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REVERSIBILITY OVER PRIME RADICALS

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Abstract. The studies of reversible and 2-primal rings have done important roles in noncommutative ring theory. We in this note introduce the concept of *quasi-reversible-over-prime-radical* (simply, *QRPR*) as a generalization of the 2-primal ring property. A ring is called *QRPR* if $ab = 0$ for $a, b \in R$ implies that ab is contained in the prime radical. In this note we study the structure of QRPR rings and examine the QRPR property of several kinds of ring extensions which have roles in noncommutative ring theory.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. $N_*(R)$, $N^*(R)$, and $N(R)$ (resp. $N_2(R)$) denote the lower nilradical (i.e., prime radical), the upper nilradical (i.e., sum of nil ideals), and the set of all nilpotent elements (resp. all nilpotent elements of index two) in R, respectively. Note $N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x)$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for

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the matrix with (i,j)-entry 1 and elsewhere 0. $\oplus$ is used to express direct sums.

According to Cohn [8], a ring R is called *reversible* if $ab=0$ implies $ba=0$ for $a,b\in R$. Anderson and Camillo [1], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible. Due to Bell [5], a ring R is called to satisfy the *Insertion-of-Factors-Property* if $ab=0$ implies $aRb=0$ for $a,b\in R$. Narbonne [18] and Shin [21] used the terms *semicommutative* and *SI* for the IFP, respectively. Here we choose "a semicommutative ring" among them, so as to cohere with other related references. A ring is usually called *reduced* if it has no nonzero nilpotent elements. Commutative rings clearly are semicommutative, and it is easily checked that any reduced ring is semicommutative. There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^l}$ for $n,l\geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is called *Abelian* if every idempotent is central. Semicommutative rings are Abelian through a simple computation.

A ring R is called 2-*primal* if $N_*(R)=N(R)$, following Birkenmeier, Heatherly, and E.K. Lee [6]. Note that a ring R is reduced if and only if R is both semiprime and 2-primal. Following the literature, a prime ideal P of a ring R is called *completely prime* if R/P is a domain. A ring R is 2-primal if and only if every minimal prime ideal of R is completely prime, by [21, Proposition 1.11]. Semicommutative rings are 2-primal through a simple computation, but the converse need not hold as can be seen by $U_2(D)$ over a 2-primal ring D, noting that $U_2(D)$ is 2-primal but non-Abelian.

Let R be a ring. According to Marks [17], R is called *NI* if $N^*(R)=N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. Following Rowen [20, Definition 2.6.5], an ideal P of R is called *strongly prime* if P is prime and R/P has no nonzero nil ideals. Maximal ideals are clearly strongly prime, but there exist many strongly prime ideals which are not maximal (e.g., the zero ideals of non-simple domains). An ideal P of R is called *minimal strongly prime* if P is minimal in the space of strongly prime ideals in R. $N^*(R)$ of R is the unique maximal nil ideal of R by [20, Proposition 2.6.2], and we have $N^*(R)=\{a\in R\mid RaR \text{ is a nil ideal of } R\}=\bigcap\{P\mid P \text{ is a strongly prime ideal of } R\}=\bigcap\{P\mid P \text{ is a minimal strongly prime ideal of } R\}$ by help of [20, Proposition 2.6.7]. 2-primal rings are clearly NI, but not conversely by Birkenmeier et al. [6, Example 3.3], [12, Example 1.2], or Marks [17, Example 2.2].

Due to Lambek [15], a ring R is called *symmetric* if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$; while, Anderson and Camillo [1] took the the term ZC_3 for this notion. Lambek proved that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with n any positive integer, implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ and $r_i \in R$ [15, Proposition 1]; while, Anderson and Camillo proved this result independently in [1, Theorem I.1]. Reduced rings are shown directly to be symmetric by the definition. We will use these facts freely in the remainder of this note.

Lemma 1.1. *For a ring R the following conditions are equivalent:*
(1) *R is 2-primal;*
(2) *$a^2 \in N_*(R)$ for $a \in R$ implies $a \in N_*(R)$;*
(3) *$abc \in N_*(R)$ for $a, b, c \in R$ implies $acb \in N_*(R)$;*
(4) *$ab \in N_*(R)$ for $a, b \in R$ implies $ba \in N_*(R)$;*
(5) *$ab \in N_*(R)$ for $a, b \in R$ implies $aRb \subseteq N_*(R)$;*
(6) *$r_1 r_2 \cdots r_n \in N_*(R)$ for $r_i \in R$ implies $Rr_{\sigma(1)} Rr_{\sigma(2)} R \cdots Rr_{\sigma(n)} R \subseteq N_*(R)$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 2$.*

Proof. The proof is obtained by the relations among the concepts above and the fact that $R/N_*(R)$ is reduced if and only if R is 2-primal. □

Lemma 1.2. *For a ring R the following conditions are equivalent:*
(1) *R is NI;*
(2) *$a^2 \in N^*(R)$ for $a \in R$ implies $a \in N^*(R)$;*
(3) *$abc \in N^*(R)$ for $a, b, c \in R$ implies $acb \in N^*(R)$;*
(4) *$ab \in N^*(R)$ for $a, b \in R$ implies $ba \in N^*(R)$;*
(5) *$ab \in N^*(R)$ for $a, b \in R$ implies $aRb \subseteq N^*(R)$;*
(6) *$R/N^*(R)$ is 2-primal;*
(7) *$r_1 r_2 \cdots r_n \in N^*(R)$ for $r_i \in R$ implies $Rr_{\sigma(1)} Rr_{\sigma(2)} R \cdots Rr_{\sigma(n)} R \subseteq N^*(R)$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 2$.*

Proof. The proof is obtained by the relations among the concepts above, using the facts that $R/N^*(R)$ is reduced if and only if R is NI and that $R/N^*(R)$ being 2-primal means $N(R) = N^*(R)$. □

We start our study by the following induced from Lemma 1.1.

Definition 1.3. A ring is called *quasi-reversible-over-prime-radical* (simply, *QRPR*) if $ab = 0$ for $a, b \in R$ implies $ba \in N_*(R)$.

The following is an immediate consequence of the definition.

Lemma 1.4. *For a ring R the following conditions are equivalent:*
(1) *R is QRPR;*
(2) *$ab = 0$ for $a, b \in R$ implies $baR \subseteq N_*(R)$;*
(3) *$ab = 0$ for $a, b \in R$ implies $Rba \subseteq N_*(R)$;*
(4) *$ab = 0$ for $a, b \in R$ implies $RbaR \subseteq N_*(R)$.*

We will use Lemma 1.4 freely. 2-primal rings are QRPR by Lemma 1.1. But the converse need not hold by the following.

Example 1.5. We use the ring and argument in [10, Example 1]. Let F be a field and $A = F\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over F. Let $R = A/(x^2)^2$, where (x^2) is the ideal of A generated by x^2. Then $N_*(R) = Rx^2R = N_2(R)$ and $N(R) = xRx + Rx^2R + Fx$, where x and y are identified with $x + (x^2)^2$ and $y + (x^2)^2$, respectively. Then $N_*(R) \neq N(R)$, entailing R is not 2-primal. If $ab = 0$ for $a, b \in R$, then $(ba)^2 = 0$. This yields $ba \in N_2(R) = N_*(R)$ by the computation in [7, Example 2.2]. Thus R is QRPR.

Following Liang et al. [16], a ring R is called *weakly semicommutative* if $ab = 0$ implies $aRb \subseteq N(R)$ for $a, b \in R$. This notion is a proper generalization of semicommutative rings as can be seen by $U_2(R)$ over a semicommutative ring R. To see this, let $A = \begin{pmatrix} a_1 & c_1 \\ 0 & b_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & c_2 \\ 0 & b_2 \end{pmatrix} \in U_2(R)$, over a semicommutative ring R, such that $AB = 0$. Then $a_1a_2 = 0, b_1b_2 = 0$, and since R is semicommutative, we have $a_1Ra_2 = 0, b_1Rb_2 = 0$. This yields $AU_2(R)B \subseteq \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = N(U_2(R))$, and so $U_2(R)$ is weakly semicommutative. But $U_2(R)$ is non-Abelian (hence not semicommutative).

Proposition 1.6. *QRPR rings are weakly semicommutative.*

Proof. Let R be a QRPR ring and suppose that $ab = 0$ for $a, b \in R$. Then for $r \in R$, we have

$$(arbarba)(barb) = 0.$$

Since R is QRPR, we also get

$$(barb)(arbarba) \in N_*(R) \text{ and } (barb)(arbarba)r \in N_*(R).$$

This yields

$$(arb)^4 = ((arbarba)r)\,(barb) \in N_*(R) \subseteq N(R),$$

entailing $arb \in N(R)$. Thus R is weakly semicommutative. $\square$

NI rings are weakly semicommutative by Lemma 1.2, so one may conjecture that NI rings may be QRPR. But we have a negative answer by the following example, entailing that the converse of Proposition 1.6 need not be true.

Example 1.7. We use the ring and argument in [12, Example 1.2]. Let S be a 2-primal ring, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over S, i.e., $R_n = U_{2^n}(S)$. Each R_n is a 2-primal (hence NI) ring by [6, Proposition 2.5]. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of D. Note $R = \cup_{i=1}^{\infty} R_n$. Then R is an NI (but not 2-primal) ring with $N_*(R) = 0$, by the argument in [12, Example 1.2]. Since R is NI, R is weakly semicommutative by lemma 1.2.

We next show that R is not QRPR. To see that, let $a = e_{23}$ and $b = e_{12}$ in R. Then $e_{23}, e_{12} \in R_k$ for some $k \geq 1$. We get $ab = 0$, but $ba = e_{13} \notin N_*(R) = 0$. So R is not QRPR.

The following provides a method by which we can examine the QRPR property of given rings.

Theorem 1.8. *Let R be a ring and I be a proper ideal of R such that R/I is QRPR. If I is 2-primal as a ring without identity then R is QRPR.*

Proof. Let $ab = 0$ for $a, b \in R$. Then bIa is a nil subset of I, and $\bar{b}\bar{a} \in N_*(R/I)$ since R is QRPR.

Assume that I is 2-primal as a ring without identity. Then $I/N_*(I)$ is a reduced ring (i.e., $N(I) = N_*(I)$), entailing that $N(I) = N_*(I)$ is a nil ideal of R by help of Andrunakievic [9, Lemma 61] (i.e., $(RN_*(I)R)^3 \subseteq$

$IRN_*(I)RI = IN_*(I)I \subseteq N_*(I)$). This yields $bIa \subseteq N(I) = N_*(I) \subseteq N_*(R)$, so

$$(baI)(baI)\cdots(baI) = b(aIbaI\cdots b)aI \subseteq bIaI \subseteq N_*(I).$$

Since $I/N_*(I)$ is a reduced ring, we get $baI \subseteq N_*(I) \subseteq N_*(R)$. Then $baI \subseteq P$ for any minimal prime ideal P of R. But P is prime, so $ba \in P$ or $I \subseteq P$. Here assume $ba \notin P$. Then $I \subseteq P$, and so $\bar{b}\bar{a} \in N_*(R/I) \subseteq P/I$. This yields $ba \in P$, a contradiction. Consequently $ba \in P$, entailing $ba \in N_*(R)$. This concludes that R is QRPR. □

As an application of Theorem 1.8, consider $E = U_n(R)$ for $n \geq 2$ over a 2-primal ring R. Then

$$N(E) = \{(a_{ij}) \in U_n(R) \mid a_{ii} \in N_*(R) \text{ for all } i\} = N_*(E),$$

entailing $\frac{E}{N_*(E)} \cong \underbrace{\frac{R}{N_*(R)} \oplus \cdots \oplus \frac{R}{N_*(R)}}_{n-\text{times}}$. Since $\frac{R}{N_*(R)}$ is a reduced ring, $\frac{E}{N_*(E)}$ is QRPR by Proposition 1.9 to follow. But $N_*(E)$ is 2-primal, so E is QRPR by Theorem 1.8, letting $I = N_*(E)$. Theorem 1.8 is also applicable to the case of setting $I = \{(a_{ij}) \in U_n(R) \mid a_{ii} = 0 \text{ for all } i \in N_*(R)\} = N_*(E)$, noting that $\frac{R}{I} \cong \underbrace{R \oplus \cdots \oplus R}_{n-\text{times}}$.

Then one may ask whether R is QRPR if I is NI in Theorem 1.8. However the answer is negative by Example 1.7. Let R be the ring in Example 1.7, $N^*(R)$ is not 2-primal by the argument in Example 1.7. But $R/N^*(R)$ is 2-primal (hence QRPR).

Proposition 1.9. *The class of QRPR rings is closed under subrings and direct sums.*

Proof. Let R be a QRPR ring and S be a subring of R. Take $ab = 0$, for $a, b \in S$. Then $ba \in N_*(R)$. Since $N_*(R) \cap S \subseteq N_*(S)$, we have $ba \in N_*(S)$.

Suppose that R_i is a QRPR ring for each i in a nonempty index set I, and let D be the direct sum of R_i's. Let $a = (a_i), b = (b_i) \in D$ with $ab = 0$. Then $a_ib_i = 0$ for each $i \in I$ and so $b_ia_i \in N_*(R_i)$. Notice that $N_*(\oplus_{i\in I}R_i) = \oplus_{i\in I}N_*(R_i)$. So we have $ba \in N_*(\oplus_{i\in I}R_i)$, entailing D is QRPR. □

As in the proof of Proposition 1.9, $N_*(R) \cap S \subseteq N_*(S)$ holds for any ring R. But the converse inclusion need not hold as can be seen by the ring R in Example 1.7. In fact, consider the subring $R_1 = U_2(A)$ over a 2-primal ring A. Then $N_*(R_1) = \begin{pmatrix} N_*(A) & A \\ 0 & N_*(A) \end{pmatrix} \neq 0$ and $N_*(R) = 0$.

Proposition 1.10. *A ring R is QRPR if and only if $U_n(R)$ is a QRPR ring for $n \geq 2$.*

Proof. It suffices to establish necessity by Proposition 1.9 since R is a subring of $U_n(R)$. Let R be a QRPR ring, and suppose $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in U_n(R)$. Then $a_{ii}b_{ii} = 0$ for all $i \in \{1, \cdots, n\}$. Since R is QRPR, we have $b_{ii}a_{ii} \in N_*(R)$. Notice that

$$N_*(U_n(R)) = \begin{pmatrix} N_*(R) & R & \cdots & R \\ 0 & N_*(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & N_*(R) \end{pmatrix}.$$

Thus $BA \in N_*(U_n(R))$. □

From [11], given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$, and the usual matrix operations are used. Propositions 1.9 and 1.10 provide the following.

Corollary 1.11. *A ring R is QRPR if and only if the trivial extension $T(R, R)$ is QRPR.*

One may suspect that if a ring R is QRPR, then $Mat_n(R)$ is QRPR (or weakly semicommutative) for $n \geq 2$. But the following example shows that $Mat_n(R)$ cannot be weakly semicommutative (hence cannot be QRPR).

Example 1.12. Let R be any ring and consider $Mat_2(R)$. We first have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$, but

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin N(Mat_2(R)).$$

So $Mat_2(R)$ is not weakly semicommutative.

For the general case, let $A = e_{ij}, B = e_{k1} + \cdots + e_{kn} \in Mat_n(R)$ with $j \neq k$. Then $AB = 0$ but

$$A(e_{j1} + \cdots + e_{jk} + \cdots + e_{jn})B = e_{ik}B = e_{i1} + \cdots + e_{in} \notin N(Mat_n(R)).$$

Following Rege and Chhawchharia [19, Definition 1.1], a ring R is called *Armendariz* if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, we have $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Every reduced ring is Armendariz by [4, Lemma 1]. Armendariz rings are Abelian by the proof of [2, Theorem 6] (or [13, Lemma 7]).

The concepts of Armendariz and QRPR are independent of each other by the following.

Example 1.13. (1) Let F be a field and $A = F\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over F. Let $R = A/(b^2)$, where (b^2) is the ideal of A. Then R is Armendariz by [3, Theorem 4.7]. But R is not QRPR as can be seen by the computation that $\bar{b}(\bar{b}\bar{a}) = 0$ and $(\bar{b}\bar{a})\bar{b}\bar{a} \notin N(R)$.

(2) $U_2(R)$ is a QRPR ring by Proposition 1.10, but this ring is non-Abelian. So $U_2(R)$ is not Armendariz since Armendariz rings are Abelian.

Lemma 1.14. [2, Proposition 1] Suppose that R is an Armendariz ring. If $f_1, \cdots, f_n$ are polynomials in $R[x]$ such that $f_1 \cdots f_n = 0$, then $a_1 \cdots a_n = 0$ where a_i is a coefficient of f_i.

Proposition 1.15. *Let R be an Armendariz ring. If R is QRPR, then $R[x]$ is QRPR.*

Proof. Let R be a QRPR ring. Assume $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $fg = 0$. Then since R is Armendariz, $a_i b_j = 0$ for all i and j. But since R is QRPR, we have $b_j a_i \in N_*(R)$ for all i, j. We already have $N_*(R)[x] = N_*(R[x])$ by [14, Theorem 10.19] for any ring R. Thus we get $gf \in N_*(R[x])$ from the fact that $b_j a_i \in N_*(R)$ for all i, j. This implies that $R[x]$ is QRPR. $\square$

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