Korean J. Math. **21** (2013), No. 1, pp. 35–39
http://dx.doi.org/10.11568/kjm.2013.21.1.35

CONTINUITY OF THE SPECTRUM ON $(\text{class}\mathcal{A})^*$

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Abstract. Let $(\text{class}\mathcal{A})^*$ denotes the class of operators satisfying $|T^2| \geq |T^*|^2$. In this paper, we show that the spectrum is continuous on $(\text{class}\mathcal{A})^*$.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denotes the algebra of bounded linear operators on a complex infinite dimensional Hilbert space $\mathscr{H}$. Recall [1] that $T \in \mathscr{L}(\mathscr{H})$ is called *hyponormal* if $T^*T \geq TT^*$, and T is called $*$*-paranormal* if

$$||T^2x|| \geq ||T^*x||^2$$

for all unit vector $x \in \mathscr{H}$. Recently, B. P. Duggal, I. H. Jeon, and I. H. Kim [7] consider a following class of operators; we say that an operator $T \in \mathscr{L}(\mathscr{H})$ belongs to $(\text{class}\mathcal{A})^*$ if

$$|T^2| \geq |T^*|^2.$$

For brevity, we shall denote classes of hyponormal operators, $*$-paranormal operators, and $(\text{class}\mathcal{A})^*$ operators by $\mathcal{H}$,$\mathcal{PN}^*$, and $(\text{class}\mathcal{A})^*$, respectively. From [7] it is well known that

$$\mathcal{H} \subset (\text{class}\mathcal{A})^* \subset \mathcal{PN}^*. \tag{1.1}$$

Received December 20, 2012. Revised January 22, 2013. Accepted January 25, 2013.

2010 Mathematics Subject Classification: 47B20.

Key words and phrases: $*$-class A operator, spectral continuity.

This paper was supported by Research Fund, Kumoh National Institute of Technology.

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Let $\mathcal{K}$ denote the set of all compact subsets of the complex plane $\mathbb{C}$. Equipping $\mathcal{K}$ with the Hausdorff metric, one may consider the spectrum σ as a function $\sigma : \mathscr{L}(\mathscr{H}) \to \mathcal{K}$ mapping operators $T \in \mathscr{L}(\mathscr{H})$ into their spectrum $\sigma(T)$. It is known that the function σ is upper semi-continuous, but has points of discontinuity [8, p.56]. Studies identifying sets $\mathcal{C}$ of operators for which σ becomes continuous when restricted to $\mathcal{C}$ has been carried out by a number authors (see, for example, [3, 4, 5, 6, 9]).

Given an operator $T \in \mathscr{L}(\mathscr{H})$, let $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathscr{H} \setminus T\mathscr{H})$. T is upper semi-Fredholm if $T\mathscr{H}$ is closed and $\alpha(T) < \infty$, and then the index of T, $\operatorname{ind}(T)$, is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. T is said to be Fredholm if $T\mathscr{H}$ is closed and the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite.

Let $T^\circ \in \mathscr{L}(\mathscr{K})$ denote the Berberian extension of an operator $T \in \mathscr{L}(\mathscr{H})$. Then the Berberian extension theorem [2] says that given an operator $T \in \mathscr{L}(\mathscr{H})$ there exists a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and an isometric $*$-isomorphism $T \to T^\circ \in \mathscr{L}(\mathscr{K})$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_p(T^\circ) = \sigma_a(T^\circ) = \sigma_a(T)$. Here σ_p and σ_a denote, respectively, the point spectrum and the approximate point spectrum. In the following, we shall denote the set of accumulation points (resp. isolated points) of $\sigma(T)$ by $\operatorname{acc}\sigma(T)$(resp. $\operatorname{iso}\sigma(T)$).

The aim of this paper is to give a proof of the following theorem.

Theorem 1.1. *The spectrum σ is continuous on* (class$\mathcal{A}$)*.

To prove the theorem we adopt the Berberian technique used in [6] and we, in a sense, try to approach in a little different way.

2. Proof of Theorem 1.1

Since the function σ is upper semi-continuous [8], if $\{A_n\} \subset \mathscr{L}(\mathscr{H})$ is a sequence which converges in the operator norm topology to $A \in \mathscr{L}(\mathscr{H})$ then

$$\limsup_n \sigma(A_n) \subseteq \sigma(A). \tag{2.1}$$

Thus to prove the theorem it would suffice to prove that if $\{A_n\} \subset$ (class$\mathcal{A}$)* is a sequence of operators such that $\lim_{n\to\infty} ||A_n - A|| = 0$ for some operator $A \in (\text{class}\mathcal{A})^*$, then

$$\sigma(A) \subseteq \liminf_n \sigma(A_n). \tag{2.2}$$

We first consider the following lemma, which actually is proved in [9, Lemma 2], but for the completeness we give a proof.

Lemma 2.1. *Let* $\{A_n\} \subset \mathscr{L}(\mathscr{H})$ *be a sequence which converges in the operator norm topology to* $A \in \mathscr{L}(\mathscr{H})$. *Then*

$$\sigma_a(A) \subseteq \liminf_n \sigma(A_n) \Rightarrow \sigma(A) \subseteq \liminf_n \sigma(A_n). \tag{2.3}$$

Proof. Suppose that $\lambda \notin \liminf_n \sigma(A_n)$. Then there exists a $\delta > 0$, a neighbourhood $\mathcal{N}_\delta(\lambda)$ of λ and a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\sigma(A_{n_k}) \cap \mathcal{N}_\delta(\lambda) = \emptyset$ for every $k \geq 1$. This implies that $A_{n_k} - \mu$ is Fredholm and $\text{ind}(A_{n_k} - \mu) = 0$ for every $\mu \in \mathcal{N}_\delta(\lambda)$. Since $\lambda \notin \sigma_a(A)$ by the assumption, then $A - \lambda$ is left invertible, hence upper semi-Fredholm with $\alpha(A - \lambda) = 0$. Then

$$||(A_{n_k} - \lambda) - (A - \lambda)|| \to 0 \text{ as } \text{n} \to 0$$

and the continuity of the index implies that $\text{ind}(A - \lambda) = 0$, and so $A - \lambda$ is Weyl. Since $\alpha(A - \lambda) = 0$, it follows that $\lambda \notin \sigma(A)$. □

It is well known that, from an argument of Newburgh [10, Lemma 3],

$$\lambda \in \text{iso}\sigma(A) \;\Rightarrow\; \lambda \in \liminf_n \sigma(A_n). \tag{2.4}$$

Indeed, if $\lambda \in \text{iso}\sigma(A)$, then for every neighbourhood $\mathcal{N}(\lambda)$ of λ there exists a positive integer N such that $\sigma(A_n) \cap \mathcal{N}(\lambda) \neq \emptyset$ for all $n > N$.

Now, we consider corresponding the Berberian extensions to A and the sequence $\{A_n\}$ as mensioned above, and then have that

$$\sigma(A) = \sigma(A^\circ), \sigma(A_n) = \sigma(A_n^\circ) \text{ and } \sigma_a(A) = \sigma_a(A^\circ) = \sigma_p(A^\circ).$$

Since if $T \in (\text{class}\mathcal{A})^*$ then $T^\circ \in (\text{class}\mathcal{A})^*$, we have that

$$\sigma(A) \subseteq \liminf_n \sigma(A_n) \Longleftrightarrow \sigma(A^\circ) \subseteq \liminf_n \sigma(A_n^\circ). \tag{2.5}$$

To complete the proof of the theorem we show the following lemma in the view of Lemma 2.1.

Lemma 2.2. *Let* $\{A_n\} \subset (\text{class}\mathcal{A})^*$ *be a sequence which converges in the operator norm topology to* $A \in (\text{class}\mathcal{A})^*$. *Then*

$$\sigma_a(A^\circ) \subseteq \liminf_n \sigma(A_n^\circ). \tag{2.6}$$

Proof. If $\lambda \in \sigma_a(A^\circ) = \sigma_p(A^\circ)$, then $(A^\circ - \lambda)^{-1}(0)$ is a reducing subspace of A° [7, Lemma 2.2], and so we have a representation of A°,

$$A^\circ = \lambda \oplus B \quad \text{on} \quad \mathscr{K} = (\mathrm{A}^\circ - \lambda)^{-1}(0) \oplus \{(\mathrm{A}^\circ - \lambda)^{-1}(0)\}^\perp$$

Evidently, $B - \lambda$ is upper semi-Fredholm and $\alpha(B - \lambda) = 0$. There exists an $\epsilon > 0$ such that $B - (\lambda - \mu_o)$ is upper semi-Fredholm with $\mathrm{ind}(B - (\lambda - \mu_o)) = \mathrm{ind}(B - \lambda)$ and $\alpha(B - (\lambda - \mu_o)) = 0$ for every μ_o satisfying $0 < |\mu_o| < \epsilon$. Choose $0 < \epsilon < \delta$ and set $\mu = \lambda - \mu_o$ $(0 < |\mu_o| < \epsilon)$. (Here $\delta > 0$ as in proof of Lemma 2.1) Then $B - \mu$ is upper semi–Fredholm, $\mathrm{ind}(B - \mu) = \mathrm{ind}(B - \lambda)$ and $\alpha(B - \mu) = 0$. This implies that

$$A^\circ - \mu = \lambda - \mu \oplus B - \mu$$

is upper semi-Fredholm ,

$$\mathrm{ind}(A^\circ - \mu) = \mathrm{ind}(B - \mu) \text{ and } \alpha(A^\circ - \mu) = 0.$$

Assume to the contrary that $\lambda \notin \liminf_n \sigma(A_n^\circ)$, then evidently, $A_{n_k}^\circ - \mu$ is Fredholm, with $\mathrm{ind}(A_{n_k}^\circ - \mu) = 0$, and

$$\lim_{n \to \infty} ||(A_{n_k}^\circ - \mu) - (A^\circ - \mu)|| = 0.$$

It follows from the continuity of the index that $\mathrm{ind}(A^\circ - \mu) = 0$ and $A^\circ - \mu$ is Fredholm. Since $\alpha(A^\circ - \mu) = 0$, $\mu \notin \sigma(A^\circ)$ for every μ in a deleted ϵ-neighbourhood of λ. This contradicts to (2.4). Hence we must have that $\lambda \in \liminf_n \sigma(A_n^\circ)$. □

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