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REFLEXIVE PROPERTY SKEWED BY RING ENDOMORPHISMS

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Abstract. Mason extended the reflexive property for subgroups to right ideals, and examined various connections between these and related concepts. A ring was usually called *reflexive* if the zero ideal satisfies the reflexive property. We here study this property skewed by ring endomorphisms, introducing the concept of an α-*skew reflexive ring*, where α is an endomorphism of a given ring.

1. Introduction

Throughout this paper, all rings are associative with identity. We denote by $R[x]$ the polynomial ring with an indeterminate x over R. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively. Denote the n by n full matrix ring over a ring R by $Mat_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. Use E_{ij} for the matrix unit with (i, j)-entry 1 and elsewhere 0.

Recall that a ring is *reduced* if it has no nonzero nilpotent elements, and a ring R is called *reversible* [4] if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and a ring R is called to satisfy the *Insertion-of-Factors-Property* (simply, an *IFP* ring) [3] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. A ring is called *Abelian* if every idempotent is central. Commutative rings and reduced rings are clearly reversible. A simple computation gives that

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reduced rings are reversible, and reversible rings are IFP and IFP rings are Abelian, but the converses do not hold in general. We will freely use these facts without reference.

Another generalization of reduced rings is a reflexive ring. Due to Mason [11], a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$ for $a, b \in R$. Semiprime rings and reversible rings are reflexive by a simple computation. In [10], it is shown that the reflexive property is Morita invariant, and the authors constructed a reflexive ring that is not semiprime from any semiprime ring.

Generalized reduced rings were extended by ring endomorphisms. According to Krempa [9], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called *α-rigid* [7] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by [7, Proposition 5].

In [2, Definition 2.1], an endomorphism α of a ring R is called *right skew reversible* if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$, and the ring R is called *right α-skew reversible* if there exists a right skew reversible endomorphism α of R. Similarly, left α-skew reversible rings are defined. A ring R is *α-skew reversible* if it is both right and left α-reversible. Note that R is an α-rigid ring if and only if R is semiprime and right α-skew reversible for a monomorphism α of R by [2, Proposition 2.5(iii)].

Motivated by above, in this paper, we extend the reflexive property to the skewed reflexive property by ring endomorphisms. We introduce the notation of *an α-skew reflexive ring* for an endomorphism α of a ring as a generalization of α-rigid rings and an extension of reflexive rings, and then study the structure of α-skew reflexive rings and their related properties. Consequently, several known results are obtained as corollaries of our results.

We change over from "an α-reversible ring" in [2] to "an α-skew reversible ring", so as to cohere with other related definitions.

Throughout this paper, α denotes a nonzero endomorphism of given rings, unless specified otherwise.

2. Basic properties of right α-skew reflexive rings

We begin with the following definition.

Definition 2.1. An endomorphism α of a ring R is called *right* (resp., *left*) *skew reflexive* if whenever $aRb = 0$ for $a, b \in R$, $bR\alpha(a) = 0$ (resp., $\alpha(b)Ra = 0$). A ring R is called *right* (resp., *left*) *α-skew reflexive* if there exists a right (resp., left) skew reflexive endomorphism α of R. A ring is called *α-skew reflexive* if it is both left and right α-skew reflexive.

A ring R is reflexive if R is 1_R-reflexive where 1_R denotes the identity endomorphism of R. Any domain R is obviously α-skew reflexive for any endomorphism α of R, but the converse need not hold by the following example which also shows that the α-skew reflexive property is not left-right symmetric.

Example 2.2. Let S be a reflexive ring. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in S \right\}.$$

(1) Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

For

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R,$$

assume that $ARB = 0$. Then for any $\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in R$, we have $aua' = 0$ and so $aSa' = 0$. Since S is reflexive, $a'Sa = 0$. This entails that $BR\alpha(A) = 0$, and hence R is right α-skew reflexive.

However, R is not left α-skew reflexive. To see this, let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

Then $ARB = 0$, but

$$0 \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^2 = \alpha(B)A^2 \in \alpha(B)RA,$$

showing that R is not left α-skew reflexive.

(2) Let $\beta : R \to R$ be an endomorphism defined by

$$\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By the similar method to (1), we can show that R is left β-skew reflexive which is not right β-skew reflexive.

We have the basic equivalences for right α-skew reflexive rings as follows. For a nonempty subset X of a ring R, we write $r_R(X) = \{c \in R \mid Xc = 0\}$ which is called the *right annihilator* of X in R. Similarly, $\ell_R(X)$ denotes the left annihilator of X in R.

Proposition 2.3. *For a ring R with an endomorphism α, the following are equivalent:*

(1) *R is right α-skew reflexive.*

(2) *For $a \in R$, $r_R(aR) \subseteq \ell_R(R\alpha(a))$.*

(3) *For nonempty subsets A and B of R, $ARB = 0$ implies $BR\alpha(A) = 0$.*

(4) *$IJ = 0$ implies $J\alpha(I) = 0$ for right (or, left) ideals I, J of R.*

(5) *$IJ = 0$ implies $J\alpha(I) = 0$ for ideals I, J of R.*

Proof. (1)⇒(2) and (3)⇒(4)⇒(5) and are straightforward.

(2)⇒(3): Let A and B be nonempty subsets satisfying $ARB = 0$. Then $aRb = 0$ for all $a \in A$ and $b \in B$, and hence $bR\alpha(a) = 0$ by the condition. Thus $BR\alpha(A) = \sum_{a \in A, b \in B} bR\alpha(a) = 0$.

(5)⇒(1): Let $aRb = 0$ for $a, b \in R$. Then $RaRRbR = 0$ and so the condition (5) implies that $bR\alpha(a) \subseteq RbR\alpha(RaR) = 0$. We conclude that R is a right α-skew reflexive ring. □

Proposition 2.4. *Let α be an endomorphism of a ring R.*

(1) *If R is a right α-skew reversible ring, then R is right α-skew reflexive. The converse holds if R is an IFP ring.*

(2) *Let R be a reversible ring. Then R is right α-skew reflexive if and only if R is left α-skew reflexive.*

(3) *If R is an α-skew reflexive ring, then $aRb = 0$ for $a, b \in R$ implies $aR\alpha^{2k}(b) = 0$ and $\alpha^{2k-1}(b)Ra = 0$; moreover, $\alpha^{2k}(a)Rb = 0$ and $bR\alpha^{2k-1}(a) = 0$ for any $k \geq 1$.*

(4) *Let S be a ring and suppose that $\sigma : R \to S$ be a ring isomorphism. Then R is a right α-skew reflexive ring if and only if S is a right $\sigma\alpha\sigma^{-1}$-skew reflexive ring.*

Proof. (1) Assume that R is a right α-skew reversible ring and $aRb = 0$ for $a, b \in R$. Then $abr = 0$ for any $r \in R$ and so $(br)\alpha(a) = 0$. Thus $bR\alpha(a) = 0$, showing that R is right α-skew reflexive.

Conversely, assume that R is right α-skew reflexive and IFP. Let $ab = 0$ for $a, b \in R$. Since R is IFP. $aRb = 0$ and so $bR\alpha(a) = 0$. Hence $b\alpha(a) = 0$, showing that R is right α-skew reversible.

(2) Let R be a reversible ring. Suppose that R is right α-skew reflexive and $aRb = 0$ for $a, b \in R$. Then

$ab = 0 \Rightarrow ba = 0 \Rightarrow bRa = 0 \Rightarrow aR\alpha(b) = 0 \Rightarrow a\alpha(b) = 0 \Rightarrow \alpha(b)a = 0 \Rightarrow \alpha(b)Ra = 0$,

and so R is left α-skew reflexive. The converse can be shown by an almost same argument as above.

(3) Suppose that R is α-skew reflexive and $aRb = 0$ for $a, b \in R$. Then

$$\alpha(b)Ra = 0 \Rightarrow aR\alpha^2(b) = 0 \Rightarrow \alpha^3(b)Ra = 0 \Rightarrow aR\alpha^4(b) = 0, \cdots;$$

and so we get $aR\alpha^{2k}(b) = 0$ and $\alpha^{2k-1}(b)Ra = 0$ for $k \geq 1$ inductively.

The remainder of the proof is similar to above.

(4) Let R be right α-skew reflexive and $a'Sb' = 0$ for $a', b' \in S$. Then there exist $a, b \in R$ such that $\sigma(a) = a'$ and $\sigma(b) = b'$. So $a'Sb' = 0$ implies $aRb = 0$ and hence $bR\alpha(a) = 0$ by hypothesis. This entails that

$$0 = \sigma(bR\alpha(a)) = \sigma(bR\alpha(\sigma^{-1}\sigma)(a)) = b'S\sigma\alpha\sigma^{-1}(a'),$$

showing that S is right $\sigma\alpha\sigma^{-1}$-skew reflexive. The converse can be similarly obtained. $\square$

Proposition 2.4(1) leads to the next corollary.

Corollary 2.5. *A ring R is reversible if and only if R is reflexive and IFP.*

Notice that the condition "R is an IFP ring" is not superfluous in Proposition 2.4(1). For example, consider a ring $R = Mat_2(\mathbb{Z})$. Then R is 1_R-skew reflexive by [10, Theorem 2.6(2)], but it is not IFP and so not 1_R-skew reversible, either.

Proposition 2.6. *A ring R is reduced and right α-skew reflexive for a monomorphism α of R if and only if R is α-rigid.*

Proof. It follows from [2, Proposition 2.5(iii)] and Proposition 2.4(1), since reduced rings are semiprime and IFP. $\square$

The following example shows that each condition in Proposition 2.6 is not superfluous.

EXAMPLE 2.7. (1) Consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a reduced ring. Now, let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism. For $a = (1, 0), b = (0, 1) \in R$, we have $aRb = 0$ but $0 \neq b = b^2\alpha(a) \in bR\alpha(a)$, showing that R is not right α-skew reflexive. Note that $a\alpha(a) = 0$ for $0 \neq a = (1, 0) \in R$ and so R is not α-rigid.

(2) Let F be a division ring and $R = F[x]$ the polynomial ring over F. Define $\alpha : R \to R$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. Then R is a domain and so reduced. Thus R is clearly right α-skew reflexive. But α is not a monomorphism, and so R is not α-rigid.

(3) Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}$$

and an endomorphism $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Note that α is an automorphism. Clearly, R is not reduced and so not α-rigid. However, R is α-skew reflexive by Proposition 2.11 to follow.

For an endomorphism α and an idempotent e of a ring R such that $\alpha(e) = e$, we have an endomorphism $\bar{\alpha} : eRe \to eRe$ defined by $\bar{\alpha}(ere) = e\alpha(r)e$.

PROPOSITION 2.8. *Let R be a ring with an endomorphism α such that $\alpha(e) = e$ for $e^2 = e \in R$.*

(1) *If R is right α-skew reflexive then eRe is right $\bar{\alpha}$-skew reflexive.*

(2) *If e is a central idempotent R, then eR and $(1 - e)R$ are right $\bar{\alpha}$-skew reflexive if, and only if R is right α-skew reflexive.*

Proof. (1) For $eae, ebe \in eRe$, suppose that $eae(eRe)ebe = 0$. Since R is right α-skew reflexive,

$$0 = (ebe)R\alpha(eae) = ebe(eRe)e\alpha(a)e = (ebe)R\bar{\alpha}(eae).$$

Thus eRe is right $\bar{\alpha}$-skew reflexive.

(2) It is enough to show the necessity by (1). Suppose that eR and $(1 - e)R$ are right $\bar{\alpha}$-skew reflexive for a central idempotent $e \in R$. Let $aRb = 0$ for $a, b \in R$. Then $ea(eR)eb = 0$ and $(1-e)a((1-e)R)(1-e)b = 0$. By hypothesis, $0 = eb(eR)\bar{\alpha}(ea) = eb(eR)e\alpha(a) = ebR\alpha(a) = 0$ and $0 = (1 - e)b((1 - e)R)\bar{\alpha}((1 - e)a) = (1 - e)b((1 - e)R)(1 - e)\alpha(a) = (1 - e)bR\alpha(a) = 0$. Thus $bR\alpha(a) = 0$, completing the proof. $\square$

Let α_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. For the product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ and the endomorphism $\bar{\alpha} : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)) = (\alpha_\gamma(a_\gamma))$, it can be easily checked that $\prod_{\gamma\in\Gamma} R_\gamma$ is right $\bar{\alpha}$-skew reflexive if and only if each R_γ is right α_γ-skew reflexive.

Recall that for a ring R with an endomorphism α and an ideal I of R, if I is an α-ideal (i.e., $\alpha(I) \subseteq I$) of R, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of a factor ring R/I. The class of right α-skew reflexive rings is not closed under homomorphic images and vice versa in general, by help of [10, Example 2.8 and Example 2.9].

Proposition 2.9. *Let R be a ring with an automorphism α and I an α-ideal of R. If R/I be a right $\bar{\alpha}$-skew reflexive ring and I is α-rigid as a ring without identity, then R is right α-skew reflexive.*

Proof. Let $aRb = 0$ with $a, b \in R$. Since R/I is right $\bar{\alpha}$-skew reflexive, we have $bR\alpha(a) \subseteq I$, and so $bR\alpha(a)R\alpha(bR\alpha(a)R) = bR\alpha(aRb)\alpha(R\alpha(a)R)$ $= 0$ implies $bR\alpha(a) = 0$ since I is α-rigid. Therefore R is right α-skew reflexive. □

The next example illuminates that the condition "I is α-rigid as a ring without identity" of Proposition 2.9 cannot be weakened by the condition "I is right α-skew reflexive as a ring without identity".

Example 2.10. Consider a ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

and an automorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

where F is a division ring.

For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have

$$ARB = 0, \text{ but } 0 \neq \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = B^2\alpha(A) = BR\alpha(A),$$

showing that R is not right α-skew reflexive.

Clearly, the ideal $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of R is right α-skew reflexive but not α-rigid (as a ring without identity), and the factor ring

$$R/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \mid a, c \in F \right\}$$

is reduced and $\bar{\alpha}$ is an identity map on R/I. Thus R/I is right $\bar{\alpha}$-skew reflexive.

We give other kinds of examples of α-skew reflexive rings as follows.

Proposition 2.11. *Let A be a commutative ring satisfying a condition that $ab = 0$ for $a, b \in A$ implies $a = -a$ or $b = -b$. Then the ring*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in A \right\}$$

is α-skew reflexive, where α is an automorphism of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Proof. Let $ARB = 0$ for nonzero $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \in R$. Then $AB = 0$, so we have $aa_1 = 0$ and $ab_1 + ba_1 = 0$.

`Case 1.` $a = 0$ or $a_1 = 0$.

Let $a = 0$. Then $b \neq 0$ and $ba_1 = 0$, entailing $a_1 r(-b) = 0$ for all $r \in R$. This yields

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 r(-b) \\ 0 & 0 \end{pmatrix} = 0$$

for every $\begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \in R$, and hence $BR\alpha(A) = 0$. The computation for the case of $a_1 = 0$ is similar, also obtaining $BR\alpha(A) = 0$.

`Case 2.` $a \neq 0$ and $a_1 \neq 0$.

We have $a = -a$ or $a_1 = -a_1$ by the condition of A. Since A is commutative, we get

$$a_1 ra = 0, \quad a_1 rb + b_1 ra = (-a_1)r(-b) + b_1 ra = 0$$

$$\text{and } a_1 r(-b) + b_1 r(-a) = (-a_1)rb + b_1 r(-a) = 0.$$

for all $r \in R$. If $a = -a$, then we get

$$0 = a_1 r(-b) + b_1 r(-a) = a_1 r(-b) + b_1 ra = 0.$$

If $a_1 = -a_1$, then we have
$0 = -((-a_1)rb + b_1 r(-a)) = -(-a_1)rb - b_1 r(-a) = (-a_1)r(-b) + b_1 ra$
$= a_1 r(-b) + b_1 ra = 0$.
Consequently we get $BR\alpha(A)$ in any case.

Therefore R is right α-skew reflexive ring. Similarly R can be shown to be left α-skew reflexive.

□

Note that the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in A \right\}$ with $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$ for $a, b \in A$ is α-skew reflexive by Proposition 2.11, when A is a commutative domain, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

3. Extensions of right α-skew reflexive rings

In this section we examine several kinds of ring extensions which have roles in ring theory, being concerned with right α-skew reflexive rings.

Let R be an algebra over a commutative ring S. Following Dorroh [5], the *Dorroh extension* of R by S is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

In the following which extends the result of [10, Theorem 2.14], we give some other example of right α-skew reflexive rings.

Theorem 3.1. *Let R be an algebra over a commutative ring S and α an endomorphism of R with $\alpha(1) = 1$. Then R is right α-skew reflexive if and only if the Dorroh extension D of R by S is right $\bar{\alpha}$-skew reflexive.*

Proof. Note that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$. Suppose that R is right α-skew reflexive. Let $(r_1, s_1)D(r_2, s_2) = 0$ for $(r_1, s_1), (r_2, s_2) \in D$. Then $(r_1, s_1)(r, s)(r_2, s_2) = 0$ for any $(r, s) \in D$. This yields $r_1rr_2 + s_1rr_2 + sr_1r_2 + s_2r_1r + s_1sr_2 + s_1s_2r +$

$ss_2r_1 = 0$ and $s_1ss_2 = 0$. So $(r_1, s_1)(r, s)(r_2, s_2) = 0$ is equivalent to $(r_1 + s_1)(r + s)(r_2 + s_2) = 0$ and $s_1ss_2 = 0$. This also implies $(r_1+s_1)R(r_2+s_2) = 0$ with $s_1Ss_2 = 0$. Since R is right α-skew reflexive and S is commutative, we have $(r_2+s_2)R\alpha(r_1+s_1) = 0$ with $s_2Ss_1 = 0$, entailing that $(r_2 + s_2)(r + s)(\alpha(r_1) + s_1) = 0$ with $s_2ss_1 = 0$, since $\alpha(1) = 1$. Thus $(r_2, s_2)(r, s)(\alpha(r_1), s_1) = (r_2, s_2)(r, s)\bar{\alpha}((r_1, s_1)) = 0$, and hence $(r_2, s_2)D\bar{\alpha}((r_1, s_1)) = 0$. Thus D is right $\bar{\alpha}$-skew reflexive.

Conversely, assume that D is right $\bar{\alpha}$-skew reflexive. Let $aRb = 0$ for $a, b \in R$. Then $a(r + s)b = 0$ for any $(r, s) \in D$. This implies $(a, 0)(r, s)(b, 0) = 0$ for any $(r, s) \in D$. Since D is right $\bar{\alpha}$-skew reflexive, we have $(b, 0)(r, s)\bar{\alpha}((a, 0)) = 0$ and hence $b(r+s)\alpha(a) = 0$, proving that $bR\alpha(a) = 0$. Hence R is right α-skew reflexive. □

Note that the condition "$\alpha(1) = 1$" in Theorem 3.1 cannot be dropped by the next example.

Example 3.2. Take a reduced ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication and let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (0, b)$. Then R is right α-skew reflexive. Indeed, assume that $(a, b)R(c, d) = 0$ for nonzero $(a, b), (c, d) \in R$. Then we have $a = 0$ and $d = 0$, or $b = 0$ and $c = 0$. These imply that $(c, d)R\alpha((a, b)) = 0$.

Note that R is not α-rigid since α is not a monomorphism. Now, consider the Dorroh extension D of R by the ring of integers $\mathbb{Z}$. For $a = ((1, 0), -1), b = ((1, 0), 0) \in D$, we get

$$aDb = 0 \ \text{ but } \ 0 \neq (-(1, 0), 0) = b^2\bar{\alpha}(a) \in bD\bar{\alpha}(a),$$

and so D is not right $\bar{\alpha}$-skew reflexive.

For a ring R and $n \geq 2$, consider the rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\},$$

which are subrings of $Mat_n(R)$.

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note that $T(R, R) = D_2(R)$ and $V_n(R) \cong R[x]/(x^n)$, where (x^n) is an ideal of the polynomial ring $R[x]$ over R generated by x^n.

For a ring R with an endomorphism α, the corresponding $(a_{ij}) \mapsto (\alpha(a_{ij}))$ induces endomorphisms of $Mat_n(R)$, $D_n(R)$, $V_n(R)$ and $T(R, R)$. We denote them by $\bar{\alpha}$.

THEOREM 3.3. *Let R be a ring with an endomorphism α. Then R is a right α-skew reflexive if and only if $Mat_n(R)$ is a right $\bar{\alpha}$-skew reflexive ring for all $n \geq 2$.*

Proof. Suppose that R is a right α-skew reflexive ring. Let $M = Mat_n(R)$ and $AB = 0$ for ideals A, B of M. Using an elementary ring theoretic argument, there exist ideals I and J of R such that $A = Mat_n(I)$ and $B = Mat_n(J)$. Then $Mat_n(IJ) = Mat_n(I)Mat_n(J) = AB = 0$ implies $IJ = 0$. Since R is right α-skew reflexive, $J\alpha(I) = 0$ by Proposition 2.3. This yields $B\bar{\alpha}(A) = Mat_n(J)\bar{\alpha}(Mat_n(I)) = Mat_n(J\alpha(I)) = 0$, and so M is right $\bar{\alpha}$-skew reflexive for all $n \geq 2$ by Proposition 2.3.

Conversely, suppose that $Mat_n(R)$ is right $\bar{\alpha}$-skew reflexive for all $n \geq 2$. Let $aRb = 0$ for $a, b \in R$. For $A = a\sum_{i=1}^{n} E_{ii}, B = b\sum_{i=1}^{n} E_{ii} \in Mat_n(R)$, we have $AMat_n(R)B = 0$, and so $BMat_n(R)\bar{\alpha}(A) = 0$ by hypothesis. This implies that $bR\alpha(a) = 0$, showing that R is right α-skew reflexive. □

EXAMPLE 3.4. (1) Let α be an endomorphism of a ring R with $\alpha(1) = 1$. If R is a right α-skew reflexive ring, then $Mat_n(R)$ for $n \geq 2$ is right $\bar{\alpha}$-skew reflexive by Theorem 3.3, but the subring $U_n(R)$ of is not right $\bar{\alpha}$-skew reflexive. For, $E_{22}U_n(R)E_{11} = 0$ but $E_{11}U_n(R)\bar{\alpha}(E_{22}) = RE_{12} \neq 0$. In conclusion, the class of right α-skew reflexive rings is not closed under subrings. Moreover, another subring $D_n(R)$ for $n \geq 3$ is not right $\bar{\alpha}$-skew reflexive: In fact, For any ring A with an endomorphism α such that $\alpha(1) = 1$, let $R = D_n(A)$ for $n \geq 3$. For $E_{(n-1)n}, E_{(n-2)(n-1)} \in R$, $E_{(n-1)n}RE_{(n-2)(n-1)} = 0$, but $E_{(n-2)(n-1)}R\bar{\alpha}(E_{(n-1)n}) \neq 0$, showing that $R = D_n(A)$ for $n \geq 3$ is not $\bar{\alpha}$-skew reflexive.

(2) Let R be an α-rigid ring. Then the ring $D_3(R)$ is not $\bar{\alpha}$-skew reflexive by (1). For $a = E_{23} = b \in D_3(R)$ and any $n \geq 0$, we get $aD_3(R)\bar{\alpha}^n(b) = 0$. This illuminates that the converses of Proposition 2.4(3) does not hold in general.

By the similar argument to the proof of the sufficient condition in Theorem 3.3, we get the following result.

Proposition 3.5. *For an endomorphism α of a ring R and $n \geq 2$, if $U_n(R)$ ($D_n(R)$) is right $\bar{\alpha}$-skew reflexive then R is right α-skew reflexive.*

For a right α-skew reflexive ring R, the trivial extension $T(R, R)$ of R need not to be $\bar{\alpha}$-skew reflexive by the next example.

Example 3.6. Consider the right α-skew reflexive ring (that is not semiprime).

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}.$$

with the endomorphism α defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix},$$

in Example 2.7(3).

For

$$A = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right) \text{ and } B = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \in T(R, R),$$

we have $AT(R, R)B = O$ but

$$\begin{aligned} 0 \neq &\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right) \\ &= B\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)\bar{\alpha}(A) \in BT(R, R)\bar{\alpha}(A). \end{aligned}$$

Thus the trivial extension $T(R, R)$ is not right $\bar{\alpha}$-skew reflexive.

However, we have the following result.

Lemma 3.7. [10, Proposition 2.5(ii)] *Let R be a semiprime ring. Then $aRbRb = 0$ if and only if $aRb = 0$ for $a, b \in R$.*

THEOREM 3.8. *Let R be a ring with an endomorphism α and $n \geq 2$.*

(1) *If R is a semiprime and right α-skew reflexive ring, then $V_n(R)$ is right $\bar{\alpha}$-skew reflexive.*

(2) *If $V_n(R)$ is right $\bar{\alpha}$-skew reflexive, then R is right α-skew reflexive.*

Proof. (1) Suppose that R is a semiprime and right α-skew reflexive ring. We use $(a_1, a_2, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

Let $AV_n(R)B = 0$ for $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n) \in V_n(R)$. For any $r \in R$, $A(r, 0, \ldots, 0)B = 0$. Thus we have the following equations:

$$a_1 r b_1 = 0. \tag{1}$$

$$a_1 r b_2 + a_2 r b_1 = 0. \tag{2}$$

$$a_1 r b_3 + a_2 r b_2 + a_3 r b_1 = 0. \tag{3}$$

$$\vdots$$

$$a_1 r b_{n-1} + a_2 r b_{n-2} + \cdots + a_{n-1} r b_1 = 0. \tag{4}$$

$$a_1 r b_n + a_2 r b_{n-1} + \cdots + a_{n-1} r b_2 + a_n r b_1 = 0. \tag{5}$$

From Eq.(1), we see

$$a_1 R b_1 = 0 \text{ and } b_1 R \alpha(a_1) = 0. \tag{6}$$

If we multiply Eq.(2) on the right-hand side by sb_1 for any $s \in R$, then $a_1 r b_2 s b_1 + a_2 r b_1 s b_1 = 0$ and hence $a_2 R b_1 = 0$ by Lemma 3.7 and Eq.(6), and $a_1 R b_2 = 0$. Thus

$$b_1 R \alpha(a_2) = 0 \text{ and } b_2 R \alpha(a_1) = 0. \tag{7}$$

If we multiply Eq.(3) on the right-hand side by sb_1 for any $s \in R$, then $a_1 r b_3 s b_1 + a_2 r b_2 s b_1 + a_3 r b_1 s b_1 = 0$ and so $a_3 r b_1 = 0$ by Lemma 3.7 and the above. Then Eq.(3) becomes

$$a_1 r b_3 + a_2 r b_2 = 0. \tag{8}$$

If we multiply Eq.(8) on the right-hand side by sb_2 for any $s \in R$, then $a_2rb_2 = 0$ and $a_1rb_3 = 0$ by the similar argument to above. Thus, we have

$$a_iRb_j = 0, \text{ and } b_jR\alpha(a_i) = 0 \text{ for all } 2 \leq i + j \leq 4.$$

Inductively, we assume that

$$a_iRb_j = 0, \text{ and } b_jR\alpha(a_i) = 0 \text{ for all } i + j \leq n.$$

If we multiply Eq. (5) on the right-hand side by $s_1b_1, s_2b_2, \ldots, s_{n-1}b_{n-1}$ for any $s_1, s_2, \ldots, s_{n-1} \in R$, in turn, then

$$a_nRb_1 = 0, a_{n-1}Rb_2 = 0, \ldots, a_2Rb_{n-1} = 0 \text{ and } a_1Rb_n = 0$$

by the similar computation to above, and so

$$b_iR\alpha(a_j) = 0 \text{ for all } i + j = n + 1.$$

Consequently, we get $BV_n(R)\bar{\alpha}(A) = 0$ and therefore $V_n(R)$ is right $\bar{\alpha}$-skew reflexive.

(2) It follows from the similar computation to the proof of the sufficient condition in Theorem 3.3. □

Corollary 3.9. *Let R be a semiprime ring with an endomorphism α. Then the following are equivalent:*

(1) *R is right α-skew reflexive.*

(2) *The trivial extension $T(R, R)$ of R is right $\bar{\alpha}$-skew reflexive.*

(3) *$R[x]/(x^n)$ is right $\bar{\alpha}$-skew reflexive for $n \geq 2$.*

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor). A multiplicatively closed (m.c. for short) subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is shown, by [12, Theorem 2.1.12], that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring of R with respect to S exists.

Suppose that the right quotient ring Q of R exists. For an automorphism α of R and any $au^{-1} \in Q(R)$ where $a \in R$ and $u \in S$, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(au^{-1}) = \alpha(a)\alpha(u)^{-1}$ is also an endomorphism.

Note that the right quotient ring Q of an α-rigid ring R is $\bar{\alpha}$-rigid, where α is an automorphism of R. As a parallel result to this, we have the following result whose proof is modified from the proof of [10, Theorem 2.11].

THEOREM 3.10. *Let S be an m.c. subset of a ring R and α an automorphism of R. Suppose that S satisfies the right Ore condition and S consists of regular elements. If R is right α-skew reflexive, then the right quotient ring Q of R with respect to S is right $\bar{\alpha}$-skew reflexive.*

Proof. Suppose that R is right α-skew reflexive. Let $AQB = 0$ where $A = au^{-1}$ and $B = bv^{-1}$ with $a, b \in R$ and $u, v \in S$. Then we have $0 = AQB = aQ(bv^{-1})$, since $Q = u^{-1}Q$. Thus $a(rs^{-1})(bv^{-1}) = 0$ for any $rs^{-1} \in Q$. By hypothesis, there exist $c \in R$ and $w \in S$ such that $s^{-1}b = cw^{-1}$. Hence, $0 = a(rs^{-1})(bv^{-1}) = arcw^{-1}v^{-1}$ for any $r \in R$ and so we have $aRc = 0$ and $cR\alpha(a) = 0$. From $aRc = 0$ and $bw = sc$, we get $0 = arsc = arbw$ for any $r \in R$ and hence $aRb = 0$ and $bR\alpha(a) = 0$. Since $v^{-1}Q = Q$, $BQ\bar{\alpha}(A) = bQ(\alpha(a)\alpha^{-1}(u))$. Consider $b(rt^{-1})\alpha(a)\alpha^{-1}(u)$ for any $rt^{-1} \in Q$. For $\alpha(a)$ and t, there exist $d \in R$ and $l \in S$ such that $\alpha(a)l = td$ and $t^{-1}\alpha(a) = dl^{-1}$. Since α is an automorphism, there exist l', t' and $d' \in R$ such that $l = \alpha(l'), t = \alpha(t')$ and $d = \alpha(d')$, and hence $al' = t'd'$. The facts that $aRb = 0$ and $al' = t'd'$ imply $0 = al'rb = t'd'rb$ for any $r \in R$, and so $d'Rb = 0$ and $bR\alpha(d') = bRd = 0$. Since $bRd = 0$, $0 = brdl^{-1}\alpha^{-1}(u) = b(rt^{-1})\alpha(a)\alpha^{-1}(u) = b(rt^{-1})\bar{\alpha}(au^{-1})$ for any $rt^{-1} \in Q$ and thus $BQ\bar{\alpha}(A) = 0$. Therefore Q is right $\bar{\alpha}$-skew reflexive. □

The following proposition is obtained by applying the method in the proof of Theorem 3.10.

PROPOSITION 3.11. *Let M be an m.c. subset of a ring R consisting of central regular elements and α an automorphism of R. Then R is right α-skew reflexive if and only if $M^{-1}R$ is right $\bar{\alpha}$-skew reflexive.*

Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by

$$\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i) x^i$$

is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall denote the extended map $R[x] \to R[x]$ by $\bar{\alpha}$ and the image of $f \in R[x]$ by $\alpha(f)$.

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} r_i x^i$ with the usual addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$.

For an endomorphism α of R, we define the map $R[x; x^{-1}] \to R[x; x^{-1}]$ by the same endomorphism as in the polynomial ring $R[x]$ above.

The following result extends the class of right α-skew reflexive rings.

COROLLARY 3.12. *For a ring R with an automorphism α, $R[x]$ is right α-skew reflexive if and only if $R[x; x^{-1}]$ is right $\bar{\alpha}$-skew reflexive.*

Proof. It directly follows from Proposition 3.11. For, let $M = \{1, x, x^2, \ldots\}$, then clearly M is a multiplicatively closed subset of $R[x]$ and $R[x; x^{-1}] = M^{-1}R[x]$. □

Let R be a ring with an endomorphism α. Suppose that $R[x]$ is a right $\bar{\alpha}$-skew reflexive ring and $aRb = 0$ for $a, b \in R$. Then $aR[x]b = 0$ by [6, Lemma 2.1], and so $bR[x]\bar{\alpha}(a) = 0$ and $bR\alpha(a) = 0$. Thus R is right α-skew reflexive. However, we actually do not know whether the right α-skew reflexive property can go up the polynomial ring.

Question. Is the polynomial ring $R[x]$ over a right α-skew reflexive ring R right $\bar{\alpha}$-skew reflexive?

Recall that a ring R is called *quasi-Armendariz* [6] provided that $a_iRb_j = 0$ for all i, j whenever $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. Semiprime rings are quasi-Armendariz by [6, Corollary 3.8], but the converse does not hold in general.

PROPOSITION 3.13. *Let R be a quasi-Armendariz ring with an endomorphism α. The following are equivalent:*

(1) *R is right α-skew reflexive.*

(2) *$R[x]$ is right $\bar{\alpha}$-skew reflexive.*

(3) *$R[x; x^{-1}]$ is right $\bar{\alpha}$-skew reflexive.*

Proof. It suffices to show (1)⇒(2) by Corollary 3.12 and the above argument. Assume that R is right α-skew reflexive. Let $f(x)R[x]g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Since R is quasi-Armendariz and right α-skew reflexive, we have $a_iRb_j = 0$ for all i, j and hence $b_jR\alpha(a_i) = 0$. This entails that $g(x)R[x]\bar{\alpha}(f(x)) = 0$ and so $R[x]$ is right $\bar{\alpha}$-skew reflexive.

□

Observe that for an α-rigid ring R, the ring $D_3(R)$ is not right α-skew reflexive by Example 3.4(2), but it is quasi-Armendariz by help of [8, Proposition 2].

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