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STABILITY OF HOMOMORPHISMS IN BANACH MODULES OVER A C^*-ALGEBRA ASSOCIATED WITH A GENERALIZED JENSEN TYPE MAPPING AND APPLICATIONS

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Abstract. Let X and Y be vector spaces. It is shown that a mapping $f : X \to Y$ satisfies the functional equation

$$\begin{aligned} &2df(\frac{x_1 + \sum_{j=2}^{2d}(-1)^j x_j}{2d}) - 2df(\frac{x_1 + \sum_{j=2}^{2d}(-1)^{j+1} x_j}{2d}) \\ (\ddagger) \qquad &= 2\sum_{j=2}^{2d}(-1)^j f(x_j) \end{aligned}$$

if and only if the mapping $f : X \to Y$ is additive, and prove the Cauchy-Rassias stability of the functional equation ($\ddagger$) in Banach modules over a unital C^*-algebra, and in Poisson Banach modules over a unital Poisson C^*-algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, Poisson C^*-algebras, Poisson JC^*-algebras or Lie JC^*-algebras. As an application, we show that every almost homomorphism $h : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism when $h(d^n uy) = h(d^n u)h(y)$ or $h(d^n u \circ y) = h(d^n u) \circ h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in C^*-algebras, Poisson C^*-algebras, Poisson JC^*-algebras or Lie JC^*-algebras, and of Lie JC^*-algebra derivations in Lie JC^*-algebras.

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1. Introduction

In 1940, Ulam [23] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [4] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$(*) \qquad \|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [15] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. The inequality (*) that was introduced for the first time by Th.M. Rassias [15] we call Cauchy-Rassias inequality and the stability of the functional equation *Cauchy-Rassias stability*. This inequality has provided a lot of influence in the development of what we now call *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [5], [12], [14], [17], [18], [19], [20], [21], [22]). Th.M. Rassias [16] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [1] following the same approach as in Th.M. Rassias [15], gave an affirmative solution to this question for $p > 1$.

Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x,y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in G$. C. Park [11] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [6] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0,\infty)$ a function such that

$$\widetilde{\varphi}(x,y) = \sum_{j=0}^{\infty} 3^{-j}\varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)| \leq \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [13] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Throughout this paper, assume that d is an integer greater than 1.

In this paper, we solve the following functional equation

(1.i)

$$2df(\frac{x_1 + \sum_{j=2}^{2d}(-1)^j x_j}{2d}) - 2df(\frac{x_1 + \sum_{j=2}^{2d}(-1)^{j+1} x_j}{2d}) = 2\sum_{j=2}^{2d}(-1)^j f(x_j).$$

We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital C^*-algebra. The main purpose of this paper is to investigate homomorphisms between C^*-algebras, between Poisson C^*-algebras, between Poisson JC^*-algebras and between Lie JC^*-algebras, and to prove their Cauchy-Rassias stability.

2. A generalized Jensen type mapping

Throughout this section, assume that X and Y are linear spaces.

LEMMA 2.1. *A mapping* $f : X \to Y$ *satisfies* (1.i) *for all* $x_1, x_2, \cdots, x_{2d} \in X$ *and* $f(0) = 0$ *if and only if* f *is additive.*

Proof. Assume that $f : X \to Y$ satisfies (1.i) for all $x_1, x_2, \cdots, x_{2d} \in X$. Putting $x_3 = \cdots = x_{2d} = 0$ in (1.i), we get

$$2df(\frac{x_1 + x_2}{2d}) - 2df(\frac{x_1 - x_2}{2d}) = 2f(x_2) \tag{2.1}$$

for all $x_1, x_2 \in X$. Putting $x_2 = x_1$ in (2.1), we get

$$2df(\frac{x_1}{d}) = 2f(x_1)$$

for all $x_1 \in X$. So we get

$$2f(\frac{x_1 + x_2}{2}) - 2f(\frac{x_1 - x_2}{2}) = 2f(x_2) \tag{2.2}$$

for all $x_1, x_2 \in X$. Putting $\frac{x_1 - x_2}{2} = x$ and $x_2 = y$ in (2.2), we get

$$2f(x + y) = 2f(\frac{x_1 + x_2}{2}) = 2f(\frac{x_1 - x_2}{2}) + 2f(x_2) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Thus f is additive.

The converse is obviously true. $\square$

3. Cauchy-Rassias stability of the generalized Jensen type mapping in Banach modules over a C^*-algebra

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Banach modules over $\mathcal{A}$ with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \to Y$, we set

$$\begin{aligned} D_u f(x_1, \cdots, x_{2d}) :=& 2df(\frac{ux_1 + \sum_{j=2}^{2d}(-1)^j ux_j}{2d}) \\ & - 2df(\frac{ux_1 + \sum_{j=2}^{2d}(-1)^{j+1} ux_j}{2d}) - 2\sum_{j=2}^{2d}(-1)^j uf(x_j) \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{2d} \in X$.

THEOREM 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^{2d} \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}) := \sum_{j=1}^{\infty} \frac{1}{d^j}\varphi(d^j x_1, \cdots, d^j x_{2d}) < \infty, \tag{3.i}$$

$$\|D_u f(x_1, \cdots, x_{2d})\| \leq \varphi(x_1, \cdots, x_{2d}) \tag{3.ii}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen type mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}) \tag{3.iii}$$

for all $x \in X$.

Proof. Let $u = 1 \in \mathcal{U}(\mathcal{A})$. Putting $x_1 = \cdots = x_{2d} = x$ in (3.ii), we have

$$\|2df(\frac{x}{d}) - 2f(x)\| \leq \varphi(\underbrace{x, \cdots, x}_{2d \text{ times}}) \tag{3.0}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{d}f(dx)\| \leq \frac{1}{2d}\varphi(\underbrace{dx, \cdots, dx}_{2d \text{ times}})$$

for all $x \in X$. Hence

$$\|\frac{1}{d^n}f(d^n x) - \frac{1}{d^{n+1}}f(d^{n+1}x)\| \leq \frac{1}{2d^{n+1}}\varphi(\underbrace{d^{n+1}x, \cdots, d^{n+1}x}_{2d \text{ times}}) \tag{3.1}$$

for all $x \in X$ and all positive integers n. By (3.1), we have

$$\|\frac{1}{d^m}f(d^m x) - \frac{1}{d^n}f(d^n x)\| \leq \sum_{k=m}^{n-1}\frac{1}{2d^{k+1}}\varphi(\underbrace{d^{k+1}x, \cdots, d^{k+1}x}_{2d \text{ times}}) \tag{3.2}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{\frac{1}{d^n}f(d^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{d^n}f(d^n x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}\frac{1}{d^n}f(d^n x)$$

for all $x \in X$. We get

$$\begin{aligned}\|D_1 L(x_1, \cdots, x_{2d})\| &= \lim_{n\to\infty} \frac{1}{d^n} \|D_1 f(d^n x_1, \cdots, d^n x_{2d})\| \\ &\le \lim_{n\to\infty} \frac{1}{d^n} \varphi(d^n x_1, \cdots, d^n x_{2d}) = 0\end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$. By Lemma 2.1, L is additive. Putting $m = 0$ and letting $n \to \infty$ in (3.2), we get (3.iii).

Now, let $L' : X \to Y$ be another generalized Jensen type mapping satisfying (3.iii). Then we have

$$\begin{aligned}\|L(x) - L'(x)\| &= \frac{1}{d^n} \|L(d^n x) - L'(d^n x)\| \\ &\le \frac{1}{d^n} (\|L(d^n x) - f(d^n x)\| + \|L'(d^n x) - f(d^n x)\|) \\ &\le \frac{2}{2d^{n+1}} \widetilde{\varphi}(\underbrace{d^n x, \cdots, d^n x}_{2d \text{ times}}),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $L(x) = L'(x)$ for all $x \in X$. This proves the uniqueness of L.

By the assumption, for each $u \in \mathcal{U}(\mathcal{A})$, we get

$$\begin{aligned}\|D_u L(x, x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}})\| &= \lim_{n\to\infty} \frac{1}{d^n} \|D_u f(d^n x, d^n x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}})\| \\ &\le \lim_{n\to\infty} \frac{1}{d^n} \varphi(d^n x, d^n x, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}) = 0\end{aligned}$$

for all $x \in X$. So

$$2dL(\frac{ux}{d}) = 2uL(x)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$. Since L is additive,

$$L(ux) = uL(x) \tag{3.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$.

Now let $a \in \mathcal{A}$ ($a \neq 0$) and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [**?**, 7], there exist three elements $u_1, u_2, u_3 \in$

$\mathcal{U}(\mathcal{A})$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So by (3.3)

$$\begin{aligned} L(ax) &= L(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot L(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}L(3\frac{a}{M}x) \\ &= \frac{M}{3}L(u_1x + u_2x + u_3x) = \frac{M}{3}(L(u_1x) + L(u_2x) + L(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)L(x) = \frac{M}{3} \cdot 3\frac{a}{M}L(x) \\ &= aL(x) \end{aligned}$$

for all $a \in \mathcal{A}$ and all $x \in X$. Hence

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in \mathcal{A}(a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the generalized Jensen type mapping $L : X \to Y$ is an $\mathcal{A}$-linear mapping, as desired. □

Corollary 3.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\|D_u f(x_1, \cdots, x_{2d})\| \leq \theta \sum_{j=1}^{2d} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen type mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{d^{p+1}}{d - d^p}\theta||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 3.1. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^{2d} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}) := \sum_{j=0}^{\infty} d^j \varphi(\frac{x_1}{d^j}, \cdots, \frac{x_{2d}}{d^j}) < \infty, \tag{3.iv}$$

$$\|D_u f(x_1, \cdots, x_{2d})\| \leq \varphi(x_1, \cdots, x_{2d}) \tag{3.v}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen type mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \ times}) \tag{3.vi}$$

for all $x \in X$.

Proof. It follows from (3.0) that

$$\|d^n f(\frac{x}{d^n}) - d^{n+1} f(\frac{x}{d^{n+1}})\| \le \frac{d^n}{2}\varphi(\underbrace{\frac{x}{d^n}, \cdots, \frac{x}{d^n}}_{2d \text{ times}}) \tag{3.4}$$

for all $x \in X$ and all positive integers n. By (3.4), we have

$$\|d^m f(\frac{x}{d^m}) - d^n f(\frac{x}{d^n})\| \le \sum_{k=m}^{n-1} \frac{d^k}{2}\varphi(\underbrace{\frac{x}{d^k}, \cdots, \frac{x}{d^k}}_{2d \text{ times}}) \tag{3.5}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{d^n f(\frac{x}{d^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{d^n f(\frac{x}{d^n})\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} d^n f(\frac{x}{d^n})$$

for all $x \in X$. Also, we get

$$\begin{aligned}\|D_1 L(x_1, \cdots, x_{2d})\| &= \lim_{n\to\infty} d^n \|D_1 f(\frac{x_1}{d^n}, \cdots, \frac{x_{2d}}{d^n})\| \\ &\le \lim_{n\to\infty} d^n \varphi(\frac{x_1}{d^n}, \cdots, \frac{x_{2d}}{d^n}) = 0\end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$. By Lemma 2.1, L is additive. Putting $m = 0$ and letting $n \to \infty$ in (3.5), we get (3.vi).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let θ and $p > 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\|D_u f(x_1, \cdots, x_{2d})\| \le \theta \sum_{j=1}^{2d} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ *and all* $x_1, \cdots, x_{2d} \in X$. *Then there exists a unique* $\mathcal{A}$*-linear generalized Jensen type mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\| \leq \frac{d^{p+1}}{d^p - d}\theta||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 3.3. □

4. Isomorphisms between unital C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $|| \cdot ||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and that $\mathcal{B}$ is a unital C^*-algebra with norm $\| \cdot \|$.

We are going to investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a bijective mapping satisfying* $h(0) = 0$ *and* $h(d^n uy) = h(d^n u)h(y)$ *for all* $u \in \mathcal{U}(\mathcal{A})$, *all* $y \in \mathcal{A}$, *and* $n = 0, 1, 2, \cdots$, *for which there is a function* $\varphi : \mathcal{A}^{2d} \to [0, \infty)$ *satisfying* (3.i) *such that*

$$\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d})$$

$$-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\| \leq \varphi(x_1, \cdots, x_{2d}), \tag{4.i}$$

$$\|h(d^n u^*) - h(d^n u)^*\| \leq \varphi(\underbrace{d^n u, \cdots, d^n u}_{2d \text{ times}}) \tag{4.ii}$$

for all $u \in \mathcal{U}(\mathcal{A})$, *all* $x_1, \cdots, x_{2d} \in \mathcal{A}$, *all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$ *and* $n = 0, 1, 2, \cdots$. *Assume that* (4.iii) $\lim_{n\to\infty} \frac{h(d^n e)}{d^n}$ *is invertible. Then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a* C^**-algebra isomorphism.*

Proof. We consider a C^*-algebra as a Banach module over a unital C^*-algebra $\mathbb{C}$. By Theorem 3.1, there exists a unique $\mathbb{C}$-linear generalized Jensen type mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}) \tag{4.iv}$$

for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n x) \tag{4.1}$$

for all $x \in \mathcal{A}$.

By (3.i) and (4.ii), we get

$$\begin{aligned} H(u^*) &= \lim_{n\to\infty} \frac{h(d^n u^*)}{d^n} = \lim_{n\to\infty} \frac{h(d^n u)^*}{d^n} = (\lim_{n\to\infty} \frac{h(d^n u)}{d^n})^* \\ &= H(u)^* \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [?, 8]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $h(d^n uy) = h(d^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n uy) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n u)h(y) = H(u)h(y) \tag{4.2}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (4.2),

$$d^n H(uy) = H(d^n uy) = H(u(d^n y)) = H(u)h(d^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{d^n} H(u)h(d^n y) = H(u) \frac{1}{d^n} h(d^n y) \tag{4.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (4.3) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.4}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in$

$\mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (4.4) that

$$(1) \qquad H(xy) = H(\sum_{j=1}^{m} \lambda_j u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j y) = \sum_{j=1}^{m} \lambda_j H(u_j) H(y)$$
$$= H(\sum_{j=1}^{m} \lambda_j u_j) H(y) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$.

By (4.2) and (4.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(d^n e)}{d^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism. □

Corollary 4.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(d^n uy) = h(d^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d})$$
$$-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\| \leq \theta \sum_{j=1}^{2d} ||x_j||^p,$$
$$\|h(d^n u^*) - h(d^n u)^*\| \leq 2d \cdot d^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_{2d} \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(d^n e)}{d^n}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 4.1. □

Theorem 4.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(d^n uy) = h(d^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and*

$n = 0, 1, 2, \cdots$, *for which there is a function* $\varphi : \mathcal{A}^{2d} \to [0, \infty)$ *satisfying* (3.i), (4.ii), *and* (4.iii) *such that*

$$\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d}) \tag{4.v}$$
$$-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\| \le \varphi(x_1, \cdots, x_{2d})$$

for all $x_1, \cdots, x_{2d} \in \mathcal{A}$ *and* $\mu = 1, i$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then the bijective mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a* C^**-algebra isomorphism.*

Proof. Put $\mu = 1$ in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Jensen type mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv). By the same reasoning as in the proof of [?, 15], the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(d^n ix)}{d^n} = \lim_{n\to\infty} \frac{ih(d^n x)}{d^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

Now we prove the Cauchy-Rassias stability of C^*-algebra homomorphisms in unital C^*-algebras.

THEOREM 4.4. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping satisfying* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{2d} \to [0, \infty)$ *satisfying* (3.i), (4.i) *and* (4.ii) *such that*

$$\|h(d^n u \cdot d^n v) - h(d^n u)h(d^n v)\| \le \varphi(d^n u, d^n v, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}) \tag{4.vi}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ *and* $n = 0, 1, 2, \cdots$. *Then there exists a unique* C^**-algebra homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *satisfying* (4.iv)

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive generalized Jensen type mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv).

By (4.vi),

$$\begin{aligned}\frac{1}{d^{2n}}\|h(d^n u \cdot d^n v) - h(d^n u)h(d^n v)\| &\le \frac{1}{d^{2n}}\varphi(d^n u, d^n v, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}) \\ &\le \frac{1}{d^n}\varphi(2^n u, 2^n v, \underbrace{0, \cdots, 0}_{2d-2 \text{ times}}),\end{aligned}$$

which tends to zero by (3.i) as $n \to \infty$. By (4.1),

$$\begin{aligned}H(uv) &= \lim_{n\to\infty} \frac{h(d^n u \cdot d^n v)}{d^{2n}} = \lim_{n\to\infty} \frac{h(d^n u)h(d^n v)}{d^{2n}} \\ &= \lim_{n\to\infty} \frac{h(d^n u)}{d^n}\frac{h(d^n v)}{d^n} = H(u)H(v)\end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned}H(xv) &= H(\sum_{j=1}^{m} \lambda_j u_j v) = \sum_{j=1}^{m} \lambda_j H(u_j v) = \sum_{j=1}^{m} \lambda_j H(u_j)H(v) \\ &= H(\sum_{j=1}^{m} \lambda_j u_j)H(v) = H(x)H(v)\end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. So the mapping $H : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism. □

THEOREM 4.5. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping satisfying* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{2d} \to [0, \infty)$ *satisfying* (3.i), (4.ii), (4.v) *and* (4.vi). *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then there exists a unique* C^**-algebra homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *satisfying* (4.iv).

Proof. The proof is similar to the proofs of Theorems 4.3 and 4.4. □

5. Homomorphisms between Poisson C^*-algebras

A *Poisson C^*-algebra* $\mathcal{A}$ is a C^*-algebra with a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, called a *Poisson bracket*, such that $(\mathcal{A},\{\cdot,\cdot\})$ is a complex Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in \mathcal{A}$. Poisson algebras have played an important role in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra (see [3], [9], [10], [25]).

Throughout this section, let $\mathcal{A}$ be a unital Poisson C^*-algebra with norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and $\mathcal{B}$ a unital Poisson C^*-algebra with norm $\|\cdot\|$.

Definition 5.1. A C^*-algebra homomorphism $H : \mathcal{A}\to\mathcal{B}$ is called a *Poisson C^*-algebra homomorphism* if $H : \mathcal{A}\to\mathcal{B}$ satisfies

$$H(\{z, w\}) = \{H(z), H(w)\}$$

for all $z, w \in \mathcal{A}$.

We are going to investigate Poisson C^*-algebra homomorphisms between Poisson C^*-algebras.

Theorem 5.2. *Let $h : \mathcal{A}\to\mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(d^n uy) = h(d^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^{2d+2} \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x_1,\cdots,x_{2d},z,w) := \sum_{j=1}^{\infty}\frac{1}{d^j}\varphi(d^jx_1,\cdots,d^jx_{2d},d^jz,d^jw) < \infty, \tag{5.i}$$

$$\begin{aligned}&\|2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^j\mu x_j+\{z,w\}}{2d}) - 2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^{j+1}\mu x_j}{2d})\\ &-2\sum_{j=2}^{2d}(-1)^j\mu h(x_j) - \{h(z),h(w)\}\| \le \varphi(x_1,\cdots,x_{2d},z,w),\end{aligned} \tag{5.ii}$$

$$\|h(d^nu^*) - h(d^nu)^*\| \le \varphi(\underbrace{d^nu,\cdots,d^nu}_{2d\ times},0,0) \tag{5.iii}$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $x_1, \cdots, x_{2d}, z, w \in \mathcal{A}$, all $\mu \in \mathbb{T}^1$ and $n = 0, 1, 2, \cdots$. Assume that (5.iv) $\lim_{n\to\infty} \frac{h(d^n e)}{d^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}, 0, 0) \tag{5.v}$$

for all $x \in \mathcal{A}$. In the proof of Theorem 4.1, we showed that the C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is exactly the mapping $h : \mathcal{A} \to \mathcal{B}$.

It follows from (4.1) that

$$H(x) = \lim_{n\to\infty} \frac{h(d^{2n}x)}{d^{2n}} \tag{5.1}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{2d} = 0$ in (5.ii). Then we get

$$\|2dh(\frac{\{z, w\}}{2d}) - \{h(z), h(w)\}\| \leq \varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. So

$$\begin{aligned}
\frac{1}{d^{2n}}\|2dh(\frac{\{d^n z, d^n w\}}{2d}) - \{h(d^n z), h(d^n w)\}\| &\leq \frac{1}{d^{2n}}\varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, d^n z, d^n w) \\
&\leq \frac{1}{d^n}\varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, d^n z, d^n w) \qquad (5.2)
\end{aligned}$$

for all $z, w \in \mathcal{A}$. By (5.i), (5.1) and (5.2),

$$\begin{aligned}
2dH(\frac{\{z, w\}}{2d}) &= \lim_{n\to\infty} \frac{2dh(d^{2n}\frac{\{z,w\}}{2d})}{d^{2n}} = \lim_{n\to\infty} \frac{2dh(\frac{\{d^n z, d^n w\}}{2d})}{d^{2n}} \\
&= \lim_{n\to\infty} \frac{1}{d^{2n}}\{h(d^n z), h(d^n w)\} = \lim_{n\to\infty}\{\frac{h(d^n z)}{d^n}, \frac{h(d^n w)}{d^n}\} \\
&= \{H(z), H(w)\}
\end{aligned}$$

for all $z, w \in \mathcal{A}$. So

$$H(\{z, w\}) = 2dH(\frac{\{z, w\}}{2d}) = \{H(z), H(w)\}$$

for all $z, w \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism. □

Now we prove the Cauchy-Rassias stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras.

THEOREM 5.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{2d+2} \to [0, \infty)$ satisfying* (5.i), (5.ii) *and* (5.iii) *such that*

$$\|h(d^n u \cdot d^n v) - h(d^n u)h(d^n v)\| \leq \varphi(d^n u, d^n v, \underbrace{0, \cdots, 0}_{2d \text{ times}}) \tag{5.vi}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Then there exists a unique Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.v)

Proof. The proof is similar to the proofs of Theorems 4.4 and 5.1. □

REMARK 5.4. If each Poisson bracket $\{\cdot, \cdot\}$ in this section is replaced by the Lie product $[\cdot, \cdot]$, which is defined in Section 8, one can obtain a result for *'Lie C^*-algebra homomorphism'*.

6. Cauchy-Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra

A *Poisson Banach module* X over a Poisson C^*-algebra $\mathcal{A}$ is a left Banach $\mathcal{A}$-module endowed with a $\mathbb{C}$-bilinear map $\{\cdot, \cdot\} : \mathcal{A} \times X \to X$ such that

$$\{\{a, b\}, x\} = \{a, \{b, x\}\} - \{b, \{a, x\}\},$$
$$\{a, b\} \cdot x = a \cdot \{b, x\} - \{b, a \cdot x\}$$

for all $a, b \in \mathcal{A}$ and all $x \in X$ (see [3], [9], [10], [25]). Here $\cdot$ denotes the associative module action.

Throughout this section, assume that $\mathcal{A}$ is a unital Poisson C^*-algebra with unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Poisson Banach $\mathcal{A}$-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

DEFINITION 6.1. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *Poisson module homomorphism* if $H : X \to Y$ satisfies

$$H(\{\{a, b\}, x\}) = \{\{a, b\}, H(x)\},$$
$$H(\{a, b\} \cdot x) = \{a, b\} \cdot H(x)$$

for all $a, b \in \mathcal{A}$ and all $x \in X$.

We are going to prove the Cauchy-Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra.

THEOREM 6.2. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : X^{2d} \to [0, \infty)$ satisfying (3.i) such that*

$$\begin{aligned} &\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d}) \\ &-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\| \le \varphi(x_1, \cdots, x_{2d}), \end{aligned} \tag{6.i}$$

$$\|h(\{\{u, v\}, x\}) - \{\{u, v\}, h(x)\}\| \le \varphi(\underbrace{x, \cdots, x}_{2d \text{ times}}), \tag{6.ii}$$

$$\|h(\{u, v\} \cdot x) - \{u, v\} \cdot h(x)\| \le \varphi(\underbrace{x, \cdots, x}_{2d \text{ times}}) \tag{6.iii}$$

for all $\mu \in \mathbb{T}^1$, all $x, x_1, \cdots, x_{2d} \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Then there exists a unique Poisson module homomorphism $H : X \to Y$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}) \tag{6.iv}$$

for all $x \in X$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : X \to Y$ satisfying (6.iv). The $\mathbb{C}$-linear mapping $H : X \to Y$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n x) \tag{6.1}$$

for all $x \in X$.

By (6.ii),

$$\begin{aligned} \|\frac{1}{d^n}h(d^n\{\{u, v\}, x\}) - \{\{u, v\}, \frac{1}{d^n}h(d^n x)\}\| \\ &= \frac{1}{d^n}\|h(\{\{u, v\}, d^n x\}) - \{\{u, v\}, h(d^n x)\}\| \\ &\le \frac{1}{d^n}\varphi(\underbrace{d^n x, \cdots, d^n x}_{2d \text{ times}}), \end{aligned}$$

which tends to zero for all $x \in X$ by (3.i). So

$$\begin{aligned} H(\{\{u,v\},x\}) &= \lim_{n\to\infty} \frac{1}{d^n} h(d^n\{\{u,v\},x\}) = \lim_{n\to\infty} \{\{u,v\}, \frac{1}{d^n} h(d^n x)\} \\ &= \{\{u,v\}, H(x)\} \end{aligned}$$

for all $x \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and $\{\cdot,\cdot\}$ is $\mathbb{C}$-bilinear and since each $a \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(\{\{a,v\},x\}) &= H(\{\{\sum_{j=1}^m \lambda_j u_j, v\},x\}) = \sum_{j=1}^m \lambda_j H(\{\{u_j,v\},x\}) \\ &= \sum_{j=1}^m \lambda_j \{\{u_j,v\}, H(x)\} = \{\{\sum_{j=1}^m \lambda_j u_j, v\}, H(x)\} = \{\{a,v\}, H(x)\} \end{aligned}$$

for all $x \in X$ and all $v \in \mathcal{U}(\mathcal{A})$. Similarly, one can show that

$$H(\{\{a,b\},x\}) = \{\{a,b\}, H(x)\}$$

for all $x \in X$ and all $a, b \in \mathcal{A}$.

By (6.iii),

$$\begin{aligned} (2) \quad & \|\frac{1}{d^n} h(d^n\{u,v\}\cdot x) - \{u,v\}\cdot \frac{1}{d^n} h(d^n x)\| \\ &= \frac{1}{d^n}\|h(\{u,v\}\cdot d^n x) - \{u,v\}\cdot h(d^n x)\| \le \frac{1}{d^n}\varphi(\underbrace{d^n x, \cdots, d^n x}_{2d \text{ times}}), \end{aligned}$$

which tends to zero for all $x \in X$ by (3.i). So

$$\begin{aligned} H(\{u,v\}\cdot x) &= \lim_{n\to\infty} \frac{1}{d^n} h(d^n\{u,v\}\cdot x) = \lim_{n\to\infty} (\{u,v\}\cdot \frac{1}{d^n} h(d^n x)) \\ &= \{u,v\}\cdot H(x) \end{aligned}$$

for all $x \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and $\{\cdot,\cdot\}$ is $\mathbb{C}$-bilinear and since each $a \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(\{a,v\}\cdot x) &= H(\{\sum_{j=1}^m \lambda_j u_j, v\}\cdot x) = \sum_{j=1}^m \lambda_j H(\{u_j,v\}\cdot x) \\ &= \sum_{j=1}^m \lambda_j \{u_j,v\}\cdot H(x) = \{\sum_{j=1}^m \lambda_j u_j, v\}\cdot H(x) = \{a,v\}\cdot H(x) \end{aligned}$$

for all $x \in X$ and all $v \in \mathcal{U}(A)$. Similarly, one can show that

$$H(\{a,b\} \cdot x) = \{a,b\} \cdot H(x)$$

for all $x \in X$ and all $a, b \in \mathcal{A}$. Thus $H : X \to Y$ is a Poisson module homomorphism.

Therefore, there exists a unique Poisson module homomorphism $H : X \to Y$ satisfying (6.iv). □

COROLLARY 6.3. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\begin{aligned} \|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1}\mu x_j}{2d}) \\ -2\sum_{j=2}^{2d}(-1)^j \mu h(x_j)\| \leq \theta \sum_{j=1}^{2d} ||x_j||^p, \\ \|h(\{\{u,v\},x\}) - \{\{u,v\},h(x)\}\| \leq 2d\theta||x||^p, \\ \|h(\{u,v\} \cdot x) - \{u,v\} \cdot h(x)\| \leq 2d\theta||x||^p \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u, v \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x, x_1, \cdots, x_{2d} \in X$. Then there exists a unique Poisson module homomorphism $H : X \to Y$ such that

$$\|h(x) - H(x)\| \leq \frac{d^{p+1}}{d - d^p}\theta||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$, and apply Theorem 6.1. □

7. Homomorphisms between Poisson JC^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [24]). Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra*. A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a

JC^*-algebra. A Poisson C^*-algebra, endowed with the anticommutator product, is called a *Poisson JC^*-algebra.*

Throughout this section, assume that $\mathcal{A}$ is a unital Poisson JC^*-algebra with unit e, norm $||\cdot||$ and unitary group $\mathcal{U}(\mathcal{A})$, and that $\mathcal{B}$ is a unital Poisson JC^*-algebra with unit e' and norm $\|\cdot\|$.

DEFINITION 7.1. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *Poisson JC^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$H(x \circ y) = H(x) \circ H(y),$$
$$H(\{x, y\}) = \{H(x), H(y)\}$$

for all $x, y \in \mathcal{A}$.

We are going to investigate Poisson JC^*-algebra homomorphisms between Poisson JC^*-algebras.

THEOREM 7.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(d^n u \circ y) = h(d^n u) \circ h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^{2d+2} \to [0, \infty)$ satisfying* (5.i) *such that*

$$\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j + \{z, w\}}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d})$$
$$\text{(7.i)} \quad -2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) - \{h(z), h(w)\}\| \le \varphi(x_1, \cdots, x_{2d}, z, w),$$

for all $x_1, \cdots, x_{2d}, z, w \in \mathcal{A}$, and all $\mu \in \mathbb{T}^1$. Assume that (7.ii) $\lim_{n\to\infty} \frac{h(d^n e)}{d^n} = e'$. *Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson JC^*-algebra homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.v).

Since $h(d^n u \circ y) = h(d^n u) \circ h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$,

$$\text{(7.1)} \quad H(u \circ y) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n u \circ y) = \lim_{n\to\infty} \frac{1}{d^n} h(d^n u) \circ h(y) = H(u) \circ h(y)$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. By the additivity of H and (7.1),

$$d^n H(u \circ y) = H(d^n u \circ y) = H(u \circ (d^n y)) = H(u) \circ h(d^n y)$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Hence

$$H(u \circ y) = \frac{1}{d^n} H(u) \circ h(d^n y) = H(u) \circ \frac{1}{d^n} h(d^n y) \tag{7.2}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Taking the limit in (7.2) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{7.3}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(x \circ y) &= H(\sum_{j=1}^m \lambda_j u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j) \circ H(y) \\ &= H(\sum_{j=1}^m \lambda_j u_j) \circ H(y) = H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (7.ii), (7.1) and (7.3),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$. So

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 7.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(d^n u \circ y) = h(d^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j + \{z, w\}}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d}) \\ &-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) - \{h(z), h(w)\}\| \leq \theta \sum_{j=1}^{2d} ||x_j||^p + ||z||^p + ||w||^p \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_{2d}, z, w \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(d^n e)}{d^n} = e'$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson JC^*-algebra homomorphism.

Proof. Define $\varphi(x_1, \cdots, x_{2d}, z, w) = \theta \sum_{j=1}^{2d} ||x_j||^p + ||z||^p + ||w||^p$, and apply Theorem 7.1. □

THEOREM 7.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(dx) = dh(x)$ for all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{2d+2} \to [0, \infty)$ satisfying* (5.i), (7.i) *and* (7.ii) *such that*

$$\|h(d^n u \circ y) - h(d^n u) \circ h(y)\| \leq \varphi(u, y, \underbrace{0, \cdots, 0}_{2d \text{ times}}) \tag{7.iii}$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson JC^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.v).

By (7.iii) and the assumption that $h(dx) = dh(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned}
&\|h(d^n u \circ y) - h(d^n u) \circ h(y)\| \\
&\qquad = \frac{1}{d^{2m}} \|h(d^m d^n u \circ d^m y) - h(d^m d^n u) \circ h(d^m y)\| \\
&\leq \frac{1}{d^{2m}} \varphi(d^m u, d^m y, \underbrace{0, \cdots, 0}_{2d \text{ times}}) \leq \frac{1}{d^m} \varphi(d^m u, d^m y, \underbrace{0, \cdots, 0}_{2d \text{ times}}),
\end{aligned}$$

which tends to zero as $m \to \infty$ by (5.i). So

$$h(d^n u \circ y) = h(d^n u) \circ h(y)$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. But by (4.1),

$$H(x) = \lim_{n \to \infty} \frac{1}{d^n} h(d^n x) = h(x)$$

for all $x \in \mathcal{A}$.

The rest of the proof is similar to the proof of Theorem 5.1. □

Now we are going to show the Cauchy-Rassias stability of homomorphisms in Poisson JC^*-algebras.

THEOREM 7.5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{2d+4} \to [0, \infty)$ such that*

$$\begin{aligned}
&\widetilde{\varphi}(x_1, \cdots, x_{2d}, z, w, a, b) \\
&\qquad := \sum_{j=1}^{\infty} \frac{1}{d^j} \varphi(d^j x_1, \cdots, d^j x_{2d}, d^j z, d^j w, d^j a, d^j b) < \infty,
\end{aligned} \tag{7.iv}$$

$$\begin{aligned}(7.\text{v})\qquad &\|2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^j\mu x_j+\{z,w\}+a\circ b}{2d})\\ &\quad -2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^{j+1}\mu x_j}{2d})-2\sum_{j=2}^{2d}(-1)^j\mu h(x_j)\\ &\quad -\{h(z),h(w)\}-h(a)\circ h(b)\|\\ &\leq \varphi(x_1,\cdots,x_{2d},z,w,a,b)\end{aligned}$$

for all $\mu\in\mathbb{T}^1$ *and all* $x_1,\cdots,x_{2d},z,w,a,b\in\mathcal{A}$. *Then there exists a unique Poisson* JC^**-algebra homomorphism* $H:\mathcal{A}\to\mathcal{B}$ *such that*

$$(7.\text{vi})\qquad \|h(x)-H(x)\|\leq\frac{1}{2}\widetilde{\varphi}(\underbrace{x,\cdots,x}_{2d\ times},0,0,0,0)$$

for all $x\in\mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ satisfying (7.vi).

The rest of the proof is similar to the proofs of Theorems 4.1 and 5.1. □

8. Homomorphisms between Lie JC^*-algebras

A unital C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x,y]=\frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *Lie* C^**-algebra*. A unital C^*-algebra $\mathcal{C}$, endowed with the Lie product $[\cdot,\ \cdot]$ and the anticommutator product $\circ$, is called a *Lie* JC^**-algebra* if $(\mathcal{C},\circ)$ is a JC^*-algebra and $(\mathcal{C},[\cdot,\ \cdot])$ is a Lie C^*-algebra (see [3], [9], [10]).

Throughout this paper, let $\mathcal{A}$ be a unital Lie JC^*-algebra with norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})=\{u\in\mathcal{A}\mid uu^*=u^*u=e\}$, and $\mathcal{B}$ a unital Lie JC^*-algebra with norm $\|\cdot\|$ and unit e'.

Definition 8.1. A $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ is called a *Lie* JC^**-algebra homomorphism* if $H:\mathcal{A}\to\mathcal{B}$ satisfies

$$\begin{aligned}H(x\circ y)&=H(x)\circ H(y),\\ H([x,y])&=[H(x),H(y)],\\ H(x^*)&=H(x)^*\end{aligned}$$

for all $x,y\in\mathcal{A}$.

Remark 8.2. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism if and only if the mapping $H : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism.

Assume that H is a Lie JC^*-algebra homomorphism. Then

$$\begin{aligned} H(xy) &= H([x,y] + x \circ y) = H([x,y]) + H(x \circ y) \\ &= [H(x), H(y)] + H(x) \circ H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. So H is a C^*-algebra homomorphism.

Assume that H is a C^*-algebra homomorphism. Then

$$H([x,y] = H(\frac{xy - yx}{2}) = \frac{H(x)H(y) - H(y)H(x)}{2} = [H(x), H(y)],$$
$$H(x \circ y) = H(\frac{xy + yx}{2}) = \frac{H(x)H(y) + H(y)H(x)}{2} = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$. So H is a Lie JC^*-algebra homomorphism.

We are going to investigate Lie JC^*-algebra homomorphisms between Lie JC^*-algebras.

Theorem 8.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(d^n u \circ y) = h(d^n u) \circ h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^{2d+2} \to [0, \infty)$ satisfying* (5.i) *and* (5.iii) *such that*

$$\begin{aligned} &\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j + [z,w]}{2d}) - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d}) \\ \text{(8.i)}\quad &-2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) - [h(z), h(w)]\| \le \varphi(x_1, \cdots, x_{2d}, z, w), \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, and all $x_1, \cdots, x_{2d}, z, w \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(d^n e)}{d^n} = e'$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.v).

In the proof of Theorem 7.1, we showed that

$$H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$, and that the mapping $H : \mathcal{A} \to \mathcal{B}$ is exactly the mapping $h : \mathcal{A} \to \mathcal{B}$.

It follows from (4.1) that

$$H(x) = \lim_{n\to\infty} \frac{h(d^{2n}x)}{d^{2n}} \tag{8.1}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{2d} = 0$ in (8.i). Then we get

$$\|2dh(\frac{[z,w]}{2d}) - [h(z), h(w)]\| \le \varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. So

$$\begin{aligned}\frac{1}{d^{2n}}\|2dh(\frac{[d^n z, d^n w]}{2d}) - [h(d^n z), h(d^n w)]\| &\le \frac{1}{d^{2n}}\varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, d^n z, d^n w)\\ &\le \frac{1}{d^n}\varphi(\underbrace{0, \cdots, 0}_{2d \text{ times}}, d^n z, d^n w)\end{aligned} \tag{8.2}$$

for all $z, w \in \mathcal{A}$. By (5.i), (8.1), and (8.2),

$$\begin{aligned}2dH(\frac{[z,w]}{2d}) &= \lim_{n\to\infty} \frac{2dh(d^{2n}\frac{[z,w]}{2d})}{d^{2n}} = \lim_{n\to\infty} \frac{2dh(\frac{[d^n z, d^n w]}{2d})}{d^{2n}}\\ &= \lim_{n\to\infty} \frac{1}{d^{2n}}[h(d^n z), h(d^n w)] = \lim_{n\to\infty}[\frac{h(d^n z)}{d^n}, \frac{h(d^n w)}{d^n}]\\ &= [H(z), H(w)]\end{aligned}$$

for all $z, w \in \mathcal{A}$. So

$$H([z,w]) = 2dH(\frac{[z,w]}{2d}) = [H(z), H(w)]$$

for all $z, w \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism. □

We are going to show the Cauchy-Rassias stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras.

THEOREM 8.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{2d+4} \to [0, \infty)$ satisfying (7.iv) such* *that*

$$\begin{aligned}(8.\mathrm{ii})\qquad &\|2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^j\mu x_j+[z,w]+a\circ b}{2d})\\ &-2dh(\frac{\mu x_1+\sum_{j=2}^{2d}(-1)^{j+1}\mu x_j}{2d})-2\sum_{j=2}^{2d}(-1)^j\mu h(x_j)\\ &\qquad\qquad-[h(z),h(w)]-h(a)\circ h(b)\|\\ &\le\varphi(x_1,\cdots,x_{2d},z,w,a,b),\end{aligned}$$

$$(8.\mathrm{iii})\qquad \|h(d^nu^*)-h(d^nu)^*\|\le\varphi(\underbrace{2^nu,\cdots,2^nu}_{2d\ times},0,0,0,0)$$

for all $\mu\in\mathbb{T}^1$, *all* $u\in\mathcal{U}(\mathcal{A})$, $n=0,1,2,\cdots$, *and all* $x_1,\cdots,x_{2d},z,w,a,b\in\mathcal{A}$. *Then there exists a unique Lie* JC^**-algebra homomorphism* $H:\mathcal{A}\to\mathcal{B}$ *satisfying* (7.vi).

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H:\mathcal{A}\to\mathcal{B}$ satisfying (7.vi).

The rest of the proof is similar to the proof of Theorem 8.1. □

9. Cauchy-Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras

Definition 9.1. A $\mathbb{C}$-linear mapping $D:\mathcal{A}\to\mathcal{A}$ is called a *Lie* JC^**-algebra derivation* if $D:\mathcal{A}\to\mathcal{A}$ satisfies

$$\begin{aligned}D(x\circ y)&=(Dx)\circ y+x\circ(Dy),\\ D([x,y])&=[Dx,y]+[x,Dy],\\ D(x^*)&=D(x)^*\end{aligned}$$

for all $x,y\in\mathcal{A}$.

Remark 9.2. A $\mathbb{C}$-linear mapping $D:\mathcal{A}\to\mathcal{A}$ is a C^*-algebra derivation if and only if the mapping $D:\mathcal{A}\to\mathcal{A}$ is a Lie JC^*-algebra derivation.

Assume that D is a Lie JC^*-algebra derivation. Then

$$\begin{aligned}D(xy)&=D([x,y]+x\circ y)=D([x,y])+D(x\circ y)\\ &=[Dx,y]+[x,Dy]+(Dx)\circ y+x\circ(Dy)=(Dx)y+x(Dy)\end{aligned}$$

for all $x, y \in \mathcal{A}$. So D is a C^*-algebra derivation.

Assume that D is a C^*-algebra derivation. Then

$$\begin{aligned}
D([x,y]) &= D(\frac{xy - yx}{2}) = \frac{(Dx)y + x(Dy) - (Dy)x - y(Dx)}{2} \\
&= [Dx, y] + [x, Dy], \\
D(x \circ y) &= D(\frac{xy + yx}{2}) = \frac{(Dx)y + x(Dy) + (Dy)x + y(Dx)}{2} \\
&= (Dx) \circ y + x \circ (Dy)
\end{aligned}$$

for all $x, y \in \mathcal{A}$. So H is a Lie JC^*-algebra derivation.

We prove the Cauchy–Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras.

Theorem 9.3. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{2d+4} \to [0, \infty)$ satisfying* (7.iv) *and* (8.iii) *such that*

$$\begin{aligned}
\text{(9.i)} \qquad & \|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j + [z,w] + a \circ b}{2d}) \\
& - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1}\mu x_j}{2d}) - 2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) \\
& - [h(z), w] - [z, h(w)] - h(a) \circ b - a \circ h(b)\| \\
& \le \varphi(x_1, \cdots, x_{2d}, z, w, a, b)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_{2d}, z, w, a, b \in \mathcal{A}$. Then there exists a unique Lie JC^*-algebra derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\text{(9.ii)} \qquad \|h(x) - D(x)\| \le \frac{1}{2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{2d \text{ times}}, 0, 0, 0, 0)$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying (9.ii). The mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$\text{(9.1)} \qquad D(x) = \lim_{n \to \infty} \frac{1}{d^n} h(d^n x)$$

for all $x \in \mathcal{A}$.

It follows from (9.1) that

$$D(x) = \lim_{n\to\infty} \frac{h(d^{2n}x)}{d^{2n}} \tag{9.2}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{2d} = a = b = 0$ in (9.i). Then we get

$$\|2dh(\frac{[z,w]}{2d}) - [h(z),w] - [z,h(w)]\| \le \varphi(\underbrace{0,\cdots,0}_{2d \text{ times}}, z, w, 0, 0)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{d^{2n}}\varphi(\underbrace{0,\cdots,0}_{2d \text{ times}}, d^n z, d^n w, 0, 0) \le \frac{1}{d^n}\varphi(\underbrace{0,\cdots,0}_{2d \text{ times}}, d^n z, d^n w, 0, 0),$$

$$\begin{aligned}
&\frac{1}{d^{2n}}\|2dh(\frac{[d^n z, d^n w]}{2d}) - [h(d^n z), d^n w] - [d^n z, h(d^n w)]\| \\
&\qquad \le \frac{1}{d^{2n}}\varphi(\underbrace{0,\cdots,0}_{2d \text{ times}}, d^n z, d^n w, 0, 0) \\
&\qquad \le \frac{1}{d^n}\varphi(0,0,d^n z, d^n w, 0, 0)
\end{aligned} \tag{9.3}$$

for all $z, w \in \mathcal{A}$. By (7.iv), (9.2), and (9.3),

$$\begin{aligned}
2dD(\frac{[z,w]}{2d}) &= \lim_{n\to\infty} \frac{2dh(d^{2n}\frac{[z,w]}{2d})}{d^{2n}} = \lim_{n\to\infty} \frac{2dh(\frac{[d^n z, d^n w]}{2d})}{d^{2n}} \\
&= \lim_{n\to\infty}([\frac{h(d^n z)}{d^n}, \frac{d^n w}{d^n}] + [\frac{d^n z}{d^n}, \frac{h(d^n w)}{d^n}]) \\
&= [D(z), w] + [z, D(w)]
\end{aligned}$$

for all $z, w \in \mathcal{A}$. So

$$D([z,w]) = 2dD(\frac{[z,w]}{2d}) = [D(z), w] + [z, D(w)]$$

for all $z, w \in \mathcal{A}$.

Similarly, one can obtain that

$$\begin{aligned}
2dD(\frac{a\circ b}{2d}) &= \lim_{n\to\infty} \frac{2dh(\frac{d^{2n}a\circ b}{2d})}{d^{2n}} = \lim_{n\to\infty} \frac{2dh(\frac{(d^n a)\circ(d^n b)}{2d})}{d^{2n}} \\
&= \lim_{n\to\infty}\left((\frac{h(d^n a)}{d^n}) \circ (\frac{d^n b}{d^n}) + (\frac{d^n a}{d^n} \circ (\frac{h(d^n b)}{d^n})\right) \\
&= (Da)\circ b + a \circ (Db)
\end{aligned}$$

for all $a, b \in \mathcal{A}$. So

$$D(a \circ b) = 2dD(\frac{a \circ b}{2d}) = (Da) \circ b + a \circ (Db)$$

for all $a, b \in \mathcal{A}$.

Hence the mapping $D : \mathcal{A} \to \mathcal{A}$ is a Lie JC^*-algebra derivation satisfying (9.ii), as desired. □

COROLLARY 9.4. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
&\|2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^j \mu x_j + [z, w] + a \circ b}{2d}) \\
&\quad - 2dh(\frac{\mu x_1 + \sum_{j=2}^{2d}(-1)^{j+1} \mu x_j}{2d}) - 2\sum_{j=2}^{2d}(-1)^j \mu h(x_j) \\
&\qquad - [h(z), w] - [z, h(w)] - h(a) \circ b - a \circ h(b)\| \\
&\quad \leq \theta \sum_{j=1}^{2d} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p,
\end{aligned}$$

$$\|h(d^n u^*) - h(d^n u)^*\| \leq 2d \cdot d^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_{2d}, z, w, a, b \in \mathcal{A}$. Then there exists a unique Lie JC^*-algebra derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|h(x) - D(x)\| \leq \frac{d^{p+1}}{d - d^p}\theta\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \cdots, x_{2d}, z, w, a, b) = \theta(\sum_{j=1}^{2d} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)$, and apply Theorem 9.1. □

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