

(10) $\mathcal{H}(A \to \alpha) \leq \mathcal{J}(A) \to \alpha$.

(11) $e_{L^X}(\mathcal{H}(A), B) = e_{L^X}(A, \mathcal{J}^{-1}(B))$ *and* $e_{L^X}(\mathcal{H}^{-1}(A), B) = e_{L^X}(A, \mathcal{J}(B))$.

(12) $e_{L^X}(A, B) \leq e_{L^X}(\mathcal{H}(A), \mathcal{H}(B))$.

(13) $e_{L^X}(A, B) \leq e_{L^X}(\mathcal{J}(A), \mathcal{J}(B))$.

Proof. (1) For $A = \bigvee_{x \in X}(A(x) \odot \top_x)$, we have

$$\begin{aligned} \mathcal{H}(A)(y) &= \mathcal{H}(\textstyle\bigvee_{x \in X}(A(x) \odot \top_x))(y) = \bigvee_{x \in X}(A(x) \odot \mathcal{H}(\top_x)(y)) \\ \mathcal{H}^{-1}(A)(y) &= \mathcal{H}^{-1}(\textstyle\bigvee_{x \in X}(A(x) \odot \top_x))(y) = \bigvee_{x \in X}(A(x) \odot \mathcal{H}^{-1}(\top_x)(y)) \\ &= \textstyle\bigvee_{x \in X}(A(x) \odot \mathcal{H}(\top_y)(x)) \end{aligned}$$

(2) For $A = \bigwedge_{x \in X}(A^*(x) \to \top^*_x)$, we have

$$\begin{aligned} \mathcal{J}(A)(y) &= \mathcal{J}(\textstyle\bigwedge_{x \in X}(A^*(x) \to \top^*_x))(y) = \bigwedge_{x \in X}(A^*(x) \to \mathcal{J}(\top^*_x)(y)) \\ &= \textstyle\bigwedge_{x \in X}(\mathcal{J}^*(\top^*_x)(y) \to A(x)) \end{aligned}$$

(3) $\mathcal{J}(\top)(y) = \bigwedge_x (\mathcal{J}^*(\top^*_x)(y) \to \top(x)) = \top$ and other cases are similarly proved.

(4) By Lemma 2.7(10), we have

$$\begin{aligned}(\mathcal{H}(A^*)(y))^* &= \Big(\bigvee_{x\in X}(A^*(x)\odot\mathcal{H}(\top_x)(y))\Big)^*\\ &= \bigwedge_{x\in X}(A^*(x)\odot\mathcal{H}(\top_x)(y)))^*\\ &= \bigwedge_{x\in X}(A^*(x)\to\mathcal{H}^*(\top_x)(y)) = \bigwedge_{x\in X}(A^*(x)\to\mathcal{J}(\top^*_x)(y)\\ &= \mathcal{J}(\bigwedge_{x\in X}(A^*(x)\to\top^*_x)(y) = \mathcal{J}(A)(y).\end{aligned}$$

$$\begin{aligned}(\mathcal{J}(A^*)(y))^* &= \Big(\bigwedge_{x\in X}(A(x)\to\mathcal{J}(\top^*_x)(y))\Big)^*\\ &= \bigvee_{x\in X}(A(x)\odot\mathcal{J}^*(\top^*_x)(y))\\ &= \bigvee_{x\in X}(A(x)\odot\mathcal{H}(\top_x)(y)) = \mathcal{H}(A)(y).\end{aligned}$$

(5) Since $\alpha\odot(\alpha\to A(x))\odot\mathcal{H}(\top_x)(y)\leq A(x)\odot\mathcal{H}(\top_x)(y)$ iff $(\alpha\to A(x))\odot\mathcal{H}(\top_x)(y)\leq\alpha\to A(y)\odot\mathcal{H}(\top_x)(y)$, then $\mathcal{H}(\alpha\to A)\leq\alpha\to\mathcal{H}(A)$.

Since $\mathcal{J}^*(\top^*_x)(y)\odot(\mathcal{J}^*(\top^*_x)(y)\to A(x))\odot\alpha\leq A(x)\odot\alpha$ iff $(\mathcal{J}^*(\top^*_x)(y)\to A(x))\odot\alpha\leq\mathcal{J}^*(\top^*_x)(y)\to A(x)\odot\alpha$, then $\mathcal{J}(A)\odot\alpha\leq\mathcal{J}(A\odot\alpha)$.

(6) By (4), $\mathcal{J}(\top_x\to\alpha)(y)=\mathcal{H}^*(\top_x\odot\alpha^*)(y)$ and

$$\begin{aligned}\mathcal{J}(\top_x\to\alpha)(z) &= \bigwedge_{y\in X}(\mathcal{J}^*(\top^*_y)(z)\to(\top_x\to\alpha)(y))\\ &= \mathcal{J}^*(\top^*_x)(z)\to\alpha=\mathcal{H}(\top_x)(z)\to\alpha.\end{aligned}$$

(7) Since $A(y)\odot(A(y)\to\alpha)\leq\alpha$ iff $A(y)\leq(A(y)\to\alpha)\to\alpha$, then

$$\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha)\geq A(y).$$

Put $\alpha=A(y)$. Then $\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha)\leq(A(y)\to A(y))\to A(y))=A(y)$. Hence $\bigwedge_{\alpha\in L}((A(y)\to\alpha)\to\alpha)=A(y)$.

(8)

$$\begin{aligned}&\bigwedge_{\alpha\in L}(\mathcal{H}(A\to\alpha)(x)\to\alpha)\\ &= \bigwedge_{\alpha\in L}(\bigvee_{x\in X}(\mathcal{H}(\top_x)(y)\odot(A\to\alpha)(x))\to\alpha)\\ &= \bigwedge_{\alpha\in L}\bigwedge_{x\in X}(\mathcal{H}(\top_x)(y)\to((A(x)\to\alpha)\to\alpha))\\ &= \bigwedge_{x\in X}(\mathcal{H}(\top_x)(y)\to\bigwedge_{\alpha\in L}((A(x)\to\alpha)\to\alpha))\\ &= \bigwedge_{x\in X}(\mathcal{H}(\top_x)(y)\to A(x)) = \bigwedge_{x\in X}(A^*(x)\to\mathcal{J}(\top^*_x)(y))\\ &= \mathcal{J}(A)(x).\end{aligned}$$

(9)

$$\begin{aligned}\mathcal{J}(A\to\alpha)(z) &= \bigwedge_{x\in X}((A\to\alpha)^*(x)\to\mathcal{J}(\top^*_x)(z))\\ &= \bigwedge_{x\in X}(\mathcal{J}^*(\top^*_x)(z)\to(A\to\alpha)(x))\\ &= \bigvee_{x\in X}(\mathcal{J}^*(\top^*_x)(z)\odot A(x))\to\alpha\\ &= \bigvee_{x\in X}(\mathcal{H}(\top_x)(z)\odot A(x))\to\alpha\\ &= \mathcal{H}(A)(z)\to\alpha\end{aligned}$$

(10) Since $(a \to b) \odot c \odot (c \to a) \le (a \to b) \odot a \le b$, $(a \to b) \odot c \le (c \to a) \to b$. Thus

$$\begin{array}{ll} \mathcal{H}(A \to \alpha) & = \bigvee_{x\in X}((A \to \alpha)(x) \odot \mathcal{H}(\top_x)(y)) \\ & \le \bigvee_{x\in X}((\mathcal{H}(\top_x)(y) \to A(x)) \to \alpha) \\ & = (\bigwedge_{x\in X}(\mathcal{H}(\top_x)(y) \to A(x)) \to \alpha \\ & = (\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(\top_x^*)(y))) \to \alpha = \mathcal{J}(A) \to \alpha. \end{array}$$

(11)

$$\begin{array}{ll} e_{L^X}(\mathcal{H}(A), B) & = \bigwedge_{y\in X}(\mathcal{H}(A)(y) \to B(y)) \\ & = \bigwedge_{y\in X}(\bigvee_{x\in X}(A(x) \odot \mathcal{H}(\top_x)(y)) \to B(y)) \\ & = \bigwedge_{y\in X} \bigwedge_{x\in X}(A(x) \to (\mathcal{H}(\top_x)(y)) \to B(y))) \\ & = \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_y^*)(x)) \to B(y))) \\ & = e_{L^X}(A, \mathcal{J}^{-1}(B)). \end{array}$$

(12)

$$\begin{array}{l} \mathcal{H}(A)(y) \odot e_{L^X}(A, B) = \bigvee_x(\mathcal{H}(\top_x)(y) \odot A(x)) \odot (A(x) \to B(x)) \\ \le \bigvee_x(\mathcal{H}(\top_x)(y) \odot B(x)) = \mathcal{H}(B)(y). \end{array}$$

(13)

$$\begin{array}{l} \mathcal{J}^*(\top_x^*)(y) \odot (\mathcal{J}^*(\top_x^*)(y) \to A(x)) \odot (A(x) \to B(x))) \le B(x) \\ \text{iff } (\mathcal{J}^*(\top_x^*)(y) \to A(x)) \odot (A(x) \to B(x))) \le \mathcal{J}^*(\top_x^*)(y) \to B(x) \\ \text{iff } A(x) \to B(x) \le (\mathcal{J}^*(\top_x^*)(y) \to A(y)) \to (\mathcal{J}^*(\top_x^*)(y) \to B(y)) \\ \text{iff } e_{L^X}(A, B) \le e_{L^X}(\mathcal{J}(A), \mathcal{J}(B)). \end{array}$$

□

THEOREM 3.2. *Let (L^X, e_{L^X}) be a fuzzy poset. Let $\mathcal{H}, \mathcal{H}^{-1} : L^X \to L^X$ be join preserving maps such that $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{H}(\top_y)(x)$ for all $x, y \in X$. Let $\mathcal{J}, \mathcal{J}^{-1} : L^X \to L^X$ be meet preserving maps such that $\mathcal{J}^{-1}(\top_x^*)(y) = \mathcal{J}(\top_y^*)(x)$ for all $x, y \in X$. For all $x, y, z \in X$ and $A \in L^X$, we have the following properties.*

(1) If $\top_x \le \mathcal{H}(\top_x)$ for all $x \in X$, then $A \le \mathcal{H}(A)$ and $A \le \mathcal{H}^{-1}(A)$.

(2) If $\mathcal{J}(\top_x^*) \le \top_x^*$ for all $x \in X$, then $\mathcal{J}(A) \le A$ and $\mathcal{J}^{-1}(A) \le A$.

(3) $\bigvee_{y\in X} \mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z) \le \mathcal{H}(\top_x)(z)$ for all $x, z \in X$ iff $\mathcal{H}(\mathcal{H}(\top_x)) \le \mathcal{H}(\top_x)$ iff $\mathcal{H}^{-1}(\mathcal{H}^{-1}(\top_x)) \le \mathcal{H}^{-1}(\top_x)$ iff $\mathcal{H}(\mathcal{H}(A)) \le \mathcal{H}(A)$ iff $\mathcal{H}^{-1}(\mathcal{H}^{-1}(A)) \le \mathcal{H}^{-1}(A)$.

(4) $\bigvee_{x\in X} \mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_x)(z) \le \mathcal{H}(\top_y)(z)$ for all $x, z \in X$ iff $\mathcal{H}(\mathcal{H}^{-1}(\top_x)) \le \mathcal{H}(\top_x)$ iff $\mathcal{H}(\mathcal{H}^{-1}(\top_x)) \le \mathcal{H}^{-1}(\top_x)$ iff $\mathcal{H}(\mathcal{H}^{-1}(A)) \le \mathcal{H}(A)$ iff $\mathcal{H}(\mathcal{H}^{-1}(A)) \le \mathcal{H}^{-1}(A)$.

(5) $\bigvee_{x\in X}\mathcal{H}(\top_y)(x)\odot\mathcal{H}(\top_z)(x)\leq\mathcal{H}(\top_y)(z)$ *for all* $x,z\in X$ *iff* $\mathcal{H}^{-1}(\mathcal{H}(\top_x))\leq\mathcal{H}^{-1}(\top_x)$ *iff* $\mathcal{H}^{-1}(\mathcal{H}(\top_x))\leq\mathcal{H}(\top_x)$ *iff* $\mathcal{H}^{-1}(\mathcal{H}(A))\leq\mathcal{H}^{-1}(A)$ *iff* $\mathcal{H}^{-1}(\mathcal{H}(A))\leq\mathcal{H}(A)$.

(6) $\bigvee_{x\in X}\mathcal{J}^*(\top^*_x)(z)\odot\mathcal{J}^*(\top^*_y)(x)\leq\mathcal{J}^*(\top^*_x)(z)$ *for all* $y,z\in X$ *iff* $\mathcal{J}(\top^*_x)\leq\mathcal{J}(\mathcal{J}(\top^*_x))$ *iff* $\mathcal{J}^{-1}(\top^*_x)\leq\mathcal{J}^{-1}(\mathcal{J}^{-1}(\top^*_x))$ *iff* $\mathcal{J}(A)\leq\mathcal{J}(\mathcal{J}(A))$ *iff* $\mathcal{J}^{-1}(A)\leq\mathcal{J}^{-1}(\mathcal{J}^{-1}(A))$.

(7) $\bigvee_{z\in X}\mathcal{J}^*(\top^*_z)(x)\odot\mathcal{J}^*(\top^*_z)(y)\leq\mathcal{J}^*(\top^*_x)(y)$ *for all* $x,y\in X$ *iff* $\mathcal{J}(\top^*_x)\leq\mathcal{J}(\mathcal{J}^{-1}(\top^*_x))$ *iff* $\mathcal{J}^{-1}(\top^*_x)\leq\mathcal{J}(\mathcal{J}^{-1}(\top^*_x))$ *iff* $\mathcal{J}(A)\leq\mathcal{J}(\mathcal{J}^{-1}(A))$ *iff* $\mathcal{J}^{-1}(A)\leq\mathcal{J}(\mathcal{J}^{-1}(A))$.

(8) $\bigvee_{x\in X}\mathcal{J}^*(\top^*_y)(x)\odot\mathcal{J}^*(\top^*_z)(x)\leq\mathcal{J}^*(\top^*_y)(z)$ *for all* $y,z\in X$ *iff* $\mathcal{J}^{-1}(\top^*_x)\leq\mathcal{J}^{-1}(\mathcal{J}(\top^*_x))$ *iff* $\mathcal{J}(\top^*_x)\leq\mathcal{J}^{-1}(\mathcal{J}(\top^*_x))$ *iff* $\mathcal{J}^{-1}(A)\leq\mathcal{J}^{-1}(\mathcal{J}(A))$ *iff* $\mathcal{J}(A)\leq\mathcal{J}^{-1}(\mathcal{J}(A))$.

(9) *If* $\mathcal{H}(\top_x)(y)=\mathcal{J}^*(\top^*_x)(y)$ *for all* $x,y\in X$, *then* $\bigvee_{y\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)\leq\mathcal{H}(\top_x)(z)$ *for all* $x,z\in X$ *iff* $\mathcal{H}^{-1}(\top_x)\leq\mathcal{J}(\mathcal{H}^{-1}(\top_x))$ *iff* $\mathcal{H}(\top_x)\leq\mathcal{J}^{-1}(\mathcal{H}(\top_x))$ *iff* $\mathcal{H}^{-1}(A)\leq\mathcal{J}(\mathcal{H}^{-1}(A))$ *iff* $\mathcal{H}(A)\leq\mathcal{J}^{-1}(\mathcal{H}(A))$.

(10) *If* $\mathcal{H}(\top_x)(y)=\mathcal{J}^*(\top^*_x)(y)$ *for all* $x,y\in X$, *then* $\bigvee_{x\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\leq\mathcal{H}(\top_y)(z)$ *for all* $y,z\in X$ *iff* $\mathcal{H}^{-1}(\top_x)\leq\mathcal{J}^{-1}(\mathcal{H}^{-1}(\top_x))$ *iff* $\mathcal{H}^{-1}(\top_x)\leq\mathcal{J}^{-1}(\mathcal{H}(\top_x))$ *iff* $\mathcal{H}(\mathcal{J}(\top^*_x))\leq\mathcal{J}^{-1}(\top^*_x)$ *iff* $\mathcal{H}(\mathcal{J}^{-1}(\top^*_x))\leq\mathcal{J}^{-1}(\top^*_x)$ *iff* $\mathcal{H}^{-1}(A)\leq\mathcal{J}^{-1}(\mathcal{H}^{-1}(A))$ *iff* $\mathcal{H}^{-1}(\mathcal{J}(A))\leq\mathcal{J}^{-1}(A)$ *iff* $\mathcal{H}^{-1}(\mathcal{J}^{-1}(A))\leq\mathcal{J}^{-1}(A)$.

(11) *If* $\mathcal{H}(\top_x)(y)=\mathcal{J}^*(\top^*_x)(y)$ *for all* $x,y\in X$, *then* $\bigvee_{y\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_z)(y)\leq\mathcal{H}(\top_x)(z)$ *for all* $x,z\in X$ *iff* $\mathcal{H}(\top_x)\leq\mathcal{J}(\mathcal{H}(\top_x))$ *iff* $\mathcal{H}(\top_x)\leq\mathcal{J}(\mathcal{H}^{-1}(\top_x))$ *iff* $\mathcal{H}(\mathcal{J}(\top^*_x)\leq\mathcal{J}(\top^*_x)$ *iff* $\mathcal{H}(\mathcal{J}^{-1}(\top^*_x)\leq\mathcal{J}(\top^*_x)$ *iff* $\mathcal{H}(A)\leq\mathcal{J}(\mathcal{H}(A))$ *iff* $\mathcal{H}(A)\leq\mathcal{J}(\mathcal{H}^{-1}(A))$ *iff* $\mathcal{H}(\mathcal{J}(A))\leq\mathcal{J}(A)$ *iff* $\mathcal{H}(\mathcal{J}^{-1}(A))\leq\mathcal{J}(A)$.

Proof. (1) For $A=\bigvee_{x\in X}(A(x)\odot\top_x)$, we have

$$\mathcal{H}(A)(y)=\bigvee_{x\in X}(A(x)\odot\mathcal{H}(\top_x)(y))\geq\bigvee_{x\in X}(A(x)\odot\top_x(y))=A(y).$$

Similarly, we have $\mathcal{H}^{-1}(A)\geq A$ for each $A\in L^X$.

(2) For $A=\bigwedge_{x\in X}(A^*(x)\rightarrow\top^*_x)$, we have

$$\mathcal{J}(A)(y)=\bigwedge_{x\in X}(A^*(x)\rightarrow\mathcal{J}(\top^*_x)(y))\leq\bigwedge_{x\in X}(A^*(x)\rightarrow\top^*_x(y))=A(y).$$

Similarly, we have $\mathcal{J}^{-1}(A)\leq A$ for each $A\in L^X$.

(3) Since $\bigvee_{y\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)=\mathcal{H}(\bigvee_{y\in X}\mathcal{H}(\top_x)(y)\odot\top_y)(z)=\mathcal{H}(\mathcal{H}(\top_x)(z)$ for all $x,z\in X$ and

$\mathcal{H}^{-1}(\mathcal{H}^{-1}(\top_x))(z) = \mathcal{H}^{-1}(\bigvee_{y\in X}\mathcal{H}^{-1}(\top_x)(y)\odot\top_y)(z) = \bigvee_{y\in X}(\mathcal{H}^{-1}(\top_x)(y)\odot \mathcal{H}^{-1}(\top_y)(z)) = \bigvee_{y\in X}(\mathcal{H}(\top_y)(x)\odot\mathcal{H}(\top_z)(y)) \le \mathcal{H}^{-1}(\top_x)(z) = \mathcal{H}(\top_z)(x)$, we have $\bigvee_{y\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z) \le \mathcal{H}(\top_x)(z)$ for all $x, z \in X$ iff $\mathcal{H}(\mathcal{H}(\top_x)) \le \mathcal{H}(\top_x)$ iff $\mathcal{H}^{-1}(\mathcal{H}^{-1}(\top_x)) \le \mathcal{H}^{-1}(\top_x)$.

Second, let $\mathcal{H}(\mathcal{H}(\top_x)) \le \mathcal{H}(\top_x)$ for all $x \in X$.

$$\begin{aligned}\mathcal{H}(\mathcal{H}(A))(z) &= \mathcal{H}(\bigvee_{y\in X}(\mathcal{H}(A)(y)\odot\top_y)(z) = \bigvee_{y\in X}(\mathcal{H}(A)(y)\odot\mathcal{H}(\top_y)(z)\\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A(x)\odot\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)\\ &\le \bigvee_{x\in X}(A(x)\odot\mathcal{H}(\top_x)(y)) = \mathcal{H}(A)(x).\end{aligned}$$

Other cases are similarly proved.

(4) Since $\bigvee_{x\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z) = \mathcal{H}(\bigvee_{x\in X}\mathcal{H}(\top_x)(y)\odot\top_x)(z) = \mathcal{H}(\bigvee_{x\in X}\mathcal{H}^{-1}(\top_y)(x)\odot\top_x)(z) = \mathcal{H}(\mathcal{H}^{-1}(\top_y)(z)$ for all $x, z \in X$, we have $\bigvee_{x\in X}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z) \le \mathcal{H}(\top_y)(z)$ for all $x, z \in X$ iff $\mathcal{H}(\mathcal{H}^{-1}(\top_y)) \le \mathcal{H}(\top_y)$.

Second, let $\mathcal{H}(\mathcal{H}^{-1}(\top_x)) \le \mathcal{H}(\top_x)$ for all $x \in X$.

$$\begin{aligned}\mathcal{H}(\mathcal{H}^{-1}(A))(z) &= \mathcal{H}(\bigvee_{y\in X}(\mathcal{H}^{-1}(A)(y)\odot\top_y)(z)\\ &= \bigvee_{y\in X}(\mathcal{H}^{-1}(A)(y)\odot\mathcal{H}(\top_y)(z)\\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A(x)\odot\mathcal{H}^{-1}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)\\ &= \bigvee_{x\in X}(A(x)\odot(\bigvee_{y\in X}(\mathcal{H}(\top_y)(x)\odot\mathcal{H}(\top_y)))(z))\\ &\le \bigvee_{x\in X}(A(x)\odot\mathcal{H}(\top_x)(z)) = \mathcal{H}(A)(z).\end{aligned}$$

Other cases and (5) are similarly proved.

(6) For $\mathcal{J}(\top_x^*) = \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \top_y^*)$, we have

$$\begin{aligned}\mathcal{J}(\mathcal{J}(\top_x^*))(z) &= \mathcal{J}(\bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y)\to\top_y^*)(z)\\ &= \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y)\to\mathcal{J}(\top_y^*)(z)\\ &\ge \mathcal{J}(\top_x^*)(z).\end{aligned}$$

Since $\mathcal{J}(\top_x^*)(z) \le \mathcal{J}(\mathcal{J}(\top_x^*))(z)$ iff $\mathcal{J}(\top_x^*)(z) \le \bigwedge_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z)$ iff $\mathcal{J}^*(\top_x^*)(y) \le \bigwedge_{z\in X}(\mathcal{J}(\top_x^*)(z) \to \mathcal{J}(\top_y^*)(z)$ iff $\mathcal{J}^*(\top_x^*)(y) \le \bigwedge_{z\in X}(\mathcal{J}^*(\top_y^*)(z) \to \mathcal{J}^*(\top_x^*)(z)$ iff $\bigvee_{y\in X}(\mathcal{J}^*(\top_x^*)(y)\odot\mathcal{J}^*(\top_y^*)(z)) \le \mathcal{J}^*(\top_x^*)(z)$.

$\bigvee_{x\in X}\mathcal{J}^*(\top_x^*)(z)\odot\mathcal{J}^*(\top_y^*)(x) \le \mathcal{J}^*(\top_x^*)(z)$ for all $y, z \in X$ iff $\mathcal{J}(\top_x^*) \le \mathcal{J}(\mathcal{J}(\top_x^*))$ iff $\mathcal{J}^{-1}(\top_x^*) \le \mathcal{J}^{-1}(\mathcal{J}^{-1}(\top_x^*))$ iff $\mathcal{J}(A) \le \mathcal{J}(\mathcal{J}(A))$ iff $\mathcal{J}^{-1}(A) \le \mathcal{J}^{-1}(\mathcal{J}^{-1}(A))$.

Second, let $\mathcal{J}(\mathcal{J}(\top_x^*)) \geq \mathcal{J}(\top_x^*)$ for all $x \in X$.

$$\begin{aligned}
\mathcal{J}(\mathcal{J}(A))(z) &= \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(z) \to \mathcal{J}(A)(x)) \\
&= \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(z) \to \bigwedge_{y\in X}(\mathcal{J}^*(\top_y^*)(x) \to A(y))) \\
&= \bigwedge_{y\in X}(\bigvee_{x\in X}\mathcal{J}^*(\top_x^*)(z) \odot \mathcal{J}^*(\top_y^*)(x) \to A(y))) \\
&\geq \bigwedge_{z\in X}(\mathcal{J}^*(\top_y^*)(z) \to A(y)).
\end{aligned}$$

(7) For $\mathcal{J}^{-1}(\top_x^*) = \bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_x^*)(y) \to \top_y^*)$, we have

$$\begin{aligned}
\mathcal{J}(\mathcal{J}^{-1}(\top_x^*))(z) &= \mathcal{J}(\bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_x^*)(y) \to \top_y^*)(z) \\
&= \bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z) \\
&\geq \mathcal{J}(\top_x^*)(z).
\end{aligned}$$

Since $\mathcal{J}(\top_x^*)(z) \leq \mathcal{J}(\mathcal{J}^{-1}(\top_x^*))(z)$ iff $\mathcal{J}(\top_x^*)(z) \leq \bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_x^*)(y) \to \mathcal{J}(\top_y^*)(z)$ iff $\mathcal{J}^{-1*}(\top_x^*)(y) \leq \bigwedge_{z\in X}(\mathcal{J}(\top_x^*)(z) \to \mathcal{J}(\top_y^*)(z)$ iff $\mathcal{J}^{-1*}(\top_x^*)(y) \leq \bigwedge_{z\in X}(\mathcal{J}^*(\top_y^*)(z) \to \mathcal{J}^*(\top_x^*)(z)$ iff $\bigvee_{y\in X}(\mathcal{J}^*(\top_y^*)(x) \odot \mathcal{J}^*(\top_y^*)(z)) \leq \mathcal{J}^*(\top_x^*)(z)$.

Then $\bigvee_{x\in X}\mathcal{J}^*(\top_y^*)(x) \odot \mathcal{J}^*(\top_z^*)(x) \leq \mathcal{J}^*(\top_y^*)(z)$ for all $y, z \in X$ iff $\mathcal{J}(\top_y^*) \leq \mathcal{J}(\mathcal{J}^{-1}(\top_y^*))$ iff $\mathcal{J}(A) \leq \mathcal{J}(\mathcal{J}^{-1}(A))$.

Second, let $\mathcal{J}(\mathcal{J}^{-1}(\top_x^*)) \geq \mathcal{J}(\top_x^*)$ for all $x \in X$.

$$\begin{aligned}
\mathcal{J}(\mathcal{J}^{-1}(A))(z) &= \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(z) \to \mathcal{J}^{-1}(A)(x)) \\
&= \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(z) \to \bigwedge_{y\in X}(\mathcal{J}^{-1*}(\top_y^*)(x) \to A(y))) \\
&= \bigwedge_{y\in X}(\bigvee_{x\in X}\mathcal{J}^*(\top_x^*)(z) \odot \mathcal{J}^*(\top_x^*)(y) \to A(y))) \\
&\geq \bigwedge_{z\in X}(\mathcal{J}^*(\top_y^*)(z) \to A(y)) = \mathcal{J}(A))(z).
\end{aligned}$$

Other cases and (8) are similarly proved.

(9) Since $\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z) \leq \mathcal{H}(\top_x)(z)$ iff $\mathcal{H}(\top_x)(y) \odot \mathcal{H}^{-1}(\top_z)(y) \leq \mathcal{H}^{-1}(\top_z)(x)$ iff $\mathcal{H}^{-1}(\top_z)(y) \leq \mathcal{H}(\top_x)(y) \to \mathcal{H}^{-1}(\top_z)(x) = \mathcal{H}^{-1*}(\top_z)(x) \to \mathcal{H}^*(\top_x)(y) = \mathcal{H}^{-1*}(\top_z)(x) \to \mathcal{J}(\top_x^*)(y)$, we have

$$\begin{aligned}
\mathcal{H}^{-1}(\top_z)(y) &\leq \bigwedge_{x\in X}(\mathcal{H}^{-1*}(\top_z)(x) \to \mathcal{J}(\top_x^*)(y)) \\
&= \mathcal{J}(\bigwedge_{x\in X}(\mathcal{H}^{-1*}(\top_z)(x) \to \top_x^*)(y)) \\
&= \mathcal{J}(\mathcal{H}^{-1}(\top_z)(y).
\end{aligned}$$

Since $\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_y)(z) \leq \mathcal{H}(\top_x)(z)$ iff $\mathcal{H}(\top_x)(y) \odot \mathcal{H}^{-1}(\top_z)(y) \leq \mathcal{H}^{-1}(\top_z)(x)$ iff $\mathcal{H}(\top_x)(y) \leq \mathcal{H}^{-1}(\top_z)(y) \to \mathcal{H}(\top_x)(z) = \mathcal{H}^*(\top_x)(z) \to \mathcal{H}^{-1*}(\top_z)(y) = \mathcal{H}^*(\top_x)(z) \to \mathcal{J}^{-1}(\top_z^*)(y)$, we have

$$\begin{aligned}
\mathcal{H}(\top_x)(y) &\leq \bigwedge_{z\in X}(\mathcal{H}^*(\top_x)(z) \to \mathcal{J}^{-1}(\top_z^*)(y)) \\
&= \mathcal{J}^{-1}(\bigwedge_{z\in X}(\mathcal{H}^*(\top_x)(z) \to \top_x^*)(y)) \\
&= \mathcal{J}^{-1}(\mathcal{H}(\top_z)(y).
\end{aligned}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)\odot A(x)\le\mathcal{H}(\top_x)(z)\odot A(x)\\ \text{iff }\ \mathcal{H}(\top_x)(y)\odot A(x)\le\mathcal{H}(\top_y)(z)\to\mathcal{H}(\top_x)(z)\odot A(x)\\ \text{iff }\ \mathcal{H}(A)(y)\le\mathcal{J}^{-1}(\mathcal{H}(A))(y).\end{array}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_y)(z)\odot A(z)\le\mathcal{H}(\top_x)(z)\odot A(z)\\ \text{iff }\ \mathcal{H}^{-1}(\top_z)(y)\odot A(z)\le\mathcal{H}(\top_x)(y)\to\mathcal{H}^{-1}(\top_z)(x)\odot A(z)\\ \text{iff }\ \mathcal{H}^{-1}(A)(y)\le\mathcal{J}(\mathcal{H}^{-1}(A))(y).\end{array}$$

Other cases are similarly proved.

(10) Since $\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\le\mathcal{H}(\top_y)(z)$ iff $\mathcal{H}(\top_x)(y)\le\mathcal{H}(\top_x)(z)\to\mathcal{H}(\top_y)(z)=\mathcal{J}^{-1*}(\top_z^*)(x)\to\mathcal{H}(\top_y)(z)$, we have

$$\begin{array}{ll}\mathcal{H}^{-1}(\top_y)(x) & \le\bigwedge_{z\in X}(\mathcal{J}^{-1*}(\top_z^*)(x)\to\mathcal{H}(\top_y)(z))\\ & =\mathcal{J}^{-1}(\mathcal{H}(\top_y)(x).\end{array}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\odot A(y)\le\mathcal{H}(\top_y)(z)\odot A(y)\\ \text{iff }\ \mathcal{H}(\top_x)(y)\odot A(y)\le\mathcal{H}(\top_x)(z)\to\mathcal{H}(\top_y)(z)\odot A(y)\\ \text{iff }\ \mathcal{H}^{-1}(A)(x)\le\mathcal{J}^{-1*}(\top_z^*)(x)\to\mathcal{H}(A)(z)\\ \text{iff }\ \mathcal{H}^{-1}(A)(x)\le\mathcal{J}^{-1}(\mathcal{H}(A))(x).\end{array}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\odot A(z)\le\mathcal{H}(\top_y)(z)\odot A(z)\\ \text{iff }\ \mathcal{H}(\top_x)(z)\odot A(z)\le\mathcal{H}(\top_x)(y)\to\mathcal{H}(\top_y)(z)\odot A(z)\\ \text{iff }\ \mathcal{H}^{-1}(A)(x)\le\mathcal{J}^{-1}(\mathcal{H}^{-1}(A))(y).\end{array}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\odot(\mathcal{H}(\top_y)(z)\to A(y))\\ \le\mathcal{H}(\top_y)(z)\odot(\mathcal{H}(\top_y)(z)\to A(y))\le A(y)\\ \text{iff }\ \mathcal{H}(\top_x)(z)\odot(\mathcal{H}(\top_y)(z)\to A(y))\le\mathcal{H}(\top_x)(y)\to A(y).\end{array}$$

Thus, $\mathcal{H}^{-1}(\mathcal{J}(A))(x)\le\mathcal{J}^{-1}(A)(x)$.

$$\begin{array}{l}\mathcal{H}^{-1}(\mathcal{J}(\top_y^*))(x)\le\mathcal{J}^{-1}(\top_y^*)(x)\\ \text{iff }\ \mathcal{J}(\top_y^*)(z)\odot\mathcal{H}^{-1}(\top_z)(x)=\mathcal{J}(\top_y^*)(z)\odot\mathcal{J}^{-1*}(\top_z^*)(x)\le\mathcal{J}^{-1}(\top_y^*)(x)\\ \text{iff }\ \mathcal{J}^*(\top_x^*)(y)\le\mathcal{J}^*(\top_x^*)(z)\to\mathcal{J}^*(\top_y^*)(z)\\ \text{iff }\ \mathcal{J}^*(\top_x^*)(y)\odot\mathcal{J}^*(\top_x^*)(z)\le\mathcal{J}^*(\top_y^*)(z).\end{array}$$

$$\begin{array}{l}\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_x)(z)\odot(\mathcal{H}(\top_y)(z)\to A(z))\\ \le\mathcal{H}(\top_y)(z)\odot(\mathcal{H}(\top_y)(z)\to A(z))\le A(z)\\ \text{iff }\ \mathcal{H}(\top_x)(y)\odot(\mathcal{J}^{-1}(\top_y)(z)\to A(y))\le\mathcal{H}(\top_x)(z)\to A(z).\end{array}$$

Thus, $\mathcal{H}^{-1}(\mathcal{J}^{-1}(A))(x)\le\mathcal{J}^{-1}(A)(x)$. Other cases are similarly proved.

(11) Since $\mathcal{H}(\top_x)(y)\odot\mathcal{H}(\top_z)(y) \leq \mathcal{H}(\top_x)(z)$ iff $\mathcal{H}(\top_x)(y) \leq \mathcal{H}(\top_z)(y) \rightarrow \mathcal{H}(\top_x)(z) = \mathcal{J}^*(\top_z^*)(y) \rightarrow \mathcal{H}(\top_x)(z)$, we have

$$\begin{aligned}\mathcal{H}(\top_x)(y) &\leq \bigwedge_{z\in X}(\mathcal{J}^*(\top_z^*)(y) \rightarrow \mathcal{H}(\top_x)(z))\\ &= \mathcal{J}(\mathcal{H}(\top_x)(y).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_z)(y) \odot A(x) \leq \mathcal{H}(\top_x)(z) \odot A(x)\\ \text{iff }\ &\mathcal{H}(\top_x)(y) \odot A(x) \leq \mathcal{H}(\top_z)(y) \rightarrow \mathcal{H}(\top_x)(z) \odot A(x)\\ \text{iff }\ &\mathcal{H}(A)(y) \leq \mathcal{J}^*(\top_z^*)(y) \rightarrow \mathcal{H}(A)(z)\\ \text{iff }\ &\mathcal{H}(A)(y) \leq \mathcal{J}(\mathcal{H}(A))(x).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_z)(y) \odot A(z) \leq \mathcal{H}(\top_x)(z) \odot A(z)\\ \text{iff }\ &\mathcal{H}(\top_z)(y) \odot A(z) \leq \mathcal{H}(\top_x)(y) \rightarrow \mathcal{H}(\top_x)(z) \odot A(z)\\ \text{iff }\ &\mathcal{H}(A)(y) \leq \mathcal{J}(\mathcal{H}^{-1}(A))(y).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_z)(y) \odot (\mathcal{H}(\top_x)(z) \rightarrow A(x))\\ &\leq \mathcal{H}(\top_x)(z) \odot (\mathcal{H}(\top_x)(z) \rightarrow A(x)) \leq A(x)\\ &\text{iff }\ \mathcal{H}(\top_z)(y) \odot (\mathcal{H}(\top_x)(z) \rightarrow A(x)) \leq \mathcal{H}(\top_x)(y) \rightarrow A(x).\end{aligned}$$

Thus, $\mathcal{H}(\mathcal{J}(A))(x) \leq \mathcal{J}(A)(x)$.

$$\begin{aligned}&\mathcal{H}(\top_x)(y) \odot \mathcal{H}(\top_z)(y) \odot (\mathcal{H}(\top_x)(z) \rightarrow A(z))\\ &\leq \mathcal{H}(\top_x)(z) \odot (\mathcal{H}(\top_x)(z) \rightarrow A(z)) \leq A(z)\\ &\text{iff }\ \mathcal{H}(\top_x)(y) \odot (\mathcal{J}^{-1*}(\top_z^*)(x) \rightarrow A(z)) \leq \mathcal{H}(\top_z)(y) \rightarrow A(z).\end{aligned}$$

Thus, $\mathcal{H}(\mathcal{J}^{-1}(A))(y) \leq \mathcal{J}(A)(y)$. Other cases are similarly proved.

□

EXAMPLE 3.3. Let $(L = [0,1], \odot, \rightarrow, ^*)$ be a complete residuated lattice with the law of double negation defined by

$$x \odot y = (x + y - 1) \vee 0,\ \ x \rightarrow y = (1 - x + y) \wedge 1,\ x^* = 1 - x.$$

Let $X = \{x, y, z\}$ and $A, B \in L^X$ as follows:

$A(x) = 0.9, A(y) = 0.8, A(z) = 0.3,\ \ B(x) = 0.3, A(y) = 0.7, A(z) = 0.8.$

Define $\mathcal{H}(1_x)(y) = \mathcal{J}^*(1_x^*)(y)$ as follows

$$\begin{pmatrix} \mathcal{H}(1_x)(x) = 1 & \mathcal{H}(1_x)(y) = 0.8 & \mathcal{H}(1_x)(z) = 0.6\\ \mathcal{H}(1_y)(x) = 0.7 & \mathcal{H}(1_y)(y) = 1 & \mathcal{H}(1_y)(z) = 0.3\\ \mathcal{H}(1_z)(x) = 0.5 & \mathcal{H}(1_z)(y) = 0.6 & \mathcal{H}(1_z)(y) = 1. \end{pmatrix}$$

(1) $\bigvee_{y\in X}(\mathcal{H}(1_x)(y)\odot\mathcal{H}(1_y)(z) = \mathcal{H}(1_x)(z)$ and $1_x \le \mathcal{H}(1_x)$ for all $x,y\in X$. Since $\mathcal{H}(A)(y)=\bigvee_{x\in X}(A(x)\odot(\mathcal{H}(1_x)(y))$ and $\mathcal{J}(A)(y)=\bigwedge_{x\in X}(\mathcal{J}^*(1^*_x)(y)\to A(x))$, we have

$$\mathcal{H}(\mathcal{H}(A))=\mathcal{H}(A)=(0.9,0.8,0.5),\ \mathcal{H}(\mathcal{H}(B))=\mathcal{H}(B)=(0.4,0.7,0.8),$$
$$\mathcal{H}(\mathcal{H}(A^*))=\mathcal{H}(A^*)=(0.2,0.3,0.7),\ \mathcal{H}(\mathcal{H}(B^*))=\mathcal{H}(B^*)=(0.7,0.5,0.3),$$
$$\mathcal{J}(\mathcal{J}(A))=\mathcal{J}(A)=(0.8,0.7,0.3),\ \mathcal{J}(\mathcal{J}(B))=\mathcal{J}(B)=(0.3,0.5,0.7),$$
$$\mathcal{J}(\mathcal{J}(A^*))=\mathcal{J}(A^*)=(0.1,0.2,0.5),\ \mathcal{J}(\mathcal{J}(B^*))=\mathcal{J}(B^*)=(0.6,0.3,0.2).$$
$$\mathcal{H}(A)=(\mathcal{J}(A^*))^*,\mathcal{J}(A)=(\mathcal{H}(A^*))^*,\mathcal{H}(B)=(\mathcal{J}(B^*))^*,\mathcal{J}(B)=(\mathcal{H}(B^*))^*.$$

(2) We obtain $\mathcal{H}^{-1}(1_x)(y)=\mathcal{J}^{-1*}(1^*_x)(y)=\mathcal{H}(1_y)(x)$ as follows

$$\begin{pmatrix} \mathcal{H}^{-1}(1_x)(x)=1 & \mathcal{H}^{-1}(1_x)(y)=0.7 & \mathcal{H}^{-1}(1_x)(z)=0.5\\ \mathcal{H}^{-1}(1_y)(x)=0.8 & \mathcal{H}^{-1}(1_y)(y)=1 & \mathcal{H}^{-1}(1_y)(z)=0.6\\ \mathcal{H}^{-1}(1_z)(x)=0.6 & \mathcal{H}^{-1}(1_x)(y)=0.3 & \mathcal{H}^{-1}(1_x)(y)=1. \end{pmatrix}$$

We have $\bigvee_{y\in X}(\mathcal{H}^{-1}(1_x)(y)\odot\mathcal{H}^{-1}(1_y)(z)=\mathcal{H}^{-1}(1_x)(z)$ and $1_x\le\mathcal{H}^{-1}(1_x)$ for all $x,y\in X$. Since $\mathcal{H}^{-1}(A)(y)=\bigvee_{x\in X}(A(x)\odot(\mathcal{H}^{-1}(1_x)(y))$, we have

$$\mathcal{H}^{-1}(\mathcal{H}^{-1}(A))=\mathcal{H}^{-1}(A)=(0.9,0.8,0.4),\ \mathcal{H}^{-1}(B)=(0.5,0.7,0.8),$$
$$\mathcal{J}^{-1}(A)=(0.7,0.8,0.3),\ \mathcal{J}^{-1}(B)=(0.3,0.5,0.8).$$

(3) Since $0.6=\bigvee_{x\in X}(\mathcal{H}(1_x)(y)\odot\mathcal{H}(1_x)(z)\not\le\mathcal{H}(1_y)(z)=0.3$, then

$$(0.8,1,0.6)=\mathcal{H}^{-1}(1_y)\not\le\mathcal{J}^{-1}(\mathcal{H}(1_y))=(0.7,1,0.3)$$

(4) Since $0.6=\bigvee_{x\in X}(\mathcal{H}(1_y)(x)\odot\mathcal{H}(1_z)(x)\not\le\mathcal{H}(1_y)(z)=0.3$, then

$$(0.5,0.6,1)=\mathcal{H}(1_z)\not\le\mathcal{J}(\mathcal{H}^{-1}(1_z))=(0.6,0.3,1).$$

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