

NONEXISTENCE OF POSITIVE VECTOR SOLUTIONS FOR QUASILINEAR SCHRÖDINGER SYSTEMS WITH VANISHING POTENTIALS AT INFINITY

OHSANG KWON

ABSTRACT. We study a class of quasilinear Schrödinger systems arising in non-linear optics, plasma physics, and Bose–Einstein condensates, in which the quasilinear term $\Delta(u^2)u$ induces density-dependent dispersion. Combining a change-of-variables technique for scalar quasilinear equations with the spherical-average method for elliptic systems, we prove that a nonexistence result for positive vector supersolutions of the corresponding semilinear system extends, under the same decay assumptions on the potentials, to the quasilinear system.

1. Introduction

We study the nonexistence of positive vector supersolutions for the following quasilinear elliptic system with potentials $V_1(x), V_2(x)$ decaying at infinity:

$$(1.1) \quad \begin{cases} \Delta u + \Delta(u^2)u - V_1(x)v + v^p = 0, & \text{in } \mathbb{R}^N, \\ \Delta v + \Delta(v^2)v - V_2(x)u + u^q = 0, & \text{in } \mathbb{R}^N. \end{cases}$$

System (1.1) is the natural two-component analogue of the scalar quasilinear Schrödinger equation

$$(1.2) \quad \Delta u + \Delta(u^2)u - V(x)u + u^p = 0, \quad \text{in } \mathbb{R}^N,$$

for a complex-valued standing wave $\phi(t, x) = e^{-iEt}u(x)$ of the quasilinear Schrödinger equation

$$i\phi_t + \Delta\phi - W(x)\phi + \phi l'(|\phi|^2)\Delta l(|\phi|^2) + \rho(|\phi|^2)\phi = 0,$$

where $V = W - E$. Depending on the choice of l , this equation models a variety of physical phenomena: the type $l(s) = \sqrt{1+s}$ describes the self-channeling of a high-power ultrashort laser pulse in matter [5, 6, 10, 12, 26]; $l(s) = \sqrt{s}$ arises in the study of dissipative quantum mechanics [15]; and further instances occur in plasma and fluid dynamics [14, 20], in the theory of Heisenberg ferromagnets and magnons [4], and in condensed matter theory [23]. The case $l(s) = s$ treated here, for which (1.2) is the

Received July 31, 2026. Revised August 17, 2026. Accepted August 21, 2026.

2010 Mathematics Subject Classification: 35J62, 35J10, 35B45, 35Q40.

Key words and phrases: decaying potential, quasilinear system, nonexistence, spherical average, change of variables.

* Corresponding author.

© The Kangwon-Kyungki Mathematical Society, 2026.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

standing-wave equation, arises from the superfluid film equation in plasma physics and from models of self-trapped electrons on quadratic or hexagonal lattices [7, 8, 16–19].

For the semilinear equation $\Delta u - V(x)u + u^p = 0$ (i.e. (1.2) without the quasilinear term), Bae and Byeon [2] found an almost optimal threshold on the decay of V at infinity separating existence from nonexistence of positive solutions; this threshold is what condition (V) below generalizes to the vector setting. For the quasilinear scalar equation (1.2) itself, existence of positive solutions has been obtained by a perturbation method [11], by constrained minimization [21, 24], under Ambrosetti–Rabinowitz type conditions with vanishing potentials [1], and with potentials singular at the origin [27], as well as on smooth bounded domains [29]. On the nonexistence side, results have so far been limited to almost linear or critical growth: under linear growth of the nonlinearity and a nonvanishing potential [9], and via the Pohozaev identity [28]. The change of variables g in (1.4), which removes the quasilinear term at the cost of a transformed nonlinearity, was introduced by Liu–Wang–Wang [22] and used by Kwon [17] to prove a subcritical-growth nonexistence theorem for the scalar equation (1.2), matching the threshold of [2]; the present paper extends that result, together with the spherical-average method of Bae and Kwon [3] for the semilinear system

$$(1.3) \quad \begin{cases} \Delta u - V_1(x)v + v^p = 0, \\ \Delta v - V_2(x)u + u^q = 0, \end{cases}$$

to the quasilinear system (1.1). Our goal is to show that the nonexistence result of [3] for (1.3) extends to (1.1), and to determine exactly how much (if at all) the hypotheses must be strengthened.

The two basic difficulties compared with [3] are: (i) the spherical average does not commute with the product $\Delta(u^2)u$, so the argument of [3], which relies crucially on the identity $\overline{\Delta u} = \bar{u}'' + \frac{N-1}{r}\bar{u}'$, cannot be applied directly to (1.1); and (ii) a naive attempt to control the resulting gradient term $2u|\nabla u|^2$ by an a priori decay rate is circular, since that decay rate is itself a conclusion of the very spherical-average argument one is trying to run. We resolve both difficulties by using, componentwise, the change of variables introduced by Liu–Wang–Wang [22] and used by Kwon [17] for the scalar equation (1.2): this substitution removes the quasilinear term *exactly* (Lemma 2.1 below), reducing (1.1) to a system in which the spherical average commutes with the principal part, at the cost of a multiplicative perturbation of the nonlinear terms which we show, by explicit estimates, does not weaken the hypotheses of [3].

Let g be the smooth odd function defined by

$$(1.4) \quad \begin{cases} g'(t) = (1 + 2g^2(t))^{-\frac{1}{2}} & \text{if } t > 0, \\ g(0) = 0, \\ g(t) = -g(-t) & \text{if } t < 0. \end{cases}$$

This is the change of variables used in [17, 22] to treat the scalar equation (1.2).

Assumption (V). We impose the decay bounds used in [3], together with the nonnegativity of the potentials required in the comparison estimates below.

- $V_1, V_2 \in C(\mathbb{R}^N)$ and $V_1, V_2 \geq 0$.
- When $N \geq 3$, there exist $\beta > 0$ and $\delta > \beta(N-2)$ such that

$$\limsup_{|x| \rightarrow \infty} V_2(x)|x|^{(N-2)(q-1)}(\log|x|)^{\frac{(N-2)(q-1)}{2}}(\log(\log|x|))^{-\beta(q-1)} = 0$$

and

$$\tilde{V}(r)r^2 \log r \leq \frac{(N-2)^2}{2} + \frac{N(N-2)}{4 \log r} - \frac{\delta}{\log(\log r)}$$

in a neighborhood of ∞ , where

$$\tilde{V}(r) = \max_{|x|=r} \{V_1(x), V_2(x)\}.$$

- When $N = 2$,

$$\tilde{V}(r)r^2(\log r)^2 \log(\log r) \leq \beta + \frac{\beta(\beta+1)}{\log(\log r)}$$

in a neighborhood of ∞ for some $\beta > 0$.

- When $N = 1$,

$$\tilde{V}(r)r^2 \leq \beta(\beta+1)$$

in a neighborhood of ∞ for some $0 < \beta < \frac{1}{q-1}$. Here $r = |x|$, but it suffices to consider the problem near $+\infty$, so we write $r = x$.

We call $(u, v) \in C^2(\Omega)^2$ a *positive vector supersolution* of (1.1) in an exterior domain Ω if $u, v > 0$ in Ω and

$$\Delta u + \Delta(u^2)u - V_1(x)v + v^p \leq 0, \quad \Delta v + \Delta(v^2)v - V_2(x)u + u^q \leq 0$$

pointwise in Ω . The decay at infinity needed in the proof is stated separately in Theorem 1.1.

Our main result is the following.

THEOREM 1.1. *Let $N \geq 1$. Assume (V) with $1 < p \leq q$ and*

$$2(q+1) \geq (N-2)(pq-1).$$

Then (1.1) has no positive C^2 vector supersolution (u, v) in any exterior domain satisfying $u(x), v(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Thus the same quantitative decay bounds on the potentials as in the semilinear argument of [3] suffice for the quasilinear system (1.1); in particular, no strengthening of δ is needed. We additionally assume $V_1, V_2 \geq 0$ explicitly, since this sign is used in the transformed comparison estimates. This is proved in Section 2.

REMARK 1.2 (Reduction to the critical scalar case). Take $N = 4$, $p = q = 2$, $V_1 \equiv V_2 \equiv V(x) = |x|^{-2}(\log |x|)^{-1}$ for $|x| \geq 2$ (smoothly cut off near the origin). Then $(N-2)p = N$ (the critical exponent) and

$$V(x)|x|^2 \log |x| = 1 < \frac{(N-2)^2}{2} = 2,$$

so (V) holds. The symmetric ansatz $u = v = w$ reduces (1.1) to the scalar quasilinear equation

$$\Delta w + \Delta(w^2)w - V(x)w + w^2 = 0,$$

which is exactly the quasilinear analogue, in the sense of [17, 22], of the scalar equation treated in the critical case $(N-2)p = N$ by Bae and Byeon [2]; Theorem 1.1 shows this reduced equation also has no positive supersolution in any exterior domain.

REMARK 1.3 (An explicit (super)solution in the existence regime). Outside the range of Theorem 1.1, positive supersolutions of (1.1) do exist. Take $N = 4$, $p = q = 3$, $V_1 \equiv V_2 \equiv 0$; then

$$2(q+1) = 8 < (N-2)(pq-1) = 16,$$

so the hypothesis of Theorem 1.1 fails. With the symmetric ansatz $u = v = w(r) = c/r$, and using $\Delta(r^{-1}) = -r^{-3}$ and the fact that r^{-2} is harmonic in $\mathbb{R}^4$ (so $\Delta(w^2)w \equiv 0$ identically), a direct computation gives

$$\Delta w + \Delta(w^2)w + w^3 = c(c^2 - 1)r^{-3}.$$

Hence $u(x) = v(x) = c/|x|$ is an exact solution for $c = 1$ and a genuine positive supersolution for $0 < c < 1$, consistently with Remark 1.2 and the threshold identified in Theorem 1.1; compare the analogous explicit family of solutions for the semilinear system (1.3) constructed in [25] and recorded in [3, Remark 1.3]. Note that at this particular scaling exponent the quasilinear correction term $\Delta(w^2)w$ vanishes identically, so the system coincides with its semilinear counterpart along this example.

2. Proof of Theorem 1.1

Let g be as in (1.4). We record the elementary properties of g used below; see [17, 22].

LEMMA 2.1. *Let $w \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$ open, and set $u = g(w)$. Then*

$$(2.1) \quad \Delta u + \Delta(u^2)u = \sqrt{1 + 2g^2(w)} \Delta w \quad \text{in } \Omega.$$

Proof. Since $u = g(w)$, $\Delta u = g'(w)\Delta w + g''(w)|\nabla w|^2$ and $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2 = 2g(w)g'(w)\Delta w + 2[g'(w)^2 + g(w)g''(w)]|\nabla w|^2$, so that

$$\Delta u + \Delta(u^2)u = g'(w)(1 + 2g^2(w))\Delta w + [g''(w)(1 + 2g^2(w)) + 2g(w)g'(w)^2]|\nabla w|^2.$$

From (1.4), $g'(w)(1 + 2g^2(w)) = \sqrt{1 + 2g^2(w)}$. Differentiating $g'(t)^2(1 + 2g^2(t)) = 1$ in t gives $2g'g''(1 + 2g^2) + 4g(g')^3 = 0$, i.e. $g''(1 + 2g^2) = -2g(g')^2$, which cancels exactly against the term $2g(w)g'(w)^2$ in the bracket above. Hence the coefficient of $|\nabla w|^2$ vanishes identically, and (2.1) follows. $\square$

Lemma 2.1 is the key mechanism of the paper: unlike a formal a priori gradient estimate, (2.1) holds pointwise with no error term, for any $w \in C^2$, independently of any decay information. Applying it to each component of (1.1), if (u, v) is a positive C^2 supersolution of (1.1) in an exterior domain $|x| \geq R_0$, and we set

$$W = g^{-1}(u), \quad Z = g^{-1}(v),$$

then $(W, Z) \in C^2$ and, by Lemma 2.1 applied with $w = W$ and $w = Z$,

$$(2.2) \quad \Delta W \leq \frac{1}{\sqrt{1 + 2g^2(W)}} [V_1(x)g(Z) - g^p(Z)], \quad \Delta Z \leq \frac{1}{\sqrt{1 + 2g^2(Z)}} [V_2(x)g(W) - g^q(W)]$$

in $|x| \geq R_0$. In (2.2), ΔW and ΔZ are ordinary Laplacians of single functions, so their spherical averages satisfy the exact identity $\bar{\Delta W} = \bar{W}'' + \frac{N-1}{r}\bar{W}'$ (see [13, Section 2.2]) used throughout [3].

Suppose, for contradiction, that (1.1) has a positive C^2 supersolution (u, v) in an exterior domain. By Lemma 2.1, $(W, Z) = (g^{-1}(u), g^{-1}(v))$ satisfies (2.2). We first record explicit, non-asymptotic bounds on g .

LEMMA 2.2. *For every $t \geq 0$,*

$$(2.3) \quad g(t) \leq t.$$

Moreover, for $0 \leq t \leq 1$,

$$(2.4) \quad t - \frac{t^3}{3} \leq g(t) \leq t,$$

and, for $p \geq 1$,

$$(2.5) \quad t^p \left(1 - \frac{p}{3}t^2\right) \leq g^p(t) \leq t^p.$$

Proof. Since $g'(t) = (1 + 2g^2(t))^{-1/2} \leq 1$ and $g(0) = 0$, integrating gives (2.3). By convexity of $x \mapsto (1+x)^{-1/2}$ on $[0, \infty)$, the tangent line inequality $(1+x)^{-1/2} \geq 1 - \frac{x}{2}$ holds for all $x \geq 0$; with $x = 2g(t)^2 \leq 2t^2$ (using (2.3)) this gives $g'(t) \geq (1+2t^2)^{-1/2} \geq 1 - t^2$ for $0 \leq t \leq 1$. Integrating over $[0, t]$ yields the lower bound in (2.4). Finally, (2.5) follows from (2.4) and Bernoulli's inequality $(1-x)^p \geq 1 - px$ for $x \in [0, 1]$, $p \geq 1$. $\square$

LEMMA 2.3. *For every $t \geq 0$,*

$$(2.6) \quad (1 + 2t^2)^{-1/2} \leq (1 + 2g^2(t))^{-1/2} \leq 1.$$

Proof. Immediate from (2.3) and monotonicity of $(1+2s)^{-1/2}$. $\square$

By the decay assumption in Theorem 1.1, $u(x), v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Since g^{-1} is continuous and $g^{-1}(0) = 0$, we also have $W, Z \rightarrow 0$. Choose $R_1 \geq R_0$ so large that, for $|x| \geq R_1$,

$$0 \leq W(x), Z(x) \leq 1, \quad 1 - \frac{\max\{p, q\}}{3} \rho(|x|)^2 > 0.$$

For $|x| \geq R_1$, apply Lemma 2.2 to bound the potential term and Lemmas 2.2–2.3 together to bound the nonlinear term from below:

$$\frac{g^p(Z)}{\sqrt{1 + 2g^2(W)}} \geq \left(1 - \frac{p}{3}Z^2\right)(1 + 2W^2)^{-1/2} Z^p =: \lambda_1(x) Z^p,$$

and analogously $\lambda_2(x)$ with (W, q) in place of (Z, p) . Then (2.2) yields, for $|x| \geq R_1$,

$$(2.7) \quad \Delta W \leq V_1(x)Z - \lambda_1(x)Z^p, \quad \Delta Z \leq V_2(x)W - \lambda_2(x)W^q,$$

where $\lambda_1(x), \lambda_2(x) \in (0, 1]$. Note that *the potential coefficients V_1, V_2 are unchanged*; only the nonlinear terms acquire the multiplicative factors $\lambda_1, \lambda_2 \leq 1$. Setting $\rho(r) = \sup_{|x|=r} \max(W, Z) \rightarrow 0$ as $r \rightarrow \infty$ and

$$\lambda(r) := \left(1 - \frac{\max(p, q)}{3} \rho(r)^2\right)(1 + 2\rho(r)^2)^{-1/2},$$

we have $\lambda_1(x), \lambda_2(x) \geq \lambda(|x|)$ for $|x| \geq R_1$, and $\lambda(r) \rightarrow 1$ as $r \rightarrow \infty$.

Case $N \geq 3$:

Set $t = \log r$ and let

$$\varphi(r) := r^{N-2}(\log r)^{\frac{N-2}{2}}(\log \log r)^{-\beta}, \quad S_1 := W_1 + Z_1,$$

where $W_1(t) := \varphi(r)\bar{W}(r)$ and $Z_1(t) := \varphi(r)\bar{Z}(r)$, exactly as the weight used in [3, Section 2, Case 1]. Since $\overline{\Delta W} = \bar{W}'' + \frac{N-1}{r}\bar{W}'$, spherically averaging (2.7) and adding the two resulting inequalities gives, after bounding $V_1, V_2 \leq \tilde{V}(r)$ and applying Jensen's inequality to the convex maps $t \mapsto t^p, t \mapsto t^q$ (so that $\bar{Z}^p \geq \bar{Z}^p, \bar{W}^q \geq \bar{W}^q$),

$$(2.8) \quad \bar{S}'' + \frac{N-1}{r}\bar{S}' - \tilde{V}(r)\bar{S} + \lambda(r)(\bar{Z}^p + \bar{W}^q) \leq 0, \quad \bar{S} := \bar{W} + \bar{Z}.$$

This is precisely the system analogue of [3, (2.1)], with the nonlinear term carrying the extra factor $\lambda(r) \in (0, 1]$ coming from the change of variables.

Writing $r = e^t$ and differentiating $S_1 = \varphi(r)\bar{S}(r)$ in t (using $\frac{d}{dt} = r\frac{d}{dr}$) gives, exactly as in [3, Case 1],

$$S_1' = \left(N - 2 + \frac{N-2}{2t} - \frac{\beta}{t \log t}\right) S_1 + r^{N-2} t^{\frac{N-2}{2}} (\log t)^{-\beta} r \bar{S}',$$

and differentiating once more,

$$\begin{aligned} S_1'' &= \left[-\frac{N-2}{2t^2} + \frac{\beta}{t^2 \log t} + \frac{\beta}{t^2 (\log t)^2}\right] S_1 \\ &\quad + \left(N - 2 + \frac{N-2}{2t} - \frac{\beta}{t \log t}\right) (2S_1' - (N-2)S_1) + r^{N-2} t^{\frac{N-2}{2}} (\log t)^{-\beta} r^2 \bar{S}'' \end{aligned}$$

Substituting $\bar{S}''$ and $r\bar{S}'$ from (2.8) and simplifying yields

$$(2.9) \quad \begin{aligned} 0 &\geq S_1'' - \left(N - 2 + \frac{N-2}{t} - \frac{2\beta}{t \log t}\right) S_1' + \Theta(t) S_1 - \tilde{V}(r) r^2 S_1 \\ &\quad + r^N t^{\frac{N-2}{2}} (\log t)^{-\beta} (\bar{Z}^p + \bar{W}^q) \lambda(r). \end{aligned}$$

where

$$\Theta(t) = \frac{(N-2)^2}{2t} + \frac{N(N-2)}{4t^2} - \frac{\beta(N-2)}{t \log t} - \frac{\beta(N-1)}{t^2 \log t} + \frac{\beta(\beta-1)}{t^2 (\log t)^2}.$$

Multiplying (2.9) by the integrating factor $\mu(t) := \exp(-(N-2)t) t^{-(N-2)} (\log t)^{2\beta}$ and integrating over $[T, t]$ for T large produces,

$$\begin{aligned} \mu(t) S_1'(t) &\leq \mu(T) S_1'(T) - \int_T^t \left[\Theta(s) - \tilde{V}(r(s)) r(s)^2 \right] \mu(s) S_1(s) ds \\ &\quad - \int_T^t \mu(s) r(s)^N s^{\frac{N-2}{2}} (\log s)^{-\beta} \lambda(r(s)) (\bar{Z}^p + \bar{W}^q) ds. \end{aligned}$$

The bracket $\Theta(s) - \tilde{V}r^2$ is nonnegative for $s \geq T$ once T is large, by hypothesis (V), and the last integral is nonpositive since $\lambda \geq 0$ and $\bar{Z}^p + \bar{W}^q \geq 0$; both integrals may thus be discarded to obtain the upper bound

$$S_1'(t) \leq \exp[(N-2)(t-T)] \left(\frac{t}{T}\right)^{N-2} \left(\frac{\log T}{\log t}\right)^{2\beta} S_1'(T).$$

Because the right side is increasing in t (as $N \geq 3$), if $S_1'(T) < 0$ for some large T then S_1 would reach a finite zero, contradicting $S_1 > 0$; hence $S_1'(t) \geq 0$ for all large t , i.e. S_1 is non-decreasing. This gives, as in [3],

$$(2.10) \quad \bar{W}(r) + \bar{Z}(r) \geq Cr^{-N+2} (\log r)^{-\frac{N-2}{2}} (\log \log r)^\beta$$

for some $C > 0$ and all r large. *Note that this step used only that the last integral in the displayed inequality is ≤ 0 : multiplying the nonlinear term by any $\lambda(r) \in (0, 1]$*

preserves this sign, so (2.10) holds with exactly the hypothesis (V) of [3], with no strengthening.

We first record the analogue of [3, Lemma 2.1]. The proof is included because the extra factor $\lambda(r)$ has to be tracked explicitly.

LEMMA 2.4. For $N \geq 3$,

$$\liminf_{r \rightarrow \infty} \frac{r^2(\bar{Z}^p + \bar{W}^q)}{\bar{W} + \bar{Z}} \leq \left(\frac{N-2}{2}\right)^2 + 1.$$

Proof. Assume to the contrary that the liminf is strictly larger than $((N-2)/2)^2 + 1$. Set

$$X(t) := \exp\left(-\frac{(N-2)t}{2}\right)t^{-\frac{N-2}{2}}(\log t)^\beta S_1(t).$$

A direct differentiation, exactly as in [3, Lemma 2.1], and (2.9) give

$$X'' + \left[\lambda(r)\frac{r^2(\bar{Z}^p + \bar{W}^q)}{\bar{W} + \bar{Z}} - \left(\frac{N-2}{2}\right)^2 - \tilde{V}(r)r^2\right]X \leq 0.$$

Because $\lambda(r) \rightarrow 1$ and $\tilde{V}(r)r^2 \rightarrow 0$, the assumed strict inequality implies that, for all sufficiently large t , the coefficient in brackets is larger than 1. Multiplying by $\sin t$ and integrating over $[2k\pi, (2k+1)\pi]$, with k large, yields

$$\begin{aligned} & \int_{2k\pi}^{(2k+1)\pi} \left[\lambda(r)\frac{r^2(\bar{Z}^p + \bar{W}^q)}{\bar{W} + \bar{Z}} - \left(\frac{N-2}{2}\right)^2 - \tilde{V}(r)r^2 - 1\right]X \sin t \, dt \\ & \leq -X(2k\pi) - X((2k+1)\pi) \leq 0. \end{aligned}$$

The integrand on the left is strictly positive, a contradiction. This proves the lemma. $\square$

Notice that, in contrast with the semilinear proof in [3, pp. 4029–4030], we do not need a separate argument to deduce $\bar{W} + \bar{Z} \rightarrow 0$: this is already part of the decay hypothesis on the supersolution, and hence follows from $W, Z \rightarrow 0$ pointwise. We now adapt [3, Lemma 2.2], keeping the factors λ_i in every estimate.

LEMMA 2.5. Let $N \geq 3$. There is no positive supersolution (W, Z) of (2.7) such that

$$\bar{W}(r) \geq c\psi(r) \quad \text{or} \quad \bar{Z}(r) \geq c\psi(r)$$

for all sufficiently large r , where

$$\psi(r) := r^{-N+2}(\log r)^{-\frac{N-2}{2}}(\log \log r)^\beta.$$

Proof. We give the argument when $\bar{W} \geq c\psi$; the other case is symmetric. After spherical averaging the second inequality in (2.7), using $\tilde{V}_2(r) := \max_{|x|=r} V_2(x)$ and Jensen's inequality, we obtain

$$\bar{Z}'' + \frac{N-1}{r}\bar{Z}' \leq \tilde{V}_2(r)\bar{W} - \lambda(r)\bar{W}^q.$$

The first condition in (V) is precisely $\tilde{V}_2(r) = o(\psi(r)^{q-1})$. Hence $\bar{W} \geq c\psi$ and $\lambda(r) \rightarrow 1$ imply, for all large r ,

$$\tilde{V}_2(r)\bar{W} \leq \frac{1}{2}\lambda(r)\bar{W}^q, \quad \bar{Z}'' + \frac{N-1}{r}\bar{Z}' \leq -\frac{1}{2}\lambda(r)\bar{W}^q \leq 0.$$

Thus $r^{N-1}\bar{Z}'(r)$ is nonincreasing. Since $\bar{Z}(r) \rightarrow 0$, it is negative for all sufficiently large r , and integration gives

$$(2.11) \quad \bar{Z}(r) \geq c_1 r^{-N+2}.$$

Moreover, the standard integration used in [3, Lemma 2.2] gives, for every $a > 1$ and all large r ,

$$(2.12) \quad (ar)^{N-2}\bar{Z}(ar) \geq r^{N-2}\bar{Z}(r).$$

We next show that the first equation is eventually superharmonic as well. The exponent assumption implies

$$(2.13) \quad (N-2)(p-1) \leq 2.$$

Indeed, if $(N-2)(p-1) > 2$, then $p > N/(N-2)$ and, since $q \geq p$,

$$(N-2)(pq-1) - 2(q+1) = q((N-2)p-2) - N > 0,$$

contrary to $2(q+1) \geq (N-2)(pq-1)$. From (V), $\tilde{V}_1(r) = O(r^{-2}(\log r)^{-1})$. Combining this with (2.11) and (2.13) yields $\tilde{V}_1(r) = o(\bar{Z}(r)^{p-1})$ (also in the equality case because of the factor $(\log r)^{-1}$). Therefore, for all sufficiently large r ,

$$\bar{W}'' + \frac{N-1}{r}\bar{W}' \leq -\frac{1}{2}\lambda(r)\bar{Z}^p \leq 0.$$

Consequently $r^{N-1}\bar{W}'$ is eventually nonincreasing and negative, and $\bar{W}(r) \geq c_2 r^{-N+2}$; also

$$(2.14) \quad (ar)^{N-2}\bar{W}(ar) \geq r^{N-2}\bar{W}(r), \quad a > 1,$$

for all large r .

We now repeat the quantitative part of [3, Lemma 2.2]. Since $\lambda(r) \rightarrow 1$, after increasing the initial radius we may use $\lambda(r) \geq 1/2$. From (2),

$$(r^{N-1}\bar{W}'(r))' \leq -cr^{N-1}\bar{Z}(r)^p.$$

Both $\bar{W}$ and $\bar{Z}$ are eventually decreasing. Integrating this inequality over $[r/2, r]$ and using $\bar{W}'(r/2) < 0$ and $\bar{Z}(s) \geq \bar{Z}(r)$ for $s \in [r/2, r]$, we obtain

$$-r^{N-1}\bar{W}'(r) \geq c \int_{r/2}^r s^{N-1}\bar{Z}(s)^p ds \geq cr^N\bar{Z}(r)^p,$$

and therefore

$$-\bar{W}'(r) \geq c_3 r \bar{Z}(r)^p.$$

Next, integrating from r to $2r$ and using (2.12), $\bar{Z}(s) \geq \bar{Z}(2r) \geq 2^{-(N-2)}\bar{Z}(r)$ for $s \in [r, 2r]$, gives

$$\bar{W}(r) \geq \bar{W}(r) - \bar{W}(2r) = \int_r^{2r} -\bar{W}'(s) ds \geq c_4 r^2 \bar{Z}(r)^p.$$

Applying the same argument to (2), together with (2.14), yields

$$(2.15) \quad \bar{Z}(r) \geq c_5 r^2 \bar{W}(r)^q.$$

(The constants may change from line to line.) Combining (2)–(2.15),

$$(2.16) \quad \bar{W}(r) \leq Cr^{-\frac{2(p+1)}{pq-1}}, \quad \bar{Z}(r) \leq Cr^{-\frac{2(q+1)}{pq-1}}.$$

Comparing the second estimate in (2.16) with (2.11) and using $2(q+1) \geq (N-2)(pq-1)$ forces

$$2(q+1) = (N-2)(pq-1), \quad \bar{Z}(r) = O(r^{-N+2}).$$

Indeed, strict inequality would make the upper bound in (2.16) decay strictly faster than the lower bound (2.11). In the equality case, (2) and $\bar{Z}(r) \geq c_1 r^{-N+2}$ give

$$\bar{W}(r) \geq c_6 r^{2-(N-2)p}.$$

Using this in the second transformed inequality and the already established potential–nonlinearity dominance, we obtain

$$-r^{N-1} \bar{Z}'(r) \geq c_7 \int_{r_0}^r s^{N-1} \bar{W}(s)^q ds - C \geq c_8 \int_{r_0}^r s^{N-1+2q-(N-2)pq} ds - C.$$

The critical identity gives $N-1+2q-(N-2)pq = -1$, and hence $-r^{N-1} \bar{Z}'(r) \geq c_9 \log r$ for large r . Since $\bar{Z}(\infty) = 0$,

$$\bar{Z}(r) = \int_r^\infty -\bar{Z}'(s) ds \geq c_{10} \int_r^\infty s^{1-N} \log s ds \geq c_{11} r^{-N+2} \log r.$$

contradicting $\bar{Z}(r) = O(r^{-N+2})$. If instead $\bar{Z} \geq c\psi$ for all large r , one first uses $\tilde{V}_1(r) = O(r^{-2}(\log r)^{-1}) = o(\psi(r)^{p-1})$, which follows from (2.13), to obtain $\Delta \bar{W} \leq -c\bar{Z}^p \leq 0$. Since $\bar{W}(r) \rightarrow 0$, the same radial monotonicity argument then yields $\bar{W}(r) \geq cr^{-N+2}$ for all sufficiently large r . Because $\psi(r) = o(r^{-N+2})$, this implies in particular $\bar{W}(r) \geq c'\psi(r)$. Thus the first case already proved applies, and gives a contradiction. Notice that this reduction is preferable to simply interchanging the two components, since the assumption on V_2 in (V) is stronger and the hypotheses are not completely symmetric. This proves the lemma. $\square$

Since $S_1 = W_1 + Z_1$ is non-decreasing, if $\bar{W}(r) \geq \bar{Z}(r)$ for all sufficiently large r , then $S_1 \leq 2W_1$ eventually, so $\bar{W}(r) \geq cr^{-N+2}(\log r)^{-\frac{N-2}{2}}(\log \log r)^\beta$ for r large, contradicting Lemma 2.5; symmetrically if $\bar{Z}(r) \geq \bar{W}(r)$ eventually. Hence neither inequality can hold for all large r , and there must exist a sequence $\{r_j\}_{j \in \mathbb{N}}$, $r_j \rightarrow \infty$, with

$$\bar{W}(r_j) = \bar{Z}(r_j), \quad \bar{W}(r) \geq \bar{Z}(r) \text{ for } r \in (r_{2j+1}, r_{2j+2}), \quad \bar{Z}(r) \geq \bar{W}(r) \text{ for } r \in (r_{2j}, r_{2j+1}).$$

Fix such a j and consider $r \in [r_{2j}, r_{2j+1}]$. On this interval $\bar{Z} \geq \bar{W}$. After spherical averaging, the estimate must be written with the radial majorant $\tilde{V}$ rather than the pointwise coefficient. Thus

$$\Delta \bar{W} \leq \tilde{V}(r) \bar{Z} - \lambda(r) \bar{Z}^p.$$

The nonlinear term does dominate the potential term under (V). Indeed, the exponent condition implies

$$(N-2)(p-1) \leq 2.$$

To see this, if $p > 1 + 2/(N-2)$, then, using $q \geq p$,

$$(N-2)(pq-1) - 2(q+1) = q((N-2)p-2) - N > 0,$$

contrary to the hypothesis. Moreover, equality $(N-2)(p-1) = 2$ forces $p = q = N/(N-2)$. Since $\bar{Z} \geq \bar{S}/2$ on the present interval, (2.10) yields $\bar{Z} \geq c\psi$. On the

other hand, (V) gives $\tilde{V}(r) = O(r^{-2}(\log r)^{-1})$. If $(N-2)(p-1) < 2$, this is $o(\psi^{p-1})$; in the equality case,

$$\psi(r)^{p-1} = r^{-2}(\log r)^{-1}(\log \log r)^{2\beta/(N-2)},$$

so again $\tilde{V}(r) = o(\psi^{p-1})$. Since $\lambda(r) \rightarrow 1$, for all sufficiently large crossing intervals,

$$\tilde{V}(r)\bar{Z} \leq \frac{1}{2}\lambda(r)\bar{Z}^p,$$

and hence $\Delta\bar{W} \leq -\frac{1}{2}\lambda(r)\bar{Z}^p \leq 0$.

On an interval where $\bar{W} \geq \bar{Z}$, the second equation is used instead. There $\bar{W} \geq c\psi$, and the first decay condition in (V) says precisely $V_2(x) = o(\psi(|x|)^{q-1})$; hence $\tilde{V}_2(r)\bar{W} \leq \frac{1}{2}\lambda(r)\bar{W}^q$ for large r and $\Delta\bar{Z} \leq 0$. Comparing $\bar{Z}$ with $c\psi$ on that interval gives $\bar{Z} \geq c\psi$ there; since $\bar{W} \geq \bar{Z}$, it follows that $\bar{W} \geq c\psi$ there as well. Thus the two types of crossing intervals together imply the same lower bound for $\bar{W}$ throughout the whole tail. The endpoint values on an interval of the first type satisfy

$$\bar{W}(r_{2j}) \geq cr_{2j}^{-N+2}(\log r_{2j})^{-\frac{N-2}{2}}(\log \log r_{2j})^\beta$$

and

$$\bar{W}(r_{2j+1}) \geq cr_{2j+1}^{-N+2}(\log r_{2j+1})^{-\frac{N-2}{2}}(\log \log r_{2j+1})^\beta.$$

Since the comparison function $\psi(r) := r^{-N+2}(\log r)^{-\frac{N-2}{2}}(\log \log r)^\beta$ satisfies $\Delta\psi(r) > 0$ for r large, the classical comparison principle on the annulus $r_{2j} \leq |x| \leq r_{2j+1}$ yields

$$\bar{W}(r) \geq c\psi(r), \quad r \in [r_{2j}, r_{2j+1}].$$

As j was arbitrary, this shows $\bar{W}(r) \geq c\psi(r)$ for all sufficiently large r , which contradicts Lemma 2.5. This contradiction shows that no such crossing sequence $\{r_j\}$ can exist, completing the proof for $N \geq 3$.

Cases $N = 1, 2$:

We now carry out the computations for $N = 1, 2$ explicitly, following [3, Theorem 1.4], with (2.7) in place of [3, (1.4)].

Case $N = 2$: Let $t = \log r$ and $S_1(t) := (\log t)^\beta \bar{S}(r)$, where $\bar{S} = \bar{W} + \bar{Z}$. As in [3, Case 2],

$$S_1'' - \frac{2\beta}{t \log t} S_1' + \left[\frac{\beta}{t^2 \log t} + \frac{\beta(\beta+1)}{t^2 (\log t)^2} - \tilde{V}(r)r^2 \right] S_1 + r^2 (\log t)^\beta \lambda(r) (\bar{Z}^p + \bar{W}^q) \leq 0.$$

By (V) the bracketed coefficient is nonnegative for large t . The same integrating-factor argument as in [3, Case 2] shows that S_1 is eventually nondecreasing, and therefore

$$(2.17) \quad \bar{S}(r) \geq C(\log \log r)^{-\beta}.$$

Since $\bar{W}, \bar{Z} \rightarrow 0$ and $p \leq q$, for all large r ,

$$\bar{Z}^p + \bar{W}^q \geq \bar{Z}^q + \bar{W}^q \geq 2^{1-q}(\bar{W} + \bar{Z})^q.$$

Consequently (2.17) gives

$$r^2 \frac{\bar{Z}^p + \bar{W}^q}{\bar{W} + \bar{Z}} \geq cr^2 (\log \log r)^{-\beta(q-1)} \rightarrow \infty.$$

Now set $X(t) := (\log t)^{-\beta} S_1(t) = \bar{S}(r)$. Since $\bar{S}''(r) + r^{-1} \bar{S}'(r) = r^{-2} X''(t)$, the averaged inequality (2.8) becomes

$$X'' + \left[\lambda(r)r^2 \frac{\bar{Z}^p + \bar{W}^q}{\bar{W} + \bar{Z}} - \tilde{V}(r)r^2 \right] X \leq 0.$$

By (2), $\lambda(r) \rightarrow 1$, and (V), the coefficient in brackets is larger than 1 for all sufficiently large t . Multiplication by $\sin t$ and integration over $[2k\pi, (2k+1)\pi]$ gives the same contradiction as in [3, Lemma 2.1, Case 2]. Hence no positive supersolution exists for $N = 2$.

Case $N = 1$: We work on the half-line $r = x > R_0$ and put $t = \log r$. Choose the number β from (V), so that $0 < \beta < 1/(q-1)$, and define

$$S_1(t) := r^\beta S(r), \quad S(r) := W(r) + Z(r).$$

Since $d/dr = r^{-1}d/dt$, the one-dimensional averaged inequality gives

$$S_1'' - (2\beta + 1)S_1' + [\beta(\beta + 1) - \tilde{V}(r)r^2]S_1 + r^{\beta+2}\lambda(r)(Z^p + W^q) \leq 0.$$

This is the corrected form of the computation in [3, Case 3]. By (V) the coefficient of S_1 is nonnegative for large t . Multiplication by $e^{-(2\beta+1)t}$ shows that if $S_1'(T) < 0$ at some sufficiently large T , then S_1 reaches zero in finite time. Hence S_1 is eventually nondecreasing and

$$(2.18) \quad S(r) = W(r) + Z(r) \geq Cr^{-\beta}.$$

As above, since $W, Z \rightarrow 0$ and $p \leq q$,

$$Z^p + W^q \geq 2^{1-q}(W + Z)^q.$$

Thus (2.18) and $\beta < 1/(q-1)$ imply

$$(2.19) \quad r^2 \frac{Z^p + W^q}{W + Z} \geq cr^{2-\beta(q-1)} \rightarrow \infty.$$

Finally set

$$X(t) := \exp\left(-\frac{(2\beta+1)t}{2}\right)S_1(t).$$

Using (2),

$$X'' + \left[\lambda(r)r^2 \frac{Z^p + W^q}{W + Z} - \frac{1}{4} - \tilde{V}(r)r^2\right]X \leq 0.$$

Here the cancellation $\beta(\beta+1) - (2\beta+1)^2/4 = -1/4$ is exactly the one used in [3, Lemma 2.1, Case 3]. By (2.19) and $\lambda(r) \rightarrow 1$, the bracket is larger than 1 for large t . Multiplying by $\sin t$ on $[2k\pi, (2k+1)\pi]$ again gives a contradiction. Hence no positive supersolution exists for $N = 1$. This completes the proof of Theorem 1.1.

3. Concluding remarks

Theorem 1.1 shows that the quasilinear correction $\Delta(u^2)u$ in (1.1), while it obstructs a direct spherical-average argument, does not change the nonexistence threshold for decaying supersolutions once the exact change of variables (1.4) is applied componentwise: the quantitative decay bounds in (V) are unchanged from [3], while the nonnegativity of the potentials is stated explicitly. It would be of interest to determine the precise existence/nonexistence threshold (in the spirit of Remark 1.3 of [3]) for the quasilinear system (1.1) in the regime $2(q+1) < (N-2)(pq-1)$, extending the explicit example of Remark 1.3.

References

- [1] J. F. Aires and M. A. S. Souto, *Existence of solutions for a quasilinear Schrödinger equation with vanishing potentials*, J. Math. Anal. Appl. **416** (2014), 924–946.
<https://doi.org/10.1016/j.jmaa.2014.03.018>
- [2] S. Bae and J. Byeon, *Standing waves of nonlinear Schrödinger equations with optimal conditions for potential and nonlinearity*, Commun. Pure Appl. Anal. **12** (2) (2013), 831–850.
<https://doi.org/10.3934/cpaa.2013.12.831>
- [3] S. Bae and O. Kwon, *Nonexistence of positive solutions of nonlinear elliptic systems with potentials vanishing at infinity*, Nonlinear Anal. **75** (2012), 4025–4032.
<https://doi.org/10.1016/j.na.2012.02.019>
- [4] F. G. Bass and N. N. Nasanov, *Nonlinear electromagnetic spin waves*, Phys. Rep. **189** (1990), 165–223.
- [5] A. V. Borovskii and A. L. Galkin, *Dynamical modulation of an ultrashort high-intensity laser pulse in matter*, JETP **77** (1993), 562–573.
- [6] H. S. Brandi, C. Manus, G. Mainfray, T. Lehner, and G. Bonnaud, *Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma*, Phys. Fluids B **5** (1993), 3539–3550.
- [7] Y. Brihaye, B. Hartmann, and W. J. Zakrzewski, *Spinning solitons of a modified nonlinear Schrödinger equation*, Phys. Rev. D **69** (2004), 087701.
<https://doi.org/10.1103/PhysRevD.69.087701>
- [8] L. Brizhik, A. Eremko, B. Piette, and W. J. Zakrzewski, *Electron self-trapping in a discrete two-dimensional lattice*, Physica D **159** (2001), 71–90.
[https://doi.org/10.1016/S0167-2789\(01\)00332-3](https://doi.org/10.1016/S0167-2789(01)00332-3)
- [9] J. Chen, X. Huang, B. Cheng, and C. Zhu, *Some results on standing wave solutions for a class of quasilinear Schrödinger equations*, J. Math. Phys. **60** (9) (2019), 091506, 55 pp.
<https://doi.org/10.1063/1.5093720>
- [10] X. L. Chen and R. N. Sudan, *Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse*, Phys. Rev. Lett. **70** (1993), 2082–2085.
<https://doi.org/10.1103/PhysRevLett.70.2082>
- [11] Y. Cheng and J. Wei, *Fast and slow decaying solutions for H^1 -supercritical quasilinear Schrödinger equations*, Calc. Var. Partial Differential Equations **58** (4) (2019), Art. 144, 24 pp.
<https://doi.org/10.1007/s00526-019-1594-0>
- [12] A. de Bouard, N. Hayashi, and J. C. Saut, *Global existence of small solutions to a relativistic nonlinear Schrödinger equation*, Commun. Math. Phys. **189** (1997), 73–105.
<https://doi.org/10.1007/s002200050191>
- [13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (1977).
- [14] M. V. Goldman and M. Porkolab, *Upper hybrid solitons and oscillating two-stream instabilities*, Phys. Fluids **19** (1976), 872–881.
- [15] R. W. Hasse, *A general method for the solution of nonlinear soliton and kink Schrödinger equations*, Z. Phys. B **37** (1980), 83–87.
- [16] H. Hartmann and W. J. Zakrzewski, *Electrons on hexagonal lattices and applications to nanotubes*, Phys. Rev. B **68** (2003), 184302.
<https://doi.org/10.1103/PhysRevB.68.184302>
- [17] O. Kwon, *Nonexistence of positive solutions for quasilinear equations with decaying potentials*, Mathematics **8** (3) (2020), Art. 425, 1–7.
<https://doi.org/10.3390/math8030425>
- [18] S. Kurihara, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Jpn. **50** (1981), 3262–3267.
- [19] E. W. Laedke, K. H. Spatschek, and L. Stenflo, *Evolution theorem for a class of perturbed envelope soliton solutions*, J. Math. Phys. **24** (1983), 2764–2769.
- [20] A. G. Litvak and A. M. Sergeev, *One dimensional collapse of plasma waves*, JETP Lett. **27** (1978), 517–520.

- [21] J. Liu and Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations*, Proc. Amer. Math. Soc. **131** (2003), 441–448.
<https://doi.org/10.1090/S0002-9939-02-06783-7>
- [22] J. Liu, Y.-Q. Wang, and Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations. II*, J. Differential Equations **187** (2) (2003), 473–493.
[https://doi.org/10.1016/S0022-0396\(02\)00064-5](https://doi.org/10.1016/S0022-0396(02)00064-5)
- [23] V. G. Makhankov and V. K. Fedyanin, *Non-linear effects in quasi-one-dimensional models of condensed matter theory*, Phys. Rep. **104** (1984), 1–86.
[https://doi.org/10.1016/0370-1573\(84\)90106-6](https://doi.org/10.1016/0370-1573(84)90106-6)
- [24] M. Poppenberg, K. Schmitt, and Z.-Q. Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Differential Equations **14** (2002), 329–344.
<https://doi.org/10.1007/s005260100105>
- [25] Y. Qi, *The existence and nonexistence theorems for ground states of nonlinear elliptic systems*, Comm. Partial Differential Equations **23** (1998), 1749–1780.
- [26] B. Ritchie, *Relativistic self-focusing and channel formation in laser-plasma interactions*, Phys. Rev. E **50** (1994), 687–689.
<https://doi.org/10.1103/PhysRevE.50.R687>
- [27] U. B. Severo and G. M. de Carvalho, *Quasilinear Schrödinger equations with unbounded or decaying potentials*, Math. Nachr. **291** (2–3) (2018), 492–517.
<https://doi.org/10.1002/mana.201600028>
- [28] Y. Xue, Y. Lv, and C. Tang, *Existence and nonexistence results for quasilinear Schrödinger equations with a general nonlinear term*, Ann. Polon. Math. **120** (3) (2017), 271–293.
<https://doi.org/10.4064/ap170502-2-12>
- [29] J. Zhang, X. Tang, and W. Zhang, *Existence of infinitely many solutions for a quasilinear elliptic equation*, Appl. Math. Lett. **37** (2014), 131–135.
<https://doi.org/10.1016/j.aml.2014.06.010>

Ohsang Kwon

Department of Mathematics, Chungbuk National University,
Chungdae-ro 1, Seowon-gu, Cheongju, Chungbuk 28644, Korea
E-mail: ohsangkwon@chungbuk.ac.kr