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COMBINATORIAL INTERPRETATIONS OF THE ORTHOGONALITY RELATIONS FOR SPIN CHARACTERS OF $\tilde{S}_n$

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Abstract. In 1911 Schur[6] derived degree and character formulas for projective representations of the symmetric groups remarkably similar to the corresponding formulas for ordinary representations. Morris[3] derived a recurrence for evaluation of spin characters and Stembridge[8] gave a combinatorial reformulation for Morris' recurrence. In this paper we give combinatorial interpretations for the orthogonality relations of spin characters based on Stembridge's combinatorial reformulation for Morris' rule.

1. Introduction

The projective representations of the symmetric groups were originally studied by Schur. In his fundamental paper[6], Schur derived degree and character formulas for projective representations of the symmetric groups remarkably similar in style to the corresponding formulas for ordinary representations due to Frobenius. Morris[3] derived a recurrence for evaluation of spin characters, which is an analogue of the well-known Murnaghan-Nakayama formula for ordinary characters of the symmetric group S_n. Stembridge[8] then gave a combinatorial reformulation for Morris' recurrence using shifted rim hook tableaux,

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rather than the machinery of Hall-Littlewood functions used by Morris. Stembridge[7] found a Frobenius-type characteristic map and an analogue of the Littlewood–Richardson rule. Sagan[4] and Worley[11] has developed independently a combinatorial theory of shifted tableaux parallel to the theory of ordinary tableaux. This theory includes shifted versions of the Robinson-Schensted-Knuth correspondence, Green's invariants, Knuth relation, and Sch*ü*tzenberger's jeu de taquin. In [9] and [10] White gave combinatorial proofs of the orthogonality relations for the ordinary characters of S_n. His proof is based on the Murnaghan-Nakayama formula for ordinary characters of S_n.

In this paper we give combinatorial interpretations for the orthogonality relations of spin characters of $\tilde{S}_n$ based on Stembridge's combinatorial reformulation for Morris' rule.

In section 2, we outline the definitions and notation used in this paper. Section 3 reviews the basic properties of a group $\tilde{S}_n$ and draw some relations between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions. In section 4, we give combinatorial interpretations for the orthogonality relations of spin characters of $\tilde{S}_n$.

2. Definitions

We use standard notation $\mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ for the set of all positive integers, the ring of integers, the field of rational numbers and the field of complex numbers, respectively.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of the nonnegative integer n, denoted $\lambda \vdash n$ or $|\lambda| = n$, so λ is a weakly decreasing sequence of positive integers summing to n. We say each term λ_i is a *part* of λ and n is the *weight* of λ. The number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. Let $\mathcal{P}_n$ be the set of all partitions of n and $\mathcal{P}$ be the set of all partitions. We also denote

$$\begin{aligned}
OP &= \{\mu \in \mathcal{P} \mid \text{every part of } \mu \text{ is odd}\},\\
OP_n &= \{\mu \in \mathcal{P}_n \mid \text{every part of } \mu \text{ is odd}\},\\
DP &= \{\mu \in \mathcal{P} \mid \mu \text{ has all distinct parts}\},\\
DP_n &= \{\mu \in \mathcal{P}_n \mid \mu \text{ has all distinct parts}\},\\
DP_n^+ &= \{\mu \in \mathcal{D}P_n \mid n-\ell(\mu) \text{ is even}\},\\
DP_n^- &= \{\mu \in \mathcal{D}P_n \mid n-\ell(\mu) \text{ is odd}\}.
\end{aligned}$$

We sometimes abbreviate the partition λ with the notation $1^{j_1}2^{j_2}\ldots$, where j_i is the number of parts of size i. Sizes which do not appear are omitted and if $j_i = 1$, then it is not written. Thus, a partition $(5,3,2,2,2,1) \vdash 15$ can be written $12^3 35$.

For each $\lambda \in DP$, a *shifted diagram* D'_λ of shape λ is defined by

$$D'_\lambda = \{(i,j) \in \mathbf{Z}^2 \mid i \le j \le \lambda_j + i - 1, 1 \le i \le \ell(\lambda)\}.$$

And for $\lambda, \mu \in DP$ with $D'_\mu \subseteq D'_\lambda$, a *shifted skew diagram* $D'_{\lambda/\mu}$ is defined as the set-theoretic difference $D'_\lambda \setminus D'_\mu$. Figure 2.1 and Figure 2.2 show D'_λ and $D'_{\lambda/\mu}$ respectively when $\lambda = (9,7,4,2)$ and $\mu = (5,3)$.

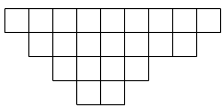

Figure 2.1

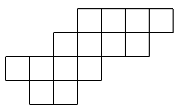

Figure 2.2

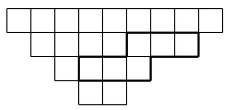

Figure 2.3

A shifted skew diagram θ is called a *single rim hook* if θ is connected and contains no 2×2 block of cells. If θ is a single rim hook, then its *head* is the upper rightmost cell in θ and its *tail* is the lower leftmost cell in θ. See Figure 2.3.

A *double rim hook* is a shifted skew diagram θ formed by the union of two single rim hooks both of whose tails are on the main diagonal. If θ is a double rim hook, we denote by $\mathcal{A}[\theta]$ (resp., $\alpha_1[\theta]$) the set of diagonals of length two (resp., one). Also let $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) be a single rim hook in θ which starts on the upper (resp., lower) of the two main diagonal cells and ends at the head of $\alpha_1[\theta]$. The tail of $\beta_1[\theta]$ (resp., $\gamma_1[\theta]$) is called the *first tail* (resp., *second tail*) of θ and the head of $\beta_1[\theta]$ or $\gamma_1[\theta]$ (resp., $\gamma_2[\theta], \beta_2[\theta]$, where $\beta_2[\theta] = \theta \setminus \beta_1[\theta]$ and $\gamma_2[\theta] = \theta \setminus \gamma_1[\theta]$) is called the *first head* (resp., *second head, third head*) of θ. Hence we have the following descriptions for a double rim hook θ:

$$\begin{aligned}\theta &= \mathcal{A}[\theta] \cup \alpha_1[\theta] \\ &= \beta_1[\theta] \cup \beta_2[\theta] \\ &= \gamma_1[\theta] \cup \gamma_2[\theta].\end{aligned}$$

A double rim hook is illustrated in Figure 2.4. We write $\mathcal{A}, \alpha_1$, etc. for $\mathcal{A}[\theta], \alpha_1[\theta]$, etc. when there is no confusion.

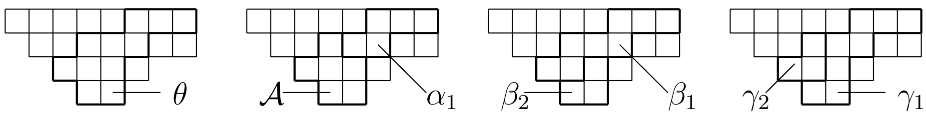

Figure 2.4

We will use the term *rim hook* to mean a single rim hook or a double rim hook.

A *shifted rim hook tableau* of shape $\lambda \in DP$ and content $\rho = (\rho_1, \ldots, \rho_m)$ is defined recursively. If $m = 1$, a rim hook with all 1's and shape λ is a shifted rim hook tableau. Suppose P of shape λ has content $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the cells containing the m's form a rim hook inside λ. If the removal of the m's leaves a shifted rim hook tableau, then P is a shifted rim hook tableau. We define a *shifted skew rim hook tableau* in a similar way. If P is a shifted rim hook tableau, we write $\kappa_P\langle r\rangle$ (or just $\kappa\langle r\rangle$) for a rim hook of P containing r.

If θ is a single rim hook then the *rank* $r(\theta)$ is one less than the number of rows it occupies and the *weight* $w(\theta) = (-1)^{r(\theta)}$; if θ is a double rim hook then the *rank* $r(\theta)$ is $|\mathcal{A}[\theta]|/2 + r(\alpha_1[\theta])$ and the *weight* $w(\theta)$ is $2(-1)^{r(\theta)}$.

The *weight* of a shifted rim hook tableau P, $w(P)$, is the product of the weights of its rim hooks. The weight of a shifted skew rim hook tableau is defined in a similar way.

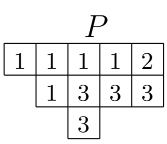

Figure 2.5

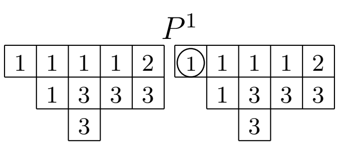

Figure 2.6

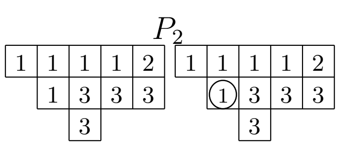

Figure 2.7

Figure 2.5 shows an example of a shifted rim hook tableau P of shape $(5,4,1)$ and content $(5,1,4)$. Here $r(\kappa\langle 1\rangle) = 1$, $r(\kappa\langle 2\rangle) = 0$ and $r(\kappa\langle 3\rangle) = 1$. Also $w(\kappa\langle 1\rangle) = -2$, $w(\kappa\langle 2\rangle) = 1$ and $w(\kappa\langle 3\rangle) = -1$. Hence $w(P) = (-2)\cdot(1)\cdot(-1) = 2$.

Let P be a shifted rim hook tableau. We denote by P^1(resp., P_2) one of the tableaux obtained from P by circling or not circling the first tail(resp., second tail) of each double rim hook in P. The P^1(resp., P_2) is called a *first* (resp., *second*) *tail circled rim hook tableau.* Similarly P_2^1 is obtained from P by circling or not circling the first tail and second tail of each double rim hook in P and is called a *tail circled rim hook tableau.* We use the notation $|\cdot|$ to refer to the uncircled version; e.g., $|P^1| = |P_2| = |P_2^1| = P$. See Figure 2.6 and Figure 2.7 for examples of first and second tail circled rim hook tableaux, respectively. Figure 2.8 shows tail circled rim hook tableaux P_2^1 when a shifted rim hook tableau P is given.

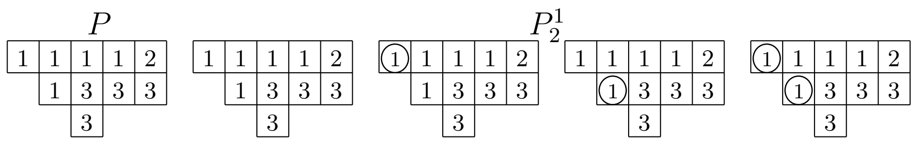

Figure 2.8

We now define a new weight function w' for first or second tail circled rim hook tableaux. If τ is a rim hook of P^1 or P_2, we define $w'(\tau) = (-1)^{r(\tau)}$. The weights $w'(P^1)$ and $w'(P_2)$ are the product of the weights of rim hooks in P^1 and P_2, respectively. For a tail circled rim hook tableau P_2^1, we define $w''(P_2^1) = 1$.

For each double rim hook τ of a rim hook tableau P, there are two first circled rim hooks τ_1, τ_2 such that $w(\tau) = w'(\tau_1) + w'(\tau_2)$. This fact implies the following:

Proposition 2.1. *Let $\gamma \in OP$. Then we have*

$$\sum_P w(P) = \sum_{P^1} w'(P^1),$$

where the left-hand sum is over all shifted rim hook tableaux P of shape λ/μ and content γ, while the right-hand sum is over all shifted first tail circled rim hook tableaux P^1 of shape λ/μ and content γ.

We can get the similar identiy using shifted second tail circled rim hook tableaux

3. Symmetric functions and irreducible spin characters of $\tilde{S}_n$

We consider the ring $\mathbb{Z}[x_1, x_2, \dots]$ of formal power series with integer coefficients in the infinite variables $x_1, x_2, \cdots$. Note that the symmetric functions form a subring of $\mathbb{Z}[x_1, x_2, \dots]$. Let $\Lambda(x)$, or simply Λ, be the ring of symmetric functions of $x_1, x_2, \cdots$. Define $\mathbb{Z}$-modules Λ^k by $\Lambda^k(x) = \Lambda^k = \{f \in \Lambda \mid f \text{ is homogeneous of degree } k\}$. Then we have $\Lambda = \prod_{k \geq 0} \Lambda^k$.

Let r be a positive integer. The rth *power sum* p_r is defined by

$$p_r = \sum_{i \geq 1} x_i^r.$$

By convention, we set $p_0 = 1$ and $p_r = 0$ for $r < 0$. Extend the definition of this symmetric function to all partitions by $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$.

We now define a group $\tilde{S}_n$ and draw some connections between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions.

For $n > 1$ let $\tilde{S}_n$ be the group generated by $t_1, t_2, \dots, t_{n-1}, -1$ subject to relations

$$t_i^2 = -1 \quad \text{for } i = 1, 2, \dots, n-1,$$
$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, 2, \dots, n-2,$$
$$t_i t_j = -t_j t_i \quad \text{for } |i - j| > 1 \ (i, j = 1, 2, \dots, n-1).$$

Note that $|\tilde{S}_n| = 2n!$. Since -1 is a central involution, Schur's lemma implies that an irreducible representation of $\tilde{S}_n$ must represent -1 by either the scalar 1 or -1. The representation of the former type is an ordinary representation of S_n, while one of the latter type will correspond to a projective representation of S_n, as we will see later. A representation T of $\tilde{S}_n$ is called a *spin representation* of $\tilde{S}_n$ if the group element -1 is represented by scalar -1, i.e., if $T(-1) = -1$.

To describe the characters of spin representations of $\tilde{S}_n$ we consider the structure of the conjugacy classes of $\tilde{S}_n$. Let $\theta_n : \tilde{S}_n \to S_n$ be an epimorphism defined by $t_i \mapsto s_i$, where s_i is an adjacent transposition

$(i\ i+1)$ in S_n. For each partition $\mu = (\mu_1, \dots, \mu_\ell)$ of n, we choose a specific element σ_μ such that $\theta_n(\sigma_\mu)$ is of cycle-type μ as follows: Define

$$\sigma_\mu = \pi_1 \pi_2 \dots \pi_\ell,$$

where $\pi_j = t_{a+1} t_{a+2} \dots t_{a+\mu_j-1}$ $(a = \sum_{i=1}^{j-1} \mu_i)$ for $1 \le j \le \ell = \ell(\mu)$. For example, if $\mu = (3,3,2) \vdash 8$, then $\sigma_\mu = t_1 t_2 t_4 t_5 t_7 \in \tilde{S}_8$ and $\theta_8(\sigma_\mu) = (123)(456)(78) \in S_8$.

Since $\ker(\theta_n) = \{\pm 1\}$, every $\sigma \in \tilde{S}_n$ is conjugate to σ_μ or $-\sigma_\mu$ for some partition μ of n.

Theorem 3.1. (Schur) *Let μ be a partition of n. Then the elements σ_μ and $-\sigma_\mu$ are not conjugate in $\tilde{S}_n$ iff either $\mu \in OP_n$ or $\mu \in DP_n^-$.*

Proof. See [1] or [7]. □

Let $\Omega_{\mathbb{Q}} = \prod_{n \ge 0} \Omega_{\mathbb{Q}}^n$ denote the graded subring of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $1, p_1, p_3, \dots$ and let $\Omega = \Lambda \cap \Omega_{\mathbb{Q}}$ denote the $\mathbb{Z}$-coefficient graded subring of $\Omega_{\mathbb{Q}}$. Clearly $\{p_\lambda \mid \lambda \in OP_n\}$ forms a basis of $\Omega_{\mathbb{Q}}^n$ and $\dim_{\mathbb{Q}} \Omega_{\mathbb{Q}}^n = |OP_n|$.

Define an inner product $[\ ,\ ]$ on $\Omega_{\mathbb{C}}^n$ by setting

$$[p_\lambda, p_\mu] = z_\lambda 2^{-\ell(\lambda)} \delta_{\lambda\mu} \quad \text{for } \lambda, \mu \in OP_n,$$

where

$$\delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. (Mac)

1. $\{Q_\lambda \mid \lambda \in DP_n\}$ *is a basis of* $\Omega_{\mathbb{Q}}^n$.
2. $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$,

where P_λ (resp., Q_λ) is the Hall-Littlewood symmetric P-function (resp., Q-function) corresponding to a partition $\lambda \in DP$.

Proof. See [2]. □

We now describe the irreducible spin characters of $\tilde{S}_n$ using the Hall-Littlewood symmetric functions P_λ and Q_λ

THEOREM 3.3. (Schur) *Define a class function* φ^λ *for each* $\lambda \in DP_n^+$ *by*

$$\varphi^\lambda(\sigma_\mu) = \begin{cases} [2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\ 0 & \text{otherwise} \end{cases}$$

and define a pair of class functions $\varphi_\pm^\lambda$ *for each* $\lambda \in DP_n^-$ *via*

$$\varphi_\pm^\lambda(\sigma_\mu) = \begin{cases} \frac{1}{\sqrt{2}}[2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\ \pm i^{(n-\ell(\lambda)+1)/2}\sqrt{\frac{1}{2}z_\lambda} & \text{if } \mu = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$ *if* $\lambda = 1^{m_1}2^{m_2}\cdots$.

Then the class functions $\varphi^\lambda(\lambda \in DP_n^+)$ *and* $\varphi_\pm^\lambda(\lambda \in DP_n^-)$ *are the irreducible spin characters of* $\tilde{S}_n$.

Proof. See [1] or [7]. □

Although Theorem 3.3 determines the irreducible spin characters φ^λ, it is difficult to use Theorem 3.3 to evaluate $\varphi^\lambda(\sigma_\mu)$ explicitly for $\mu \in OP$. But Morris has derived a recurrence for the evaluation of these characters which is similar to the well-known Murnaghan-Nakayama formula for ordinary characters of S_n.

In 1990 Stembridge[8] gave a combinatorial reformulation for Morris' recurrence using shifted tableaux, rather than the machinery of Hall-Littlewood functions used by Morris. We now describe Stembridge's interpretation for Morris' rule.

LEMMA 3.4. (Stembridge) *Let* k *be an odd number and* $|\lambda/\mu| = k$. *Then*

1. $[Q_{\lambda/\mu}, p_k] = 0$ *unless* λ/μ *is a rim hook.*
2. $[Q_{\lambda/\mu}, p_k] = (-1)^r$ *if* λ/μ *is a single rim hook of rank* r.
3. $[Q_{\lambda/\mu}, p_k] = 2(-1)^r$ *if* λ/μ *is a double rim hook of rank* r.

Proof. See [8]. □

THEOREM 3.5. (Stembridge) *For any* $\gamma \in OP$, *we have*

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_S w(S),$$

where the sum is over all shifted rim hook tableaux S *of shape* λ/μ *and content* γ.

Proof. Since the P_λ's and Q_λ's are dual bases, we have

$$p_r P_\mu = \sum_{\lambda \in DP} [p_r P_\mu, Q_\lambda] P_\lambda \quad \text{for any odd integer } r.$$

By iterating this expansion successively for $r = \gamma_1, \ldots, \gamma_\ell$, we find

$$[p_\gamma P_\mu, Q_\lambda] = \sum_{\{\lambda^j\}} [p_{\gamma_1} P_{\lambda^0}, Q_{\lambda^1}] \cdots [p_{\gamma_\ell} P_{\lambda^{\ell-1}}, Q_{\lambda^\ell}],$$

where $\mu = \lambda^0, \lambda = \lambda^\ell$. Since $[Q_{\lambda/\mu}, P_\nu] = [Q_\lambda, P_\mu P_\nu]$ and the P_ν's span $\Omega_{\mathbf{Q}}$, $[Q_{\lambda/\mu}, f] = [Q_\lambda, f P_\mu]$ for any $f \in \Omega_{\mathbf{Q}}$, and therefore

$$[Q_{\lambda/\mu}, p_\gamma] = \sum_{\{\lambda^j\}} [Q_{\lambda^1/\lambda^0}, p_{\gamma_1}] \cdots [Q_{\lambda^\ell/\lambda^{\ell-1}}, p_{\gamma_\ell}].$$

Note that $Q_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$. Thus the only nonzero contributions to $[Q_{\lambda/\mu}, p_\gamma]$ in this expansion occur when $\lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^\ell$ and $|\lambda^i| - |\lambda^{i-1}| = \gamma_i$ $(1 \le i \le \ell)$. Hence it suffices to evaluate $[Q_{\lambda/\mu}, p_k]$ for all skew shapes λ/μ of weight k (k odd), and the description of $[Q_{\lambda/\mu}, p_k]$ in Lemma 3.4 gives a complete proof of Theorem 3.5. □

Example 3.6. Consider $\lambda = (6, 3, 2, 1)$, $\gamma = (5, 3, 3, 1)$. There are four shifted rim hook tableaux of shape λ and content γ. See Figure 3.1. Since $w(T_1) = w(T_2) = w(T_3) = -2$ and $w(T_4) = 4$, $[Q_\lambda, p_\gamma] = -2$. Therefore Theorem 3.5 implies that

$$\varphi^\lambda(\sigma_\gamma) = [Q_\lambda, p_\gamma] = -2.$$

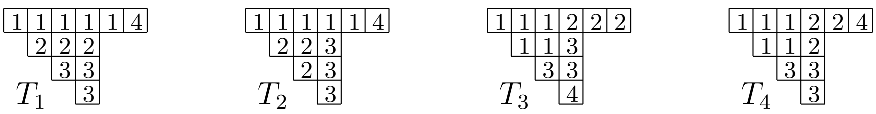

Figure 3.1

4. Combinatorial interpretations of the orthogonality relations for spin characters of $\tilde{S}_n$

Recall that there are two kinds of orthogonality relations for characters of a group G. See [5] for detail. First we give combinatorial interpretation for the orthogonality relation of the first kind for spin characters of $\tilde{S}_n$.

Theorem 4.1. (Orthogonality relation of the first kind) *Let G be a group of order g. If χ and ψ are irreducible characters of a group G. Then*

$$\frac{1}{g}\sum_{x\in G}\chi(x)\psi(x^{-1}) = \delta_{\chi\psi}.$$

Using Stembridge's combinatorial interpretation for Morris' rule given in Theorem 3.5, orthogonality relation of the first kind for $\tilde{S_n}$ in Theorem 4.1 can be reformulated as follows;

Corollary 4.2. (Orthogonality relation of the first kind for $\tilde{S_n}$) *Let $\lambda, \mu \in DP_n$. Then*

$$\sum 2^{\ell(type(\sigma))}w(P)w(Q) = \delta_{\lambda\mu}2^{\ell(\lambda)}n!,$$

where the sum is over triples (P, Q, σ), with P a shifted rim hook tableau of shape λ, Q a shifted rim hook tableau of shape μ and $\sigma \in S_n$, which satisfy type(σ) $\in OP_n$, content(P)=content(Q)=content(σ).

Given $\lambda \in DP_n$, a *permutation tableau of shape* λ is a filling of the shifted diagram D'_λ with positive integers $1, 2, \ldots, n$ and a *circled permutation tableau* is a permutation tableau with main diagonal entry either circled or uncircled. For example,

$$T = \begin{array}{cccc} 6 & 7 & 1 & 3 \\ & 5 & 8 & 4 \\ & & 7 & \end{array}$$

is a circled permutation tableau of shape $(4, 3, 1)$.

Let $\sigma \in S_n$ and write σ in cycle form, $\sigma = \sigma_1\sigma_2\ldots\sigma_m$, where the cycles σ_i are written in increasing order of the largest in the cycle. Recall that content (σ) is the sequence $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$, where $\rho_i = |\sigma_i|$ =length of the cycle σ_i. If $\sigma \in S_n$, then let $\overline{\sigma}$ be a permutation obtained from σ in which each cycle of σ is either barred or unbarred. If $\sigma = (42)(8371)$, $\overline{\sigma}$ is one of $(42)(8371)$, $\overline{(42)}(8371)$, $(42)\overline{(8371)}$ and $\overline{(42)}\,\overline{(8371)}$.

Now let $\lambda, \mu \in DP_n$ and let

$$\begin{aligned}
\pi_\lambda &= \text{the set of all circled permutation tableaux of shape } \lambda, \\
\Gamma_\lambda &= \{\, (P_2^1\ , \overline{\sigma}) \,\}, \\
\Psi^+_{\lambda\mu} &= \{\, (P^1, Q_2\ , \overline{\sigma}) \mid w'(P^1)w'(Q_2) = 1 \,\}, \\
\Psi^-_{\lambda\mu} &= \{\, (P^1, Q_2\ , \overline{\sigma}) \mid w'(P^1)w'(Q_2) = -1 \,\},
\end{aligned}$$

where P is a shifted rim hook tableau of shape λ , Q is a shifted rim hook tableau of shape μ , $\sigma \in S_n$ with type$(\sigma) \in OP_n$ and content$(P) =$ content$(Q) =$ content(σ).

Note that

$$|\pi_\lambda| = 2^{\ell(\lambda)} n!,$$
$$|\Gamma_\lambda| = \sum 2^{\ell(type(\sigma))} |P_2^1| = \sum 2^{\ell(type(\sigma))} w''(P_2^1),$$

where P is a shifted rim hook tableau of shape λ. Since the map given by $P_2^1 \to (P^1, P_2)$ is clearly a bijection, we get the following theorem from Corollary 4.2.

Theorem 4.3. *Let $\lambda, \mu \in DP_n$.*
(a) If $\lambda \neq \mu$, then there is a bijection between $\Psi^+_{\lambda\mu}$ and $\Psi^-_{\lambda\mu}$.
(b) If $\lambda = \mu$, then there is a bijection between Γ_λ and π_λ.

Let's now give combinatorial interpretation for the orthogonality relation of the second kind for spin characters of $\tilde{S}_n$.

Theorem 4.4. (Orthogonality relation of the second kind) *Let G be a group of order g. Let $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ be all irreducible characters of G. Then*

$$\sum_{i=1}^{k} \chi^{(i)}_\alpha \overline{\chi^{(i)}_\beta} = \frac{g}{h_\alpha} \delta_{\alpha\beta},$$

where h_α is the number of elements in the conjugacy class C_α of α.

Using Schur's spin character formulas described in Theorem 3.3 and Stembridge's combinatorial interpretation for spin characters of $\tilde{S}_n$ respectively, orthogonality relation of the second kind for $\tilde{S}_n$ can be described in the following ways.

Corollary 4.5. (Orthogonality relation of the second kind for $\tilde{S}_n$) *Let $\mu,\ \nu \in P_n$. Then*

$$\sum_{\lambda \in DP_n^+} \varphi^\lambda(\sigma_\mu)\varphi^\lambda(\sigma_\nu^{-1}) + \sum_{\lambda \in DP_n^-} \varphi^\lambda_\pm(\sigma_\mu)\varphi^\lambda_\pm(\sigma_\nu^{-1}) = \delta_{\mu\nu} 1^{j_1} j_1! 2^{j_2} j_2! \cdots,$$

where $\mu = 1^{j_1} 2^{j_2} \cdots$.

Corollary 4.6. *Let $\mu = 1^{j_1} 2^{j_2} \cdots \in OP_n$ and $\nu \in OP_n$. Then*

$$\sum_{(\lambda,P,Q)} 2^{n-\ell(\lambda)} w(P) w(Q) = \delta_{\mu\nu} 2^{n-\ell(\mu)} 1^{j_1} j_1! 2^{j_2} j_2! \cdots,$$

where $\lambda \in DP_n$, P is a shifted rim hook tableau of shape λ and content μ and Q is a shifted rim hook tableau of the same shape λ and content ν.

We will describe combinatorial objects whose weights represent the both sides of the identity given in Corollary 4.6.

$\mathcal{H} = (H_1, H_2, \ldots, H_m)$ is said to be a *circled hook permutation* of content $\rho = k^m$, and shape $(\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)})$ if the following conditions hold:

(1) each H_i is a hook tableau of shape $\tau^{(i)}$,
(2) $|\tau^{(i)}| = k$ and
(3) for each i, all cells of H_i except its tail can be circled or uncircled.

Figure 4.1 gives a circled hook permutation of content 5^5.

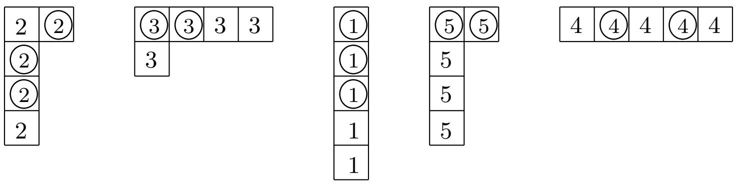

Figure 4.1

Using shifted first tail circled rim hook tableaux and circled hook permutations defined in the above, Corollary 4.6 implies the following theorem.

Theorem 4.7. *There is a bijection between positive pairs (P, Q), where P is a shifted (first tail circled) rim hook tableau of shape λ and content μ and Q is a circled shifted rim hook tableau of the same shape λ and content ν, and,*

(a) if $\mu \neq \nu$, negative pairs of $(\hat{P}, \hat{Q})$, where $\hat{P}$ is a shifted (first tail circled) rim hook tableau of shape $\hat{\lambda}$ and content μ and $\hat{Q}$ is a circled shifted rim hook tableau of the same shape $\hat{\lambda}$ and content ν, or,

(b) if $\mu = \nu$, the union of the set of negative pairs of $(\hat{P}, \hat{Q})$, where $\hat{P}$ is a shifted (first tail circled) rim hook tableau of shape $\hat{\lambda}$ and content μ and $\hat{Q}$ is a circled shifted rim hook tableau of the same shape $\hat{\lambda}$ and content ν, with the set of circled hook permutations of content μ.

It will be very interesting to construct bijections directly described in Theorem 4.3 and Theorem 4.7.

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