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ON k-QUASI-CLASS A CONTRACTIONS

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Abstract. A bounded linear Hilbert space operator T is said to be k-quasi-class A operator if it satisfy the operator inequality ${T^*}^k|T^2|T^k \geq {T^*}^k|T|^2T^k$ for a non-negative integer k. It is proved that if T is a k-quasi-class A contraction, then either T has a nontrivial invariant subspace or T is a proper contraction and the nonnegative operator $D = {T^*}^k(|T^2| - |T|^2)T^k$ is strongly stable.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. For any operator T in $B(\mathcal{H})$ set, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$ (the self-commutator of T), and consider the following standard definitions: T is *hyponormal* if $|T^*|^2 \geq |T|^2$ (i.e., if self-commutator $[T^*, T]$ is non-negative or, equivalently, if $\|T^*x\| \leq \|Tx\|$ for every x in $\mathcal{H}$), *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for some $p \in (0, 1]$, and T is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathcal{H}$. Following [13] and [4] we say that $T \in B(\mathcal{H})$ belongs to *class A* if $|T^2| \geq |T|^2$. We shall denote classes of hyponormal operators, p-hyponormal operators, paranormal operators, and class A operators by $\mathcal{H}$, $\mathcal{H}(p)$, $\mathcal{PN}$, and $\mathcal{A}$,

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respectively. It is well known that

$$\mathcal{H} \subset \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN}. \tag{1}$$

In [8] authors considered an extension of the notion of class $\mathcal{A}$ operators; we say that $T \in B(\mathcal{H})$ is *k-quasi-class* $\mathcal{A}$ operator if

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$$

for non-negative integer k; when $k = 1$, it is called the quasi-class A operator. We shall denote the set of k-quasi-class $\mathcal{A}$ operators by $\mathcal{QA}(k)$. Class $\mathcal{QA}(k)$ properly contains class $\mathcal{A}$ and quasi-class A.

It is well known that

$$\mathcal{H} \subset \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA} \subset \mathcal{QA}(k). \tag{2}$$

In view of inclusions (1) and (2), it seems reasonable to expect that the operators in class $\mathcal{QA}$ are paranormal. But there exists an example of a class $\mathcal{QA}$ operator which is not paranormal ([8]).

Recall, [10], that a contraction A (i.e., if $\|A\| \leq 1$, which means that $\|Ax\| \leq \|x\|$ for every $x \in \mathcal{H}$) is said to be a proper contraction if $||Ax|| < ||x||$ for every nonzero $x \in \mathcal{H}$. A strict contraction (i.e., a contraction A such that $||A|| < 1$) is a proper contraction, but a proper contraction is not necessarily a strict contraction. C. S. Kubrusly and N. Levan [10] have proved that if a hyponormal contraction A has no nontrivial invariant subspace, then

(a) A is a proper contraction and

(b) its self-commutator $[A^*, A]$ is a strict contraction.

Recently B. p. Duggal, I. H. Jeon and C. S. Kubrusly [2] showed that if A is a class A contraction, then either A has a nontrivial invariant subspace or A is a proper contraction and the non-negative operator $D = |A^2| - |A|^2$ is strongly stable (i.e., the power sequence $\{D^n\}$ converges strongly to 0). Very recently B. P. Duggal and authors [3] extend these results to contractions in $\mathcal{QA}$. In this paper, we extend these results to contractions in $\mathcal{QA}(k)$, which generalizes results proved for contractions in $\mathcal{QA}$ [2].

2. Results

We begin with well known following lemma;

LEMMA 2.1. *(see, [13]) An operator* $T \in \mathcal{QA}(k)$ *has a following matrix representation if* $\mathrm{ran(T^k)}$ *is not dense*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad \overline{\mathrm{ran(T^k)}} \oplus \ker(\mathrm{T^{*k}}), \tag{3}$$

where $A \in \mathcal{A}$*,* C *is a nilpotent with order* k *and* $\sigma(T) = \sigma(A) \cup \{0\}$*.*

LEMMA 2.2. *If* $T \in \mathcal{QA}(k)$ *is a contraction, then the non–negative operator* $D = T^{*k}(|T^2| - |T|^2)T^k$ *is a contraction such that the power sequence* $\{D^n\}$ *converges strongly to a projection* P *satisfying* $T^{k+1}P = 0$*.*

Proof. Set $R = D^{\frac{1}{2}}$. Then, for every $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle D^{n+1}x, x\rangle &= \langle R^{n+1}x, R^{n+1}\rangle \\ &= \langle DR^n x, R^n x\rangle \\ &= \langle T^{*k}(|T^2| - |T|^2)T^k R^n x, R^n x\rangle \\ &= \langle |T^2|T^k R^n x, T^k R^n x\rangle - \langle |T|^2 T^k R^n x, T^k R^n x\rangle \\ &\leq \| \, |T^2|^{\frac{1}{2}} T^k R^n x \, \|^2 \\ &\leq ||R^n x||^2 \qquad (T \text{ is contraction}) \\ &= \langle D^n x, x\rangle, \end{aligned}$$

which implies that D is a contraction. Evidently, the sequence $\{D^n\}$ being a monotonic decreasing sequence of non–negative contractions. Therefore $\{D^n\}$ converges strongly to a projection P. Now we should be show that $T^{k+1}P = 0$. Since

$$\begin{aligned} \|R^n x\|^2 - \|R^{n+1}x\|^2 &= \|R^n x\|^2 - \langle |T^2|T^k R^n x, T^k R^n x\rangle + \|T^{k+1}R^n x\|^2 \\ &= \langle (1 - T^{*k}|T^2|T^k)R^n x, R^n x\rangle + \|T^{k+1}R^n x\|^2 \\ &\geq \|T^{k+1}R^n x\|^2 \qquad (T \text{ is contraction}), \end{aligned}$$

we have that

$$\sum_{n=0}^{m} ||T^{k+1}R^n x||^2 \leq \sum_{n=0}^{m} ||R^n x||^2 - \sum_{n=0}^{m} ||R^{n+1}x||^2 = ||x||^2 - ||R^{m+1}x||^2 \leq ||x||^2$$

for every $x \in \mathcal{H}$ and non–negative integer m. Hence $||T^{k+1}R^n x|| \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, we have

$$T^{k+1}Px = T^{k|1} \lim_{n \longrightarrow \infty} D^n x = \lim_{n \longrightarrow \infty} T^{k+1}R^{2n}x = 0,$$

for every $x \in \mathcal{H}$. Hence $T^2 P = 0$. □

Recall that $T \in B(\mathcal{H})$ is a $C_{0.}$-contraction (resp., $C_{1.}$-contraction) if $||T^n x||$ converges to 0 for all $x \in \mathcal{H}$ (resp., does not converge to 0 for all non-trivial $x \in \mathcal{H}$); T is of class $C_{.0}$, or $C_{.1}$, if T^* is of class $C_{0.}$, respectively $C_{1.}$. All combinations are allowed, leading to the classes C_{00}, C_{01}, C_{10} and C_{11} of contractions [11, Page 72]. Duggal, Jeon and Kubrusly [2] showed that the following lemma;

Lemma 2.3. *If a class A contraction T has no nontrivial invariant subspace, then (a) T is a proper contraction and (b) the non-negative operator* $D = |T^2| - |T|^2$ *is a strongly stable contraction (so that* $D \in C_{00}$*).*

Using the above lemmas we can show that the following theorem;

Theorem 2.4. *If* $T \in \mathcal{QA}(k)$ *is a contraction with no non–trivial invariant subspace for non-negative integer k, then: (a) T is a proper contraction; (b) the non–negative operator* $D = T^{*k}(|T^2| - |T|^2)T^k$ *is a strongly stable contraction (and hence of class* C_{00}*).*

Proof. We may assume that T is non-zero.

(a) If either of $T^{-1}(0)$ or $\overline{\mathrm{ran}(T^k)}$ is non–trivial (i.e., $T^{-1}(0) \neq \{0\}$ or $\overline{\mathrm{ran}(T^k)} \neq \mathcal{H}$), then T has a non–trivial invariant subspace. Hence, if $T \in \mathcal{QA}(k)$ has no non–trivial invariant subspace, then T is injective and $\overline{\mathrm{ran}(T^k)} = \mathcal{H}$) Consequently, T must be class A operator. The proof now follows from Lemma 2.3.

(b) If $T \in \mathcal{QA}(k)$ is a contraction, then by Lemma 2.2 D is a contraction, $\{D^n\}$ converges strongly to a projection P and $T^{k+1}P = 0$. Therefore we have $PT^{*k+1} = 0$. Suppose T has no non-trivial invariant subspace. Since $P^{-1}(0)$ is a non-zero invariant subspace for T whenever $PT^{*k+1} = 0$, we must have $P^{-1}(0) = \mathcal{H}$. hence P must be zero and so $\{D^n\}$ converges strongly to 0, that is, D is a strongly stable contraction. Since D is a self-adjoint, $D \in C_{00}$. □

It is well known that a self-adjoint operator is a proper contraction if and only if it is a C_{00}-contraction. Hence, we have the following from Theorem 2.4.

Corollary 2.5. *If* $T \in \mathcal{QA}(k)$ *is a contraction with no non–trivial invariant subspace for non-negative integer k, then both T and* T^* *are proper contractions.*

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