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COINCIDENCES OF DIFFERENT TYPES OF FUZZY IDEALS IN ORDERED Γ-SEMIGROUPS

Arunothai Kanlaya and Aiyared Iampan

Abstract. The notion of Γ-semigroups was introduced by Sen in 1981 and that of fuzzy sets by Zadeh in 1965. Any semigroup can be reduced to a Γ-semigroup but a Γ-semigroup does not necessarily reduce to a semigroup. In this paper, we study the coincidences of fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy interior ideals and fuzzy ideals in regular, left regular, right regular, intra-regular, semisimple ordered Γ-semigroups.

1. Introduction and Preliminaries

A fuzzy subset of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [32] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [32], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: In 1971, Rosenfeld [20] was the first who studied fuzzy sets in the structure of groups. Fuzzy semigroups have been first considered by Kuroki [13–16], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu

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and Tsingelis [8, 9]. In 2007, Kehayopulu and Tsingelis [10] characterized the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. Dheena and Manivasan [6] gave some characterizations of regular Γ-semigroups through fuzzy ideals. In 2008, Shabir and Khan [29] defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. Zhan and Ma [33] studied fuzzy interior ideals in semigroups. In 2009, Majumder and Sardar [19] studied fuzzy ideals and fuzzy ideal extensions in ordered semigroups. Prince Williams, Latha and Chandrasekeran [31] studied fuzzy bi-ideals in Γ-semigroups. Kim [12] studied the intuitionistic fuzzification of the concept of several ideals in an ordered semigroup, and investigated some properties of such ideals. In 2010, Chinram and Malee [3] investigated some properties of L-fuzzy ternary subsemiring and L-fuzzy ideals in ternary semirings. Chinram and Saelee [4] studied fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups. Shah and Rehman [30] introduced Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids and studied some characteristics of Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids. Iampan [7] characterized the relationship between the fuzzy ordered ideals (fuzzy ordered filters) and the characteristic mappings of fuzzy ordered ideals (fuzzy ordered filters) in ordered Γ-semigroups. Majumder [17] studied the fuzzy weakly completely prime ideals in Γ-semigroups. In 2011, Saelee and Chinram [21] studied rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups. Chon [5] characterized the fuzzy bi-ideals generated by a fuzzy subset in semigroups. In 2012, Sardar, Davvaz, Majumder and Kayal [24] studied the generalized fuzzy interior ideals in Γ-semigroups. Sardar, Davvaz, Majumder and Mandal [25] studied the characteristic ideals and fuzzy characteristic Ideals of Γ-semigroups. Majumder and Mandal [18] introduced the concept of fuzzy generalized bi-ideal of a Γ-semigroup, which is an extension of the concept of a fuzzy bi-ideal of a Γ-semigroup and characterized regular Γ-semigroups in terms of fuzzy generalized bi-ideals. In 2013, Khan, Sarmin and Khan [11] introduced (λ, θ)-fuzzy bi Γ-ideal of ordered Γ-semigroups. Bashir, Amin and Shabir [2] defined prime, strongly prime and semiprime fuzzy bi-ideals of Γ-semigroups. Abdullah, Aslam, Davvaz and Naeem [1] characterized the $(\in, \in \vee q)$-fuzzy bi-ideals in ordered semigroups.

The concept of Γ-semigroups, a generalization of both the concepts of semigroups and ternary semigroups, was introduced by Sen [26] and the concept of ordered Γ-semigroups was introduced by Sen and Seth [28]. For examples of Γ-semigroups and ordered Γ-semigroups, see [11, 22, 23, 25, 27, 31]. The fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy interior ideals and fuzzy ideals play an important role in studying the structure of ordered Γ-semigroups. Therefore, we will study the coincidences of fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy interior ideals and fuzzy ideals in ordered Γ-semigroups.

Before going to prove the main results we need the following definitions that we use later.

Definition 1.1. Let M and Γ be any two nonempty sets. Then (M, Γ) is called a Γ-*semigroup* if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. A nonempty subset K of M is called a Γ-*subsemigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Definition 1.2. A partially ordered Γ-semigroup $(M, \Gamma, \leq)$ is called an *ordered* Γ-*semigroup* if for any $a, b, c \in M$ and $\gamma \in \Gamma$,

$$a \leq b \text{ implies } a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b.$$

Definition 1.3. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. For $A \subseteq M$, we define

$$(A] := \{t \in M \mid t \leq a \text{ for some } a \in A\}.$$

Definition 1.4. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of M is called a *left ideal* of M if

(1) $M\Gamma A \subseteq A$, and
(2) $(A] \subseteq A$.

A nonempty subset A of M is called a *right ideal* of M if

(1) $A\Gamma M \subseteq A$, and
(2) $(A] \subseteq A$.

A nonempty subset A of M is called an *ideal* of M if it is both a left ideal and a right ideal of M. That is,

(1) $M\Gamma A \subseteq A$ and $A\Gamma M \subseteq A$, and
(2) $(A] \subseteq A$.

Definition 1.5. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A Γ-subsemigroup A of M is called an *interior ideal* of M if

(1) $M\Gamma A\Gamma M \subseteq A$, and
(2) $(A] \subseteq A$.

DEFINITION 1.6. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A Γ-subsemigroup A of M is called a *bi-ideal* of M if

(1) $A\Gamma M\Gamma A \subseteq A$, and
(2) $(A] \subseteq A$.

DEFINITION 1.7. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of M is called a *generalized bi-ideal* of M if

(1) $A\Gamma M\Gamma A \subseteq A$, and
(2) $(A] \subseteq A$.

DEFINITION 1.8. A *fuzzy subset* of a nonempty set X is an arbitrary mapping $f \colon X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line.

DEFINITION 1.9. Let X be a set and $A \subseteq X$. The *characteristic mapping* $f_A \colon X \to [0, 1]$ defined via

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of X into $\{0, 1\} \subset [0, 1]$. Hence f_A is a fuzzy subset of X.

DEFINITION 1.10. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty fuzzy subset f of M is called a *fuzzy* Γ-*subsemigroup* of M if for any $x, y \in M$ and $\gamma \in \Gamma$,

$$f(x\gamma y) \geq \min\{f(x), f(y)\}.$$

DEFINITION 1.11. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty fuzzy subset f of M is called a *fuzzy left ideal* of M if

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\gamma y) \geq f(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

A nonempty fuzzy subset f of M is called a *fuzzy right ideal* of M if

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\gamma y) \geq f(x)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

A nonempty fuzzy subset f of M is called a *fuzzy ideal* of M if it is both a fuzzy left ideal and a fuzzy right ideal of M. That is,

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\gamma y) \geq f(y)$ and $f(x\gamma y) \geq f(x)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

DEFINITION 1.12. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy Γ-subsemigroup f of M is called a *fuzzy interior ideal* of M if

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\alpha y\beta z) \geq f(y)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

DEFINITION 1.13. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A fuzzy Γ-subsemigroup f of M is called a *fuzzy bi-ideal* of M if

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\alpha y\beta z) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

DEFINITION 1.14. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty fuzzy subset f of M is called a *fuzzy generalized bi-ideal* of M if

(1) for any $x, y \in M, x \leq y$ implies $f(x) \geq f(y)$, and
(2) $f(x\alpha y\beta z) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

REMARK 1.15. Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. We have the following statements.

(1) Every fuzzy ideal of M is a fuzzy interior ideal.
(2) Every fuzzy bi-ideal of M is a fuzzy generalized bi-ideal.

DEFINITION 1.16. An ordered Γ-semigroup $(M, \Gamma, \leq)$ is called *regular* if for each $a \in M$, there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that

$$a \leq a\alpha x\beta a.$$

DEFINITION 1.17. An ordered Γ-semigroup $(M, \Gamma, \leq)$ is called *left regular* if for each $a \in M$, there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that

$$a \leq x\alpha a\beta a.$$

DEFINITION 1.18. An ordered Γ-semigroup $(M, \Gamma, \leq)$ is called *right regular* if for each $a \in M$, there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that

$$a \leq a\alpha a\beta x.$$

DEFINITION 1.19. An ordered Γ-semigroup $(M, \Gamma, \leq)$ is called *intra-regular* if for each $a \in M$, there exist $x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$ such that

$$a \leq x\alpha a\beta a\gamma y.$$

DEFINITION 1.20. An ordered Γ-semigroup $(M, \Gamma, \leq)$ is called *semisimple* if for each $a \in M$, there exist $x, b, y \in M$ and $\gamma, \alpha, \beta, \delta \in \Gamma$ such that

$$a \leq x\gamma a\alpha b\beta a\delta y.$$

2. Propositions of Several Fuzzy Subsets

Proposition 2.1. [7] *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq A \subseteq M$. *Then* $A = (A]$ *if and only if the fuzzy subset* f_A *of* M *has the following property:*

$$x, y \in M, x \leq y \Rightarrow f_A(x) \geq f_A(y).$$

Proposition 2.2. [7] *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq K \subseteq M$. *Then* K *is a* Γ*-subsemigroup of* M *if and only if the fuzzy subset* f_K *is a fuzzy* Γ*-subsemigroup of* M.

Proposition 2.3. [7] *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq L \subseteq M$. *Then* L *is a left ideal of* M *if and only if the fuzzy subset* f_L *is a fuzzy left ideal of* M.

Proposition 2.4. [7] *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq R \subseteq M$. *Then* R *is a right ideal of* M *if and only if the fuzzy subset* f_R *is a fuzzy right ideal of* M.

Corollary 2.5. [7] *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq I \subseteq M$. *Then* I *is an ideal of* M *if and only if the fuzzy subset* f_I *is a fuzzy ideal of* M.

Proposition 2.6. *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\emptyset \neq A \subseteq M$. *Then* A *is an interior ideal of* M *if and only if the fuzzy subset* f_A *is a fuzzy interior ideal of* M.

Proof. Assume that A is an interior ideal of M. Then A is a Γ-subsemigroup of M and $A = (A]$. By Proposition 2.1 and 2.2, we have that $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$, and f_A is a fuzzy Γ-subsemigroup of M. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. If $y \notin A$, then $f_A(y) = 0$ and so $f_A(x\alpha y\beta z) \geq 0 = f_A(y)$. Let $y \in A$. Then $f_A(y) = 1$. Since $x\alpha y\beta z \in M\Gamma A\Gamma M \subseteq A$, we have $f_A(x\alpha y\beta z) = 1$. Thus $f_A(x\alpha y\beta z) = 1 \geq 1 = f_A(y)$. Hence f_A is a fuzzy interior ideal of M.

Conversely, assume that the fuzzy subset f_A is a fuzzy interior ideal of M. Then $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$, and f_A is a fuzzy Γ-subsemigroup of M. By Proposition 2.1 and 2.2, we have $(A] = A$ and A is a Γ-subsemigroup of M. Let $x, y \in M, a \in A$ and $\alpha, \beta \in \Gamma$. Since f_A is a fuzzy interior ideal of M, we have $f_A(x\alpha a\beta y) \geq f_A(a)$. Since $a \in A$, we have $f_A(a) = 1$. Thus $f_A(x\alpha a\beta y) = 1$ and so $x\alpha a\beta y \in A$. Hence A is an interior ideal of M. $\square$

PROPOSITION 2.7. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\emptyset \neq A \subseteq M$. Then A is a bi-ideal of M if and only if the fuzzy subset f_A is a fuzzy bi-ideal of M.*

Proof. Assume that A is a bi-ideal of M. Then A is a Γ-subsemigroup of M and $A = (A]$. By Proposition 2.1 and 2.2, we have that $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$, and f_A is a fuzzy Γ-subsemigroup of M. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Without loss of generality, we may assume that $\min\{f_A(x), f_A(z)\} = f_A(x)$. Then $f_A(z) \geq f_A(x)$. If $x \notin A$, then $f_A(x) = 0$ and so $f_A(x\alpha y\beta z) \geq 0 = \min\{f_A(x), f_A(z)\}$. Let $x \in A$. Then $f_A(x) = 1$. Thus $f_A(z) = 1$ and so $z \in A$. Since $x\alpha y\beta z \in A\Gamma M\Gamma A \subseteq A$, we have $f_A(x\alpha y\beta z) = 1$. Thus $f_A(x\alpha y\beta z) = 1 \geq 1 = \min\{f_A(x), f_A(z)\}$. Hence f_A is a fuzzy bi-ideal of M.

Conversely, assume that the fuzzy subset f_A is a fuzzy bi-ideal of M. Then $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$, and f_A is a fuzzy Γ-subsemigroup of M. By Proposition 2.1 and 2.2, we have $(A] = A$ and A is a Γ-subsemigroup of M. Let $x \in M, a, b \in A$ and $\alpha, \beta \in \Gamma$. Since f_A is a fuzzy bi-ideal of M, we have $f_A(a\alpha x\beta b) \geq \min\{f_A(a), f_A(b)\}$. Since $a, b \in A$, we have $f_A(a) = 1 = f_A(b)$ and so $\min\{f_A(a), f_A(b)\} = 1$. Thus $f_A(a\alpha x\beta b) = 1$ and so $a\alpha x\beta b \in A$. Hence A is a bi-ideal of M. □

PROPOSITION 2.8. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\emptyset \neq A \subseteq M$. Then A is a generalized bi-ideal of M if and only if the fuzzy subset f_A is a fuzzy generalized bi-ideal of M.*

Proof. Assume that A is a generalized bi-ideal of M. Then $A = (A]$. By Proposition 2.1, we have that $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Without loss of generality, we may assume that $\min\{f_A(x), f_A(z)\} = f_A(x)$. Then $f_A(z) \geq f_A(x)$. If $x \notin A$, then $f_A(x) = 0$ and so $f_A(x\alpha y\beta z) \geq 0 = \min\{f_A(x), f_A(z)\}$. Let $x \in A$. Then $f_A(x) = 1$. Thus $f_A(z) = 1$ and so $z \in A$. Since $x\alpha y\beta z \in A\Gamma M\Gamma A \subseteq A$, we have $f_A(x\alpha y\beta z) = 1$. Thus $f_A(x\alpha y\beta z) = 1 \geq 1 = \min\{f_A(x), f_A(z)\}$. Hence f_A is a fuzzy generalized bi-ideal of M.

Conversely, assume that the fuzzy subset f_A is a fuzzy generalized bi-ideal of M. Then $x \leq y$ implies $f_A(x) \geq f_A(y)$ for all $x, y \in M$. By Proposition 2.1, we have $(A] = A$. Let $x \in M, a, b \in A$ and $\alpha, \beta \in \Gamma$. Since f_A is a fuzzy bi-ideal of M, we have $f_A(a\alpha x\beta b) \geq \min\{f_A(a), f_A(b)\}$. Since $a, b \in A$, we have $f_A(a) = 1 = f_A(b)$ and so $\min\{f_A(a), f_A(b)\} = 1$. Thus $f_A(a\alpha x\beta b) = 1$ and so $a\alpha x\beta b \in A$. Hence A is a generalized bi-ideal of M. □

Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i\}_{i\in I}$ a nonempty family of fuzzy subsets of M. We define

$$\bigwedge_{i\in I} f_i \colon M \to [0,1] \mid x \mapsto (\bigwedge_{i\in I} f_i)(x) := \inf\{f_i(x)\}_{i\in I}.$$

Proposition 2.9. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i\}_{i\in I}$ a nonempty family of fuzzy subsets of M. Then $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M.*

Proof. Let $x \in M$. Then the set $\{f_i(x)\}_{i\in I}$ is a nonempty bounded below subset of $\mathbb{R}$. By the Infimum Property, there exists the $\inf\{f_i(x)\}_{i\in I}$ in $\mathbb{R}$. Since $0 \leq f_i(x) \leq 1$ for each $i \in I$, we have $0 \leq \inf\{f_i(x)\}_{i\in I} \leq 1$. Thus $0 \leq (\bigwedge_{i\in I} f_i)(x) \leq 1$. If $x, y \in M$ is such that $x = y$, then $\{f_i(x)\}_{i\in I} = \{f_i(y)\}_{i\in I}$. Thus $\inf\{f_i(x)\}_{i\in I} = \inf\{f_i(y)\}_{i\in I}$, so $(\bigwedge_{i\in I} f_i)(x) = (\bigwedge_{i\in I} f_i)(y)$. Hence $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. □

Proposition 2.10. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i\}_{i\in I}$ a family of fuzzy Γ-subsemigroups of M. Then $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M.*

Proof. By Proposition 2.9, we have $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy Γ-subsemigroup, $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus $f_i(x) \geq f_i(y) \geq \inf\{f_i(y)\}_{i\in I}$ for all $i \in I$, so $\inf\{f_i(y)\}_{i\in I}$ is a lower bound of $\{f_i(x)\}_{i\in I}$. Hence $\inf\{f_i(x)\}_{i\in I} \geq \inf\{f_i(y)\}_{i\in I}$, so $(\bigwedge_{i\in I} f_i)(x) \geq (\bigwedge_{i\in I} f_i)(y)$.

Let $x, y \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
(\bigwedge_{i\in I} f_i)(x\gamma y) &= \inf\{f_i(x\gamma y)\}_{i\in I} \\
&\geq \inf\{\min\{f_i(x), f_i(y)\}\}_{i\in I} \\
&= \min\{\inf\{f_i(x)\}_{i\in I}, \inf\{f_i(y)\}_{i\in I}\} \\
&= \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}.
\end{aligned}$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M. □

Proposition 2.11. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i\}_{i\in I}$ a family of fuzzy left ideals of M. Then $\bigwedge_{i\in I} f_i$ is a fuzzy left ideal of M.*

Proof. By Proposition 2.9, we have $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy left ideal of

M, $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus $f_i(x) \geq f_i(y) \geq \inf\{f_i(y)\}_{i\in I}$ for all $i \in I$, so $\inf\{f_i(y)\}_{i\in I}$ is a lower bound of $\{f_i(x)\}_{i\in I}$. Hence $\inf\{f_i(x)\}_{i\in I} \geq \inf\{f_i(y)\}_{i\in I}$, so $(\bigwedge_{i\in I} f_i)(x) \geq (\bigwedge_{i\in I} f_i)(y)$.

Let $x, y \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (\bigwedge_{i\in I} f_i)(x\gamma y) &= \inf\{f_i(x\gamma y)\}_{i\in I} \\ &\geq \inf\{f_i(y)\}_{i\in I} \\ &= (\bigwedge_{i\in I} f_i)(y). \end{aligned}$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy left ideal of M. □

Proposition 2.12. *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\{f_i\}_{i\in I}$ *a family of fuzzy right ideals of* M*. Then* $\bigwedge_{i\in I} f_i$ *is a fuzzy right ideal of* M*.*

Proof. The proof is similar to the proof of Proposition 2.11. □

Proposition 2.13. *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\{f_i\}_{i\in I}$ *a family of fuzzy bi-ideals of* M*. Then* $\bigwedge_{i\in I} f_i$ *is a fuzzy bi-ideal of* M*.*

Proof. By Proposition 2.10, we have $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M.

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} (\bigwedge_{i\in I} f_i)(x\alpha y\beta z) &= \inf\{f_i(x\alpha y\beta z)\}_{i\in I} \\ &\geq \inf\{\min\{f_i(x), f_i(z)\}\}_{i\in I} \\ &= \min\{\inf\{f_i(x)\}_{i\in I}, \inf\{f_i(z)\}_{i\in I}\} \\ &= \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(z)\}. \end{aligned}$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy bi-ideal of M. □

Proposition 2.14. *Let* $(M, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* $\{f_i\}_{i\in I}$ *a family of fuzzy generalized bi-ideals of* M*. Then* $\bigwedge_{i\in I} f_i$ *is a fuzzy generalized bi-ideal of* M*.*

Proof. By Proposition 2.9, we have $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M.

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy generalized bi-ideal of M, $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus $f_i(x) \geq f_i(y) \geq \inf\{f_i(y)\}_{i\in I}$

for all $i \in I$, so $\inf\{f_i(y)\}_{i\in I}$ is a lower bound of $\{f_i(x)\}_{i\in I}$. Hence $\inf\{f_i(x)\}_{i\in I} \geq \inf\{f_i(y)\}_{i\in I}$, so $(\bigwedge_{i\in I} f_i)(x) \geq (\bigwedge_{i\in I} f_i)(y)$.

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
(\bigwedge_{i\in I} f_i)(x\alpha y\beta z) &= \inf\{f_i(x\alpha y\beta z)\}_{i\in I} \\
&\geq \inf\{\min\{f_i(x), f_i(z)\}\}_{i\in I} \\
&= \min\{\inf\{f_i(x)\}_{i\in I}, \inf\{f_i(z)\}_{i\in I}\} \\
&= \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(z)\}.
\end{aligned}$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy generalized bi-ideal of M. □

PROPOSITION 2.15. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{f_i\}_{i\in I}$ a family of fuzzy interior ideals of M. Then $\bigwedge_{i\in I} f_i$ is a fuzzy interior ideal of M.*

Proof. By Proposition 2.10, we have $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subsemigroup of M.

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
(\bigwedge_{i\in I} f_i)(x\alpha y\beta z) &= \inf\{f_i(x\alpha y\beta z)\}_{i\in I} \\
&\geq \inf\{f_i(y)\}_{i\in I} \\
&= (\bigwedge_{i\in I} f_i)(y).
\end{aligned}$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy interior ideal of M. □

3. Main Results

In this section, we study the coincidences of fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy interior ideals and fuzzy ideals in regular, left regular, right regular, intra-regular, semisimple ordered Γ-semigroups.

THEOREM 3.1. *Let $(M, \Gamma, \leq)$ be a regular ordered Γ-semigroup. Then every fuzzy generalized bi-ideal of M is a fuzzy bi-ideal.*

Proof. Let f be a fuzzy generalized bi-ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is a regular, there exist $a \in M$ and $\alpha, \beta \in \Gamma$ such that $y \leq y\alpha a\beta y$. Thus $x\gamma y \leq x\gamma(y\alpha a\beta y) = x\gamma(y\alpha a)\beta y$. Hence

$f(x\gamma y) \geq f(x\gamma(y\alpha a)\beta y) \geq \min\{f(x), f(y)\}$. Therefore f is a fuzzy bi-ideal of M. $\square$

THEOREM 3.2. *Let $(M, \Gamma, \leq)$ be a left regular ordered Γ-semigroup. Then every fuzzy generalized bi-ideal of M is a fuzzy bi-ideal.*

Proof. Let f be a fuzzy generalized bi-ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is left regular, there exist $a \in M$ and $\alpha, \beta \in \Gamma$ such that $y \leq a\alpha y\beta y$. Thus $x\gamma y \leq x\gamma(a\alpha y\beta y) = x\gamma(a\alpha y)\beta y$. Hence $f(x\gamma y) \geq f(x\gamma(a\alpha y)\beta y) \geq \min\{f(x), f(y)\}$. Therefore f is a fuzzy bi-ideal of M. $\square$

THEOREM 3.3. *Let $(M, \Gamma, \leq)$ be a right regular ordered Γ-semigroup. Then every fuzzy generalized bi-ideal of M is a fuzzy bi-ideal.*

Proof. The proof is similar to the proof of Theorem 3.2. $\square$

THEOREM 3.4. *Let $(M, \Gamma, \leq)$ be a regular ordered Γ-semigroup. Then every fuzzy interior ideal of M is a fuzzy ideal.*

Proof. Let f be a fuzzy interior ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is a regular, there exist $a_1 \in M$ and $\alpha_1, \beta_1 \in \Gamma$ such that $x \leq x\alpha_1 a_1\beta_1 x$. Thus $x\gamma y \leq (x\alpha_1 a_1\beta_1 x)\gamma y = (x\alpha_1 a_1)\beta_1 x\gamma y$, so

$$f(x\gamma y) \geq f((x\alpha_1 a_1)\beta_1 x\gamma y) \geq f(x).$$

Hence f is a fuzzy right ideal of M. Similarly, there exist $a_2 \in M$ and $\alpha_2, \beta_2 \in \Gamma$ such that $y \leq y\alpha_2 a_2\beta_2 y$. Thus $x\gamma y \leq x\gamma(y\alpha_2 a_2\beta_2 y) = x\gamma y\alpha_2(a_2\beta_2 y)$, so

$$f(x\gamma y) \geq f(x\gamma y\alpha_2(a_2\beta_2 y)) \geq f(y).$$

Hence f is a fuzzy left ideal of M. Therefore f is a fuzzy ideal of M. $\square$

THEOREM 3.5. *Let $(M, \Gamma, \leq)$ be an intra-regular ordered Γ-semigroup. Then every fuzzy interior ideal of M is a fuzzy ideal.*

Proof. Let f be a fuzzy interior ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is intra-regular, there exist $a_1, b_1 \in M$ and $\alpha_1, \beta_1, \delta_1 \in \Gamma$ such that $x \leq a_1\delta_1 x\alpha_1 x\beta_1 b_1$. Then $x\gamma y \leq (a_1\delta_1 x\alpha_1 x\beta_1 b_1)\gamma y = (a_1\delta_1 x)\alpha_1 x\beta_1(b_1\gamma y)$, so

$$f(x\gamma y) \geq f((a_1\delta_1 x)\alpha_1 x\beta_1(b_1\gamma y)) \geq f(x).$$

Hence f is a fuzzy right ideal of M. Similarly, there exist $a_2, b_2 \in M$ and $\alpha_2, \beta_2, \delta_2 \in \Gamma$ such that $y \leq a_2\delta_2 y\alpha_2 y\beta_2 b_2$. Then $x\gamma y \leq x\gamma(a_2\delta_2 y\alpha_2 y\beta_2 b_2) = (x\gamma a_2)\delta_2 y\alpha_2(y\beta_2 b_2)$, so

$$f(x\gamma y) \geq f((x\gamma a_2)\delta_2 y\alpha_2(y\beta_2 b_2)) \geq f(y).$$

Hence f is a fuzzy left ideal of M. Therefore f is a fuzzy ideal of M. $\square$

THEOREM 3.6. *Let $(M, \Gamma, \leq)$ be a left regular ordered Γ-semigroup. Then every fuzzy interior ideal of M is a fuzzy ideal.*

Proof. Let f be a fuzzy interior ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is left regular, there exist $a_1 \in M$ and $\alpha_1, \beta_1 \in \Gamma$ such that $x \leq a_1\alpha_1 x\beta_1 x$. Then $x\gamma y \leq (a_1\alpha_1 x\beta_1 x)\gamma y = (a_1\alpha_1 x)\beta_1 x\gamma y$, so

$$f(x\gamma y) \geq f((a_1\alpha_1 x)\beta_1 x\gamma y) \geq f(x).$$

Hence f is a fuzzy right ideal of M. Similarly, there exist $a_2 \in M$ and $\alpha_2, \beta_2 \in \Gamma$ such that $y \leq a_2\alpha_2 y\beta_2 y$. Then $x\gamma y \leq x\gamma(a_2\alpha_2 y\beta_2 y) = (x\gamma a_2)\alpha_2 y\beta_2 y$, so

$$f(x\gamma y) \geq f((x\gamma a_2)\alpha_2 y\beta_2 y) \geq f(y).$$

Hence f is a fuzzy left ideal of M. Therefore f is a fuzzy ideal of M. $\square$

THEOREM 3.7. *Let $(M, \Gamma, \leq)$ be a right regular ordered Γ-semigroup. Then every fuzzy interior ideal of M is a fuzzy ideal.*

Proof. The proof is similar to the proof of Theorem 3.6. $\square$

THEOREM 3.8. *Let $(M, \Gamma, \leq)$ be a semisimple ordered Γ-semigroup. Then every fuzzy interior ideal of M is a fuzzy ideal.*

Proof. Let f be a fuzzy interior ideal of M and let $x, y \in M$ and $\gamma \in \Gamma$. Since M is semisimple, there exist $a_1, b_1, c_1 \in M$ and $\alpha_1, \beta_1, \delta_1, \xi_1 \in \Gamma$ such that $x \leq a_1\alpha_1 x\beta_1 b_1\delta_1 x\xi_1 c_1$. Then $x\gamma y \leq (a_1\alpha_1 x\beta_1 b_1\delta_1 x\xi_1 c_1)\gamma y = (a_1\alpha_1 x\beta_1 b_1)\delta_1 x\xi_1(c_1\gamma y)$, so

$$f(x\gamma y) \geq f((a_1\alpha_1 x\beta_1 b_1)\delta_1 x\xi_1(c_1\gamma y)) \geq f(x).$$

Hence f is a fuzzy right ideal of M. Similarly, there exist $a_2, b_2, c_2 \in M$ and $\alpha_2, \beta_2, \delta_2, \xi_2 \in \Gamma$ such that $y \leq a_2\alpha_2 y\beta_2 b_2\delta_2 y\xi_2 c_2$. Then $x\gamma y \leq x\gamma(a_2\alpha_2 y\beta_2 b_2\delta_2 y\xi_2 c_2) = (x\gamma a_2)\alpha_2 y\beta_2(b_2\delta_2 y\xi_2 c_2)$, so

$$f(x\gamma y) \geq f((x\gamma a_2)\alpha_2 y\beta_2(b_2\delta_2 y\xi_2 c_2)) \geq f(y).$$

Hence f is a fuzzy left ideal of M. Therefore f is a fuzzy ideal of M. $\square$

By Theorem 3.1, 3.2 and 3.3, we have Corollary 3.9.

COROLLARY 3.9. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. If M is regular, left regular, or right regular, then fuzzy generalized bi-ideals and fuzzy bi-ideals coincide.*

By Theorem 3.4, 3.5, 3.6, 3.7 and 3.8, we have Corollary 3.10.

COROLLARY 3.10. *Let $(M, \Gamma, \leq)$ be an ordered Γ-semigroup. If M is regular, intra-regular, left regular, right regular, or semisimple, then fuzzy interior ideals and fuzzy ideals coincide.*

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Arunothai Kanlaya
Department of Mathematics
University of Phayao
Phayao 56000, Thailand
E-mail: baby_zad@hotmail.com

Aiyared Iampan
Department of Mathematics
University of Phayao
Phayao 56000, Thailand
E-mail: aiyared.ia@up.ac.th