

## ON ALMOST $\omega_1$ - $p^{\omega+n}$ -PROJECTIVE ABELIAN $p$ -GROUPS

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ABSTRACT. We define the class of almost  $\omega_1$ - $p^{\omega+n}$ -projective abelian  $p$ -primary groups and investigate their basic properties. The established results extend classical achievements due to Hill (Comment. Math. Univ. Carol., 1995), Hill-Ullery (Czech. Math. J., 1996) and Keef (J. Alg. Numb. Th. Acad., 2010).

### 1. Introduction and Backgrounds

Throughout the present paper, unless specified something else, let us assume that all groups are  $p$ -torsion abelian, written additively as is the custom when studying them. Standardly, for any group  $G$ , we set  $p^i G = \{p^i g \mid g \in G\}$ , where  $i \in \mathbb{N}$ . Clearly,  $p^i G$  is a subgroup of  $G$  and the intersection over all  $i$  forms a subgroup  $p^\omega G$  which is called *first Ulm subgroup*. Usually, a group  $G$  is called *separable* if  $p^\omega G = \{0\}$ .

All other not explicitly explained herein notions and notations are well-known and mainly follow those from [6] and [7].

The next concept of Hill from [8] is crucial for our further investigation.

DEFINITION 1. ([8]) The separable group  $G$  is said to be *almost  $\Sigma$ -cyclic* if it possesses a collection  $\mathcal{C}$  consisting of nice subgroups of  $G$  which satisfies the following three conditions:

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- (i)  $\{0\} \in \mathcal{C}$ ;
- (ii)  $\mathcal{C}$  is closed with respect to ascending unions, i.e., if  $H_i \in \mathcal{C}$  with  $H_i \subseteq H_j$  whenever  $i \leq j$  ( $i, j \in I$ ) then  $\cup_{i \in I} H_i \in \mathcal{C}$ ;
- (iii) If  $K$  is a countable subgroup of  $G$ , then there is  $L \in \mathcal{C}$  (that is, a nice subgroup  $L$  of  $G$ ) such that  $K \subseteq L$  and  $L$  is countable.

Removing the requirement of separability but adding that of being reduced, the last was generalized to the notion of so-called *almost totally projective groups*. Furthermore, as in the classical theory of simply presented groups, we shall say that a group is *almost simply presented* if it is the direct sum of a divisible group and an almost totally projective group.

We now shall list some important results that will be used in the sequel without a concrete referring.

**THEOREM 1.1.** ([10]) *Suppose  $\alpha$  is an arbitrary ordinal. Then  $G$  is almost totally projective if and only if both  $p^\alpha G$  and  $G/p^\alpha G$  are almost totally projective.*

*In particular,  $G$  is almost  $\Sigma$ -cyclic if and only if  $p^n G$  is almost  $\Sigma$ -cyclic, whenever  $n$  is a natural.*

**THEOREM 1.2.** ([1]) *If  $A \leq G$  and  $G$  is almost  $\Sigma$ -cyclic, then  $A$  is almost  $\Sigma$ -cyclic.*

**THEOREM 1.3.** ([8]) *Let  $G$  be an almost  $\Sigma$ -cyclic group and  $G \leq T$  with  $p^\omega T = \{0\}$ . Then  $T$  is almost  $\Sigma$ -cyclic, provided  $T/G$  is countable.*

**THEOREM 1.4.** ([8]) *Let  $G$  be a separable group with a countable subgroup  $K$  such that  $G/K$  is almost  $\Sigma$ -cyclic. Then  $G$  is almost  $\Sigma$ -cyclic.*

The last two statements can be generally superseded via the following:

**THEOREM 1.5.** ([9] and [2]) *Let  $G$  be an almost totally projective group and  $G \leq T$  where  $T$  is reduced. Then  $T$  is almost totally projective, provided  $T/G$  is countable.*

**THEOREM 1.6.** ([2]) *Let  $G$  be a reduced group with countable nice subgroup  $N$  such that  $G/N$  is almost totally projective. Then  $G$  is almost totally projective.*

Recall that the homomorphism  $f : G \rightarrow H$  of two groups  $G$  and  $H$  is said to be  $\omega_1$ -*bijjective* if its kernel and co-kernel are both countable.

**THEOREM 1.7.** ([5]) *Let  $f : G \rightarrow H$  be an  $\omega_1$ -bijection such that both  $G$  and  $H$  are reduced groups. If  $G$  is almost totally projective, then  $H$  is almost totally projective.*

*In particular, if  $G$  is a nice- $\aleph_0$ -elongation of  $H$ , then  $G$  is almost totally projective if and only if  $H$  is almost totally projective.*

As in the proof of Theorem 2.4 from [4], with restatements of Propositions 2.5 and 2.6 again from [4] (see also Proposition 5.3 and Corollary 5.5 in [5]) for almost totally projective groups, we derive:

**THEOREM 1.8.** *Suppose that  $f : G \rightarrow H$  is an  $\omega_1$ -bijective homomorphism. If  $G$  is almost simply presented, then  $H$  is almost simply presented.*

*In particular, if  $G$  is a nice- $\aleph_0$ -elongation of  $H$ , then  $G$  is almost simply presented if and only if  $H$  is almost simply presented.*

## 2. New Definitions and Results

The above Definition 1 can be successfully generalized to the following:

**DEFINITION 2.** The group  $G$  is said to be *almost  $p^{\omega+n}$ -projective* if there exists  $B \leq G[p^n]$  such that  $G/B$  is almost  $\Sigma$ -cyclic.

Clearly,  $p^{\omega+n}G = \{0\}$ .

Our objective now is to show that the Definition 2 can be equivalently formulated as follows:

**PROPOSITION 2.1.** *The group  $G$  is almost  $p^{\omega+n}$ -projective if and only if  $G \cong H/A$  for some almost  $\Sigma$ -cyclic group  $H$  and  $A \leq H[p^n]$ .*

*Proof.* "⇒". Let  $X$  be a group with  $p^nX = G$  and let  $H = X/B$ . Consequently,  $p^nH = p^nX/B = G/B$  is almost  $\Sigma$ -cyclic and hence, by Theorem 1.1,  $H$  is almost  $\Sigma$ -cyclic too. Supposing  $A = X[p^n]/B$ , we deduce that  $H/A \cong X/X[p^n] \cong p^nX = G$ , as required.

"⇐". Assume that  $G \cong H/A$ , which we without loss of generality interpret as an equality, whence we get  $B = H[p^n]/A \subseteq G[p^n]$ . Thus, again in view of Theorem 1.1,  $G/B \cong H/H[p^n] \cong p^nH$  is almost  $\Sigma$ -cyclic, as desired. □

On another vein, Keef introduced in [11] the notion of  $\omega_1$ - $p^{\omega+n}$ -projective groups and showed that  $G$  is  $\omega_1$ - $p^{\omega+n}$ -projective if and only

if there exists a countable nice subgroup  $K$  such that  $G/K$  is  $p^{\omega+n}$ -projective. So, formulating the last in terms of almost  $p^{\omega+n}$ -projective groups, we obtain a common strengthening of Definition 2 like this:

**DEFINITION 3.** *The group  $G$  is called almost  $\omega_1$ - $p^{\omega+n}$ -projective if there is a countable subgroup  $C$  such that  $G/C$  is almost  $p^{\omega+n}$ -projective.*

Apparently,  $p^{\omega+n}G$  is countable.

So, the leitmotif of this article is by the utilization of the above material to explore certain characteristic properties of the stated new class of groups in Definition 3. Although at first glance there is an absolute analogue with [11], this is definitely untrue; the main reason is that the almost  $\Sigma$ -cyclic groups do not have the important direct decomposition property of the  $\Sigma$ -cyclic groups.

**REMARK 1.** Another way of generalizing  $\omega_1$ - $p^{\omega+n}$ -projectivity under the name *weak  $\omega_1$ - $p^{\omega+n}$ -projectivity* the interested reader can see in [3].

Besides, since there is an almost  $\Sigma$ -cyclic group which is not  $\Sigma$ -cyclic, there exists an almost  $p^{\omega+n}$ -projective group (and thus an almost  $\omega_1$ - $p^{\omega+n}$ -projective group) that is not  $p^{\omega+n}$ -projective (and so not  $\omega_1$ - $p^{\omega+n}$ -projective).

Two immediate consequences of the listed above Balof-Keef's theorem are these:

**COROLLARY 2.2.** *A subgroup of an almost  $p^{\omega+n}$ -projective group is also almost  $p^{\omega+n}$ -projective.*

*Proof.* Let  $B \leq G[p^n]$  such that  $G/B$  is almost  $\Sigma$ -cyclic, and suppose that  $A \leq G$ . Then  $(A+B)/B \subseteq G/B$  is again almost  $\Sigma$ -cyclic, and  $A/(A \cap B) \cong (A+B)/B$  with  $A \cap B \subseteq A[p^n]$ , as wanted.  $\square$

**COROLLARY 2.3.** *A subgroup of an almost  $\omega_1$ - $p^{\omega+n}$ -projective group is almost  $\omega_1$ - $p^{\omega+n}$ -projective as well.*

*Proof.* Assume that  $S \leq G$  where  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective. Thus  $G/C$  is almost  $p^{\omega+n}$ -projective for some countable subgroup  $C$ . Moreover,  $(S+C)/C \subseteq G/C$  is, in virtue of Corollary 2.2, almost  $p^{\omega+n}$ -projective too, and  $S/(S \cap C) \cong (S+C)/C$ . Since  $S \cap C \leq C$  is countable, we are done.  $\square$

The next statements refine the corresponding assertion from [8] as follows:

PROPOSITION 2.4. *Let  $G \leq A$  where  $A$  is separable with  $A/G$  countable. Then  $A$  is almost  $p^{\omega+n}$ -projective if and only if  $G$  is almost  $p^{\omega+n}$ -projective.*

*Proof.* The necessity follows from Corollary 2.2.

As for the sufficiency, write  $A = G + Z$  for some countable subgroup  $Z$ . Since there exists  $B \leq G[p^n]$  with  $G/B$  almost  $\Sigma$ -cyclic, we conclude that  $A/B = (G/B) + (Z + B)/B$  where  $(Z + B)/B \cong Z/(Z \cap B)$  is countable. Exploiting [10] or [2], we obtain that  $A/B$  is almost simply presented. Henceforth,  $(A/B)/p^\omega(A/B) \cong A/[\cap_{i<\omega}(p^i A + B)]$  is almost  $\Sigma$ -cyclic. But  $\cap_{i<\omega}(p^i A + B)$  is obviously  $p^n$ -bounded, as needed.  $\square$

LEMMA 2.5. *Let  $G$  be a separable group with a countable nice subgroup  $C$ . Then  $G$  is almost  $\Sigma$ -cyclic if and only if  $G/C$  is almost  $\Sigma$ -cyclic.*

*Proof.* Follows by a direct application of Proposition 5.3 and Corollary 5.5 of [5] (see [2] too).  $\square$

REMARK 2. *The niceness of  $C$  in  $G$  is tantamount to the requirement that  $G/C$  is separable.*

The last affirmation can be slightly improved thus:

LEMMA 2.6. *Suppose  $G/C$  is almost  $\Sigma$ -cyclic for some group  $G$  and its countable subgroup  $C$ . Then  $G$  is almost simply presented.*

*Proof.* We see that  $p^\omega G \subseteq C$  is countable and  $G/C \cong (G/p^\omega G)/(C/p^\omega G)$  is almost  $\Sigma$ -cyclic where  $C/p^\omega G$  remains countable. We therefore apply Lemma 2.5 or Hill's result from [8] quoted above (see also [2]) to infer that  $G/p^\omega G$  is almost  $\Sigma$ -cyclic. Thus  $G$  must be almost simply presented, as formulated.  $\square$

LEMMA 2.7. *A high subgroup of an almost simply presented group is almost  $\Sigma$ -cyclic.*

*Proof.* Suppose  $H_G$  is a high subgroup of the almost simply presented group  $G$ . Since  $H_G \cap p^\omega G = \{0\}$ , one may see that  $H_G \cong (H_G \oplus p^\omega G)/p^\omega G \subseteq G/p^\omega G$ . Since as noticed above  $G/p^\omega G$  is almost  $\Sigma$ -cyclic, we apply the aforementioned Balof-Keef's theorem from [1] to get that so is  $H_G$ , as claimed.  $\square$

Now Lemma 2.6 can be somewhat refined thus:

LEMMA 2.8. *Let  $G/C$  be almost  $\Sigma$ -cyclic for some group  $G$  and its countable subgroup  $C$ . Then  $G$  is the sum of a countable group and an almost  $\Sigma$ -cyclic group.*

*Proof.* Observing that  $p^\omega G \subseteq C$  is countable, we may isomorphically embed it in an essential subgroup of  $G/H_G$  where  $H_G$  is a high subgroup of  $G$  and thus  $G/H_G$  will be countable as well. In fact,  $p^\omega G \cong (p^\omega G \oplus H_G)/H_G \subseteq G/H_G$  where it is easily checked that  $(p^\omega G \oplus H_G)/H_G$  is essential in  $G/H_G$  because  $H_G$  is maximal with respect to intersecting  $p^\omega G$  trivially. This substantiates our claim.

Furthermore, one can write that  $G = H_G + K$  for some countable group  $K$ . Next, appealing to a combination of Lemmas 2.6 and 2.7, we obtain that  $H_G$  must be almost  $\Sigma$ -cyclic and so the required decomposition.  $\square$

A useful consequence is the following one:

COROLLARY 2.9. *Suppose that  $A$  is an almost  $\Sigma$ -cyclic group and  $K$  is its countable group. Then  $A/K$  is the sum of a countable group and an almost  $\Sigma$ -cyclic group.*

*Proof.* Since  $K \subseteq Z$  for some countable nice subgroup  $Z$  of  $A$ , it follows from Lemma 2.5 that  $A/Z$  remains almost  $\Sigma$ -cyclic. But  $A/Z \cong (A/K)/(Z/K)$  where  $Z/K$  is countable, so that the wanted decomposition of  $A/K$  can be deduced from Lemma 2.8.  $\square$

PROPOSITION 2.10. *Let  $C \leq G$  be a countable nice subgroup of the separable group  $G$ . Then  $G$  is almost  $p^{\omega+n}$ -projective if and only if  $G/C$  is almost  $p^{\omega+n}$ -projective.*

*Proof.* " $\Rightarrow$ ". Assume that  $G/B$  is almost  $\Sigma$ -cyclic for some  $B \leq G[p^n]$ . Observe that the two isomorphisms hold:

$$(G/C)/(B+C)/C \cong G/(B+C) \cong (G/B)/(B+C)/B.$$

Since  $(B+C)/B \cong C/(B \cap C)$  is countable, it follows from Theorem 1.8 that  $A/V$  is almost simply presented where we put  $G/C = A$  and  $(B+C)/C = V$ . Therefore,

$$A/[\cap_{i<\omega}(p^i A + V)] \cong (A/V)/[\cap_{i<\omega}(p^i A + V)]/V = (A/V)/p^\omega(A/V)$$

is almost  $\Sigma$ -cyclic with  $p^n(\cap_{i<\omega}(p^i A + V)) = p^\omega A = p^\omega(G/C) = (p^\omega G + C)/C = \{0\}$ . Thus by Definition 2 the factor-group  $A = G/C$  is almost  $p^{\omega+n}$ -projective, as claimed.

" $\Leftarrow$ ". Let us now  $G/C$  be almost  $p^{\omega+n}$ -projective. Consequently there is a quotient  $M/C$  with  $M \leq G$  and  $p^n M \subseteq C$  such that  $(G/C)/(M/C) \cong G/M$  is almost  $\Sigma$ -cyclic. Hence  $M$  is the direct sum of a countable group and a  $p^n$ -bounded group, say  $M = K \oplus P$  where  $K$  is countable and  $p^n P = \{0\}$ . We further deduce that

$$G/M = G/(K \oplus P) \cong (G/P)/(K \oplus P)/P,$$

where  $(K \oplus P)/P \cong K$  is countable. Hence, Lemma 2.6 applies to show that  $G/P$  is almost simply presented. So,  $(G/P)/p^\omega(G/P) \cong G/[\cap_{i<\omega}(p^i G + P)]$  is almost  $\Sigma$ -cyclic with  $p^n(\cap_{i<\omega}(p^i G + P)) = p^\omega G = \{0\}$ . Finally,  $G$  is almost  $p^{\omega+n}$ -projective, as expected.  $\square$

REMARK 3. Observe that the condition on niceness was not used in the proof of the sufficiency. Moreover, our proof definitely simplifies the corresponding one for  $p^{\omega+n}$ -projective groups given in [4] as it can also be successfully applied in that case.

Combining Propositions 2.4 and 2.10, one can state the following improvement of Theorem 4.2 from [4] as follows:

THEOREM 2.11. *Separable almost  $p^{\omega+n}$ -projectives are closed under the formation of  $\omega_1$ -bijections.*

PROPOSITION 2.12. *Suppose that  $G$  is a separable almost  $\omega_1$ - $p^{\omega+n}$ -projective group. Then  $G$  is almost  $p^{\omega+n}$ -projective.*

*Proof.* Let  $C$  be a countable subgroup of the group  $G$  such that  $G/C$  is almost  $p^{\omega+n}$ -projective. Exploiting Proposition 2.10 along with Remark 3, we conclude that  $G$  is almost  $p^{\omega+n}$ -projective, as asserted.  $\square$

PROPOSITION 2.13. (a) *If  $G$  is almost  $p^{\omega+n}$ -projective, then  $G/p^\lambda G$  is almost  $p^{\omega+n}$ -projective for any ordinal  $\lambda$ .*

(b) *If  $G$  is (nicely) almost  $\omega_1$ - $p^{\omega+n}$ -projective, then  $G/p^\lambda G$  is (nicely) almost  $\omega_1$ - $p^{\omega+n}$ -projective for all ordinals  $\lambda$ .*

*Proof.* Since for  $\lambda < \omega$  the statements are self-evident, we will concentrate on  $\lambda \geq \omega$ .

(a) So, write  $G/B$  is almost  $\Sigma$ -cyclic for some  $B \leq G[p^n]$ . So,  $p^\omega G \subseteq B$  and hence  $p^\lambda G \subseteq B$  for each ordinal  $\lambda \geq \omega$ . Furthermore,  $G/B \cong$

$(G/p^\lambda G)/(B/p^\lambda G)$  is almost  $\Sigma$ -cyclic with  $B/p^\lambda G \subseteq (G/p^\lambda G)[p^n]$ , as wanted.

(b) We will first be concerned with the "nicely" version. So, writing  $G/C$  is almost  $p^{\omega+n}$ -projective for some countable nice subgroup  $C$ , we infer with the aid of point (a) that

$$(G/C)/p^\lambda(G/C) = (G/C)/(p^\lambda G + C)/C \cong G/(p^\lambda G + C) \cong (G/p^\lambda G)/(p^\lambda G + C)/p^\lambda G$$

is almost  $p^{\omega+n}$ -projective. Since  $(p^\lambda G + C)/p^\lambda G \cong C/(p^\lambda G \cap C)$  is obviously countable and nice in  $G/p^\lambda G$  (see [6]), we are finished.

The variant without "niceness" seems to be more complicated. To show it, let  $G/C$  be almost  $p^{\omega+n}$ -projective for some countable group  $C$ . We claim that

$$(G/C)/(p^\lambda G + C)/C \cong G/(p^\lambda G + C) \cong (G/p^\lambda G)/(p^\lambda G + C)/p^\lambda G.$$

is almost  $p^{\omega+n}$ -projective. In fact, if  $H$  is an almost  $p^{\omega+n}$ -projective group with  $R \subseteq p^\lambda H$ , then  $H/R$  is also almost  $p^{\omega+n}$ -projective. To this goal, write  $H/S$  is almost  $\Sigma$ -cyclic for some  $S \leq H[p^n]$ . Thus  $p^\omega H \subseteq S$  whence  $R \subseteq p^\lambda H \subseteq p^\omega H \subseteq S$ . This gives that  $H/S \cong (H/R)/(S/R)$  is almost  $\Sigma$ -cyclic for  $S/R \leq (H/R)[p^n]$  and means that  $H/R$  is really as desired. We just apply this assertion to  $H = G/C$  and  $R = (p^\lambda G + C)/C \subseteq p^\lambda(G/C) = p^\lambda H$  and the claim is sustained.

Furthermore, by what we have previously shown,  $(G/p^\lambda G)/(p^\lambda G + C)/p^\lambda G$  being almost  $p^{\omega+n}$ -projective with countable  $(p^\lambda G + C)/p^\lambda G \cong C/(C \cap p^\lambda G)$  ensures that  $G/p^\lambda G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective, as formulated. □

As a direct consequence, we yield:

**COROLLARY 2.14.** *If  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective, then  $G/p^\omega G$  is almost  $p^{\omega+n}$ -projective.*

*Proof.* Follows according to Proposition 2.13 (b) accomplished with Proposition 2.12. □

For groups with countable first Ulm subgroup, we can say even more:

**THEOREM 2.15.** *Suppose  $G$  is a group whose  $p^\omega G$  is countable. Then  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $G/p^\omega G$  is almost  $p^{\omega+n}$ -projective.*



*Proof.* The "and only if" part was proved in Corollary 2.14. The remaining "if" part follows immediately via Definition 3.  $\square$

**THEOREM 2.16.** *The group  $G$  is (nicely) almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $p^{\omega+n}G$  is countable and  $G/p^{\omega+n}G$  is (nicely) almost  $\omega_1$ - $p^{\omega+n}$ -projective.*

*Proof.* The necessity follows from Proposition 2.13 (b) substituting  $\lambda = \omega + n$ .

As for the sufficiency, suppose  $(G/p^{\omega+n}G)/(T/p^{\omega+n}G) \cong G/T$  is almost  $p^{\omega+n}$ -projective for some countable (nice) quotient  $T/p^{\omega+n}G$  such that  $T \leq G$ . But  $T$  is countable (and nice) in  $G$  (cf. [6]), so that  $G$  is (nicely) almost  $\omega_1$ - $p^{\omega+n}$ -projective, as claimed.  $\square$

Utilizing the above idea, one can state the following:

**COROLLARY 2.17.** *Suppose that  $p^\lambda G$  is countable for some ordinal  $\lambda$ . Then  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $G/p^\lambda G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective.*

*Proof.* The necessity is true using Proposition 2.13 (b).

The sufficiency follows by copying the same method as that demonstrated in Theorem 2.16.  $\square$

**PROPOSITION 2.18.** *The direct sums of almost  $p^{\omega+n}$ -projective groups are almost  $p^{\omega+n}$ -projective groups.*

*Proof.* Write  $G = \bigoplus_{i \in I} G_i$  where all components  $G_i$  are almost  $p^{\omega+n}$ -projective. So,  $G_i/B_i$  are almost  $\Sigma$ -cyclic for some  $B_i \leq G_i[p^n]$ . Furthermore, putting  $B = \bigoplus_{i \in I} B_i$ , we infer that  $B \leq G[p^n]$  and that  $G/B \cong \bigoplus_{i \in I} G_i/B_i$  is almost  $\Sigma$ -cyclic owing to [8], as expected.  $\square$

The following improves (Proposition 2.4, [11]) to the new framework.

**PROPOSITION 2.19.** *Suppose  $G = \bigoplus_{i \in I} G_i$  is a group for some index set  $I$ . Then  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $G_i$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective for each index  $i \in I$ , and there exists a countable subset  $J \subseteq I$  such that  $G_i$  are almost  $p^{\omega+n}$ -projective for all  $i \in I \setminus J$ .*

*Proof.* "**Necessity**". Let  $C$  be a countable subgroup of  $G$  such that  $G/C$  is almost  $p^{\omega+n}$ -projective. That all  $G_i$  are almost  $\omega_1$ - $p^{\omega+n}$ -projective follows from Corollary 2.3. Clearly,  $C \subseteq \bigoplus_{i \in J} G_i$  for some  $J \subseteq I$  with  $|J| \leq \aleph_0$ . Therefore,

$$G/C \cong [(\oplus_{i \in J} G_i)/C] \oplus [\oplus_{i \in I \setminus J} G_i],$$

so that  $\oplus_{i \in I \setminus J} G_i$ , and hence  $G_i$ , is almost  $p^{\omega+n}$ -projective for every  $i \in I \setminus J$  in conjunction with Corollary 2.2.

"**Sufficiency**". Let all factors  $G_i/C_i$  be almost  $p^{\omega+n}$ -projective for some countable subgroups  $C_i \leq G_i$ . Set  $C = \oplus_{i \in J} C_i$ , whence  $C$  is countable. However,

$$G/C \cong [\oplus_{i \in J} (G_i/C_i)] \oplus [\oplus_{i \in I \setminus J} G_i],$$

and so Proposition 2.18 works to infer that  $G/C$  is almost  $p^{\omega+n}$ -projective, as required.  $\square$

As an immediate consequence, we derive:

**COROLLARY 2.20.** *The countable direct sum of almost  $\omega_1$ - $p^{\omega+n}$ -projective groups is an almost  $\omega_1$ - $p^{\omega+n}$ -projective group.*

Our aim now is to establish here some equivalencies that give comprehensive characterizations of almost  $\omega_1$ - $p^{\omega+n}$ -projectivity.

**THEOREM 2.21.** *The following conditions are equivalent:*

- (1)  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective;
- (2)  $G/P$  is the sum of a countable group and an almost  $\Sigma$ -cyclic group, where  $p^n P = \{0\}$ ;
- (3)  $G \cong T/V$ , where  $T$  is the sum of a countable group and an almost  $\Sigma$ -cyclic group and  $p^n V = \{0\}$ ;
- (4)  $G/M$  is almost  $\Sigma$ -cyclic, where  $p^n M$  is countable ( $M$  is the direct sum of a countable group and a  $p^n$ -bounded group);
- (5)  $G \cong S/N$ , where  $S$  is almost  $\Sigma$ -cyclic and  $p^n N$  is countable ( $N$  is the direct sum of a countable group and a  $p^n$ -bounded group);
- (6)  $G \cong A/H$ , where  $A$  is almost  $p^{\omega+n}$ -projective and  $H$  is countable;
- (7)  $G/Y$  is countable, where  $Y$  is almost  $p^{\omega+n}$ -projective.

*Proof.* Although the series of implications (2)  $\Rightarrow$  (4)  $\Rightarrow$  (1)  $\Rightarrow$  (7)  $\Rightarrow$  (6)  $\Rightarrow$  (5)  $\Rightarrow$  (3)  $\Rightarrow$  (2) are absolutely enough, we shall prove more relationships in order to demonstrate the abundance of methods and ideas.

And so, we start with:

"(1)  $\Rightarrow$  (7)". Suppose  $G/X$  is almost  $p^{\omega+n}$ -projective for some countable  $X \leq G$ . Let  $Y \leq G$  be maximal with respect to  $Y \cap X = \{0\}$ .

Clearly  $Y \cong (Y \oplus X)/X \subseteq G/X$ , so that Corollary 2.2 applies to get that  $Y$  is almost  $p^{\omega+n}$ -projective too.

On the other hand,  $X \cong (X \oplus Y)/Y$  where the latter is an essential subgroup of  $G/Y$ , and thus  $G/Y$  will be countable. In fact, for any  $Z \leq G$  with  $Z \neq Y$  we obtain by the modular law from [6] that  $[(X \oplus Y)/Y] \cap (Z/Y) = ((X \oplus Y) \cap Z)/Y = (Y + X \cap Z)/Y \neq \{0\}$  because  $X \cap Z \not\subseteq Y$  since  $X \cap Z \neq \{0\}$ . Thus point (7) follows.

“(1)  $\iff$  (4)”. Suppose first that  $G/M$  is almost  $\Sigma$ -cyclic, where  $M = X \oplus P$  for some countable subgroup  $X$  and  $p^n$ -bounded subgroup  $P$ . But  $G/M = G/(X \oplus P) \cong G/X/(X \oplus P)/X$ , and  $(X \oplus P)/X \cong P$  is  $p^n$ -bounded. Therefore,  $G/X$  is almost  $p^{\omega+n}$ -projective and (1) holds.

Conversely, let  $G/X$  be almost  $p^{\omega+n}$ -projective for some countable subgroup  $X$ . Hence there is a  $p^n$ -bounded subgroup  $M/X$  with  $M \leq G$  such that  $(G/X)/(M/X) \cong G/M$  is almost  $\Sigma$ -cyclic. Since  $p^n M \subseteq X$  is countable, we are done.

“(2)  $\iff$  (4)”. First, we note the following helpful fact: Letting  $A = K + S$ , where  $K$  is countable and  $S$  is almost  $\Sigma$ -cyclic, there exists a countable group  $C$  such that  $A/C$  is almost  $\Sigma$ -cyclic. Indeed,  $K \cap S$  being a countable subgroup of  $S$  forces that  $K \cap S \subseteq L$  where  $L$  is a countable nice subgroup of  $S$  whence by Lemma 2.5 we infer that  $S/L$  is almost  $\Sigma$ -cyclic. It therefore follows that  $A/L = [(K + L)/L] \oplus [S/L]$ . That is why,  $(A/L)/(K + L)/L \cong A/(K + L) \cong S/L$  is almost  $\Sigma$ -cyclic. Denoting  $C = K + L$ , we are done. Furthermore, applying the last observation to  $G/P$  we obtain that  $(G/P)/(M/P) \cong G/M$  is almost  $\Sigma$ -cyclic, where  $M/P$  is countable. Hence  $p^n M$  is countable, as stated.

Reciprocally, let us assume that  $G/M$  is almost  $\Sigma$ -cyclic with  $M = R \oplus P$  where  $R$  is countable and  $P$  is bounded by  $p^n$ . However,  $G/M \cong (G/P)/(M/P)$  is almost  $\Sigma$ -cyclic with countable  $M/P \cong R$ , so that Lemma 2.8 is applicable for  $G/P$  to finish the equivalence.

“(6)  $\iff$  (5)”. First, assume that  $G \cong A/H$  for some almost  $p^{\omega+n}$ -projective group  $A$  and its countable subgroup  $H$ . Utilizing Proposition 2.1, one may write that  $A = S/P$  where  $S$  is almost  $\Sigma$ -cyclic with  $p^n P = \{0\}$ , and  $H = N/P$  is countable with  $N \leq S$ . Furthermore,  $G \cong S/N$  and since  $N = P + C$  for some countable group  $C$ , one may derive that  $p^n N = p^n C$  is countable, as required.

Second, let us assume that  $G \cong S/N$  where  $S$  is almost  $\Sigma$ -cyclic and  $p^n N$  is countable. Since  $N$  is the direct sum of a countable group  $K$  and a  $p^n$ -bounded group  $B$ , say  $N = K \oplus B$ , one may deduce that  $G \cong$

$S/(K \oplus B) \cong (S/B)/(K \oplus B)/B$ . However, using Proposition 2.1,  $S/B$  is almost  $p^{\omega+n}$ -projective, whereas  $(K \oplus B)/B \cong K$  is countable. This ensures that (6) holds, thus completing the verification of the desired equivalence.

"(7)  $\Rightarrow$  (6)". Suppose that  $G/Y$  is countable for some almost  $p^{\omega+n}$ -projective subgroup  $Y$ . Let  $Z$  be a countable  $\Sigma$ -cyclic group and  $\phi : Z \rightarrow G$  be a homomorphism such that  $G = Y + \phi(Z)$ . If we set  $L = Y \oplus Z$ , then  $L$  is almost  $p^{\omega+n}$ -projective appealing to Proposition 2.18. If now we let  $id : Y \rightarrow G$  be the identity map, then we have a surjective homomorphism  $\psi : L \rightarrow G$ . If  $K$  is its kernel, then obviously  $K \cap Y = \{0\}$ ; in fact,  $x \in K \cap Y$  forces that  $\psi(x) = x = 0$ . Hence  $G \cong L/K$  and  $K$  is isomorphic to a subgroup of  $Z$ . Thus  $K$  is countable, and we are done.

"(5)  $\Rightarrow$  (3)". Write  $G \cong S/N$  where  $S$  is almost  $\Sigma$ -cyclic and  $N = R \oplus P$  where  $R$  is countable and  $P$  is bounded by  $p^n$ . Since  $G \cong (S/R)/(N/R)$  and  $N/R \cong P$  is  $p^n$ -bounded, we just take into account Corollary 2.9 to conclude that  $S/R$  is the sum of a countable group and an almost  $\Sigma$ -cyclic group, as desired.

"(3)  $\Rightarrow$  (2)". We may identify  $G$  with  $T/V$ , so that we write  $G = T/V$  where  $T = K + S$  with countable  $K$  and almost  $\Sigma$ -cyclic  $S$ . Since  $V \subseteq T[p^n]$ , set  $P = T[p^n]/V \subseteq G$ . Thus  $G/P \cong T/T[p^n] \cong p^n T = p^n K + p^n S$  remains again the sum of a countable group and an almost  $\Sigma$ -cyclic group employing the Balof-Keef's theorem from [1].

"(3)  $\Rightarrow$  (7)". Write  $G = T/V$ , where  $T = K + S$  and  $K$  is countable whereas  $S$  is almost  $\Sigma$ -cyclic. But  $T/V = (C + V)/V + (S + V)/V$ . Observing that  $(C + V)/V \cong C/(C \cap V)$  is countable and  $(S + V)/V \cong S/(S \cap V)$  is almost  $p^{\omega+n}$ -projective in conjunction with Proposition 2.1, we routinely see that  $G/Y$  is countable for  $Y = (S + V)/V$ , as needed.

"(2)  $\iff$  (7)". Suppose first that  $G/Y$  is countable for some almost  $p^{\omega+n}$ -projective group  $Y$ . So, we write  $G = Y + C$  for some countable subgroup  $C \leq G$ . Consequently,  $Y/P$  is almost  $\Sigma$ -cyclic for some  $P \leq Y[p^n]$  and thus  $G/P = [Y/P] + [(C + P)/P]$  where  $(C + P)/P \cong C/(C \cap P)$  is countable. Thus (2) is satisfied.

Conversely, let  $G/P = (A/P) + (Y/P)$  where the first summand is countable while the second is almost  $\Sigma$ -cyclic. So,  $G = A + Y$  and  $A = P + R$  where  $R$  is countable, which gives  $G = Y + R$ . But  $Y$  is almost  $\Sigma$ -cyclic and  $G/Y = (R + Y)/Y \cong R/(R \cap Y)$  is countable. Hence (7) is fulfilled.  $\square$

**THEOREM 2.22.** *Let  $G \leq A$  be groups such that  $A/G$  is countable. Then  $A$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective.*

*Proof.* The necessity follows directly with Corollary 2.3 in hand.

To attack the sufficiency, according to point (7) of Theorem 2.21, we write that  $G/Y$  is countable for some almost  $p^{\omega+n}$ -projective group  $Y$ . Therefore,  $A/G \cong (A/Y)/(G/Y)$  being countable implies that  $A/Y$  is countable and again point (7) of Theorem 2.21 yields that  $A$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective, as claimed.  $\square$

**THEOREM 2.23.** *Let  $K \leq G$  be a countable subgroup of  $G$ . Then  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if  $G/K$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective.*

*Proof.* For the "and only if" half, assume that  $G/C$  is almost  $p^{\omega+n}$ -projective for some countable subgroup  $C$ . Utilizing point (6) of Theorem 2.21, we have that  $(G/C)/(K+C)/C \cong G/(K+C) \cong (G/K)/(K+C)/K$  is almost  $p^{\omega+n}$ -projective because  $(K+C)/C \cong K/(K \cap C)$  is countable. Since  $(K+C)/K \cong C/(C \cap K)$  remains countable, an appeal to the "if" part proved below assures that  $G/K$  is almost  $p^{\omega+n}$ -projective, as asserted.

As for the "if" half, suppose  $G/K$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective for the countable subgroup  $K$ . Therefore there is a countable subgroup  $H/K$  of  $G/K$  with  $H \leq G$  such that  $(G/K)/(H/K) \cong G/H$  is almost  $p^{\omega+n}$ -projective. Since  $H$  remains countable,  $G$  must be almost  $\omega_1$ - $p^{\omega+n}$ -projective, as claimed.  $\square$

The main thesis is now the following:

**THEOREM 2.24.** *The class of almost  $\omega_1$ - $p^{\omega+n}$ -projectives is closed under taking of  $\omega_1$ -bijections, and is the minimal class containing almost  $p^{\omega+n}$ -projectives with that property.*

*Proof.* The first part follows by a combination of Theorems 2.22 and 2.23 along with ([11], Lemma 1.9).

As for the second half, we employ ([11], Proposition 1.10) together with Theorem 2.21.  $\square$

We now have the needed instruments in order to refine Definition 3 listed above in Section 2.

**THEOREM 2.25. (Main Criterion)** *The group  $G$  is almost  $\omega_1$ - $p^{\omega+n}$ -projective if and only if there exists a countable nice subgroup  $N$  which satisfies the inequalities  $p^{\omega+n}G \subseteq N \subseteq p^\omega G$  such that  $G/N$  is almost  $p^{\omega+n}$ -projective.*

*Proof.* The sufficiency being elementarily fulfilled, we will be dealing with the necessity. So, with point (3) of Theorem 2.21 at hand, we write  $G = T/V$  where  $T$  is the sum of a countable group  $C$  and an almost  $\Sigma$ -cyclic group  $S$ , say  $T = C + S$ , and  $V \leq T[p^n]$ . Setting  $N = (p^\omega T + V)/V$ , we observe that

$$G/N \cong T/(p^\omega T + V) \cong (T/p^\omega T)/(p^\omega T + V)/p^\omega T.$$

We shall prove now two things about  $T$ , that are,  $T/p^\omega T$  is almost  $\Sigma$ -cyclic, and  $p^\omega T$  is countable whence so is  $p^\omega T/(p^\omega T \cap V) \cong N$ . In fact, since  $C \cap S \subseteq S$  is countable, there is a countable nice subgroup  $K$  of  $S$  such that  $K \supseteq C \cap S$ . That is why,  $T/K = [(C + K)/K] \oplus [S/K]$ . However,  $(C + K)/K$  remains countable, whereas  $S/K$  remains separable. Consequently,  $p^\omega(T/K) = p^\omega[(C + K)/K]$  is countable and hence the same holds for its subgroup  $(p^\omega T + K)/K \cong p^\omega T/(K \cap p^\omega T)$ . But  $K \cap p^\omega T$  is countable. so that  $p^\omega T$  is countable, indeed.

As for the other claim we want to show, by Lemma 2.5 we have that  $(T/K)/(C + K)/K \cong T/(C + K) \cong S/K$  is almost  $\Sigma$ -cyclic. Since  $C + K$  remains countable, Lemma 2.6 applies to get that  $T$  is almost simply presented. Then  $T/p^\omega T$  is really almost  $\Sigma$ -cyclic referring to [10].

Furthermore, since  $(p^\omega T + V)/p^\omega T \cong V/(V \cap p^\omega T)$  is bounded by  $p^n$ , it now follows from Proposition 2.1 that  $G/N$  must be almost  $p^{\omega+n}$ -projective, as desired.

Finally, it is routinely seen that  $N \leq p^\omega G$  and that  $N \geq p^{\omega+n}G$  because  $(p^{\omega+n}G + N)/N \leq p^{\omega+n}(G/N) = \{0\}$ . It is next easily checked that such a subgroup  $N$  satisfying the above two inequalities should be nice in  $G$ , as wanted.  $\square$

### 3. Open Problems

In closing, we pose the following left-open questions:

**PROBLEM 1.** If  $G$  is a group with a countable subgroup  $H$  such that  $G/H$  is almost  $\Sigma$ -cyclic, is then  $G$  the direct sum of a countable group and an almost  $\Sigma$ -cyclic group?

If this is true, it will be a valuable generalization of the classical Charles lemma for extensions of countable subgroups by  $\Sigma$ -cyclic quotients.

**PROBLEM 2.** If  $G \leq A$  such that  $A/G$  is countable and  $G$  is almost  $\Sigma$ -cyclic, is then  $G$  the direct sum of a countable group and an almost  $\Sigma$ -cyclic group?

**PROBLEM 3.** Does it follow that a subgroup of the direct sum of an almost  $\Sigma$ -cyclic group and a countable group is again a direct sum of an almost  $\Sigma$ -cyclic group and a countable group?

Note that if Problem 1 holds in the affirmative, then a pure subgroup of the direct sum of a countable group and an almost  $\Sigma$ -cyclic group will also be a direct sum of a countable group and an almost  $\Sigma$ -cyclic group. Indeed, let  $T$  be a pure subgroup of a group  $A = C \oplus S$  where  $C$  is countable and  $S$  is almost  $\Sigma$ -cyclic. Consequently,  $p^\omega T \subseteq p^\omega A = p^\omega C$  is countable, and  $T/p^\omega T = T/(p^\omega A \cap T) \cong (T + p^\omega A)/p^\omega A \subseteq A/p^\omega A \cong (C/p^\omega C) \oplus S$  is almost  $\Sigma$ -cyclic. Thus  $T/p^\omega T$  is almost  $\Sigma$ -cyclic owing to the aforementioned Balof-Keef's result from [1].

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