

AN APPLICATION OF LINKING THEOREM TO FOURTH ORDER ELLIPTIC BOUNDARY VALUE PROBLEM WITH FULLY NONLINEAR TERM

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ABSTRACT. We show the existence of nontrivial solutions for some fourth order elliptic boundary value problem with fully nonlinear term. We obtain this result by approaching the variational method and using a linking theorem. We also get a uniqueness result.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $c \in R^1$ and $g : \bar{\Omega} \times R \rightarrow R$ be a C^1 function.

In this paper we investigate the existence of the nontrivial solutions for the following fourth order elliptic problem with Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u - bu^+ &= f(x, u) && \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 &&& \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

where $f(x, s) = |s|^{p-2}s^+ - |s|^{q-2}s^-$ with $p, q > 2$ and $p \neq q$.

Jung and Choi [6] investigated the number of the weak solutions for the following fourth order elliptic problem with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = g(x, u) \quad \text{in } \Omega, \quad (1.2)$$

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$$u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega.$$

They assumed that $g \in C^1(\bar{\Omega} \times R, R)$ satisfies the following:

- (g1) $g \in C^1(\bar{\Omega} \times R, R)$,
- (g2) $g(x, 0) = 0, g(x, \xi) = o(|\xi|)$ uniformly with respect to $x \in \bar{\Omega}$,
- (g3) there exists $C > 0$ such that $|g(x, \xi)| < C \forall (x, \xi) \in \bar{\Omega} \times R$.

Liu [9] investigated the existence of nontrivial solutions for the the semilinear beam equation

$$(1) \quad \begin{aligned} u_{tt} + u_{xxxx} + bu^+ &= f(x, t, u) && \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ u(\pm\frac{\pi}{2}, t) &= u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u &\text{ is } \pi\text{-periodic in } t \text{ and even in } x \text{ and } t, \end{aligned}$$

where $u^+ = \max\{u, 0\}$, the nonlinear term is a functions with different powers:

$$f(x, t, s) = \begin{cases} s^2, & s \geq 0 \\ s^3, & s \leq 0. \end{cases}$$

The eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

has infinitely many eigenvalues $\lambda_j, j \geq 1$ which is repeated as often as its multiplicity, and the corresponding eigenfunctions $\phi_j, j \geq 1$ suitably normalized with respect to $L^2(\Omega)$ inner product. The eigenvalue problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= \Lambda u && \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has also infinitely many eigenvalues $\Lambda_j = \lambda_j(\lambda_j - c), j \geq 1$ and corresponding eigenfunctions $\phi_j, j \geq 1$. We note that $\Lambda_1, \Lambda_2, \dots, \Lambda_j$ are negative and

$$0 < \Lambda_{j+1} \leq \Lambda_{j+2} \leq \dots \leq \Lambda_k \leq \dots, \quad \Lambda_k \rightarrow +\infty,$$

where we assume that $c \in R^1$ satisfies $\lambda_j < c < \lambda_{j+1}$.

Jung and Choi [5] proved that (1.1) has at least one nontrivial solution when $c < \lambda_1$ and g satisfies the condition (g1), (g2) and additional conditions

- (g3)' there exists $\xi \geq 0$ such that $p(x, \xi) \leq 0 \forall x \in \bar{\Omega}$,
- (g4)' there exist a constant $r > 0$ and an element $e \in H$ such that $\|e\| = r, e < \xi$ and $\frac{1}{2}r^2 - \int_{\Omega} P(x, e) < 0$,

by reducing the problem (1.1) to the problem with bounded nonlinear term and then applying the maximum principle for the elliptic operator $-\Delta$ and $-\Delta - c$ two times and the mountain pass theorem in the critical point theory. Jung and Choi [3] showed the existence of at least two solutions, one of which is bounded solution and large norm solution of (1.1) when $g(u)$ is polynomial growth or exponential growth nonlinear term. The authors proved these results by the variational method and the mountain pass theorem. For the constant coefficient semilinear case Choi and Jung [2] showed that the problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= bu^+ + s && \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

has at least two nontrivial solutions when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$ or when $\lambda_1 < c < \lambda_2$, $b < \Lambda_1$ and $s > 0$. The authors obtained these results by use of the variational reduction method. The authors [5] also proved that when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$, (1.2) has at least three nontrivial solutions by use of the degree theory. Tarantello [11] also studied the problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= b((u + 1)^+ - 1) && \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

She showed that if $c < \lambda_1$ and $b \geq \Lambda_1$, then (1.4) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [9] also proved that if $c < \lambda_1$ and $b \geq \Lambda_2$, then (1.4) has at least three solutions by the variational linking theorem and Leray-Schauder degree theory.

In this paper we are trying to find weak solutions of (1.1), that is,

$$\int_{\Omega} [\Delta^2 u \cdot v + c\Delta u \cdot v - bu^+v - f(x, u)v]dx = 0, \quad \forall v \in H,$$

where H is introduced in section 2.

We consider the associated functional of (1.1)

$$I(u) = \int_{\Omega} [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - F(x, u)]dx, \tag{1.4}$$

where $F(x, s) = \int_0^s f(x, \tau)d\tau$. By (g1), I is well defined.

Our main result is the following.

THEOREM 1.1. *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. If $\Lambda_i^- \leq -b$ then problem (1.1) has at least one nontrivial solution.*

We prove Theorem 1.1 by approaching the variational method and using a linking theorem for the reduced fourth order elliptic problem with bounded nonlinear term. The outline of the proof of Theorem 1.1 is as follows: In section 2, we prove the functional $I(u) \in C^1$ and the functional I satisfies the Palais Smale condition. In section 3, we prove the uniqueness result for problem (1.1). In section 4, we show the existence of nontrivial solutions for some fourth order elliptic boundary value problem with fully nonlinear term.

2. Variational approach

Let $L^2(\Omega)$ be a square integrable function space defined on Ω . Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) \mid \sum |\Lambda_k| h_k^2 < \infty\}. \quad (2.1)$$

Then this is a complete normed space with a norm

$$\|u\| = \left[\sum |\Lambda_k| h_k^2 \right]^{\frac{1}{2}}.$$

Since $\lambda_k \rightarrow +\infty$ and c is fixed, we have $\Lambda_k \rightarrow \infty$ and

- (i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
 - (ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
 - (iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
- which is proved in [1].

Suppose that $\lambda_j < c < \lambda_{j+1}$. We denote by $(\Lambda_i^-)_{i \geq 1}$ the sequence of the negative eigenvalues of $\Delta^2 + c\Delta$, by $(\Lambda_i^+)_{i \geq 1}$ the sequence of the positive ones, so that

$$\cdots < \Lambda_1^- < 0 < \Lambda_1^+ = \lambda_{j+1}(\lambda_{j+1} - c) < \Lambda_2^+ = \lambda_{j+2}(\lambda_{j+2} - c) < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H_+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H_- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

Then $H = H_- \oplus H_+$, for $u \in H$, $u = u^- + u^+ \in H_- \oplus H_+$. Let P_+ be the orthogonal projection from H onto H_+ and P_- be the orthogonal

projection from H onto H_- . We can write $P_+u = u^+$, $P_-u = u^-$, for $u \in H$.

By the following Lemma 2.1, the weak solutions of (1.1) coincide with the critical points of the associated functional $I(u)$.

LEMMA 2.1. *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$, and g satisfies the conditions (g1) – (g3). Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$I'(u)h = \int_{\Omega} [\Delta u \cdot \Delta h - c \nabla u \cdot \nabla h - g(x, u)h] dx.$$

If we set

$$F(u) = \frac{1}{2} \int_{\Omega} G(x, u) dx,$$

then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact and

$$F'(u)h = \int_{\Omega} g(x, u)h dx \quad \text{for all } h \in H,$$

this implies that $I \in C^1(H, R)$ and $F(u)$ is weakly continuous.

The proof of Lemma 2.1 has the similar process to that of the proof in Appendix B in [10].

Now we shall show that $I(u)$ satisfies Palais-Smale condition.

LEMMA 2.2. *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$, and g satisfies the conditions (g1) – (g3). Then the functional I satisfies Palais-Smale condition: Any sequence (u_m) in H for which $|I(u_m)| \leq M$ and $I'(u_m) \rightarrow 0$ as $m \rightarrow \infty$ possesses a convergent subsequence.*

Proof. Let us choose $u \in H$. By $g \in C^1$ and (g1), $G(x, u)$ is bounded. Then we have

$$\begin{aligned} I(u) &= \int_{\Omega} \left[\frac{1}{2} |\Delta u|^2 - \frac{c}{2} |\nabla u|^2 - G(x, u) \right] dx \\ &\geq \frac{1}{2} \{ \lambda_1 (\lambda_1 - c) \} \|u\|_{L^2(\Omega)}^2 - \int_{\Omega} G(x, u) dx. \end{aligned}$$

Since u is bounded and $\int_{\Omega} G(x, u) dx$ is bounded, $I(u)$ is bounded from below. Thus I satisfies the (PS) condition. \square

3. Uniqueness

The following theorem is the uniqueness result for problem (1.1).

LEMMA 3.1. *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. Let $b < \Lambda_1 < 0$ and*

$$f(x, s) = \begin{cases} 0, & s \geq 0 \\ |s|^{q-2}s, & s \leq 0. \end{cases}$$

Then problem (1.1) has only trivial solution.

Proof. Let $Lu = \Delta^2 u + c\Delta u$ and we rewrite (1.1) as

$$\begin{aligned} Lu - \Lambda_1 u &= f(x, u) - \Lambda_1 u + bu^+ \\ &= (u^-)^{q-2}u - \Lambda_1 u + bu^+ \\ &= (u^-)^{q-2}u - (\Lambda_1 - b)u^+ + \Lambda_1 u^-. \end{aligned}$$

Multiplying across by $-\phi_1$ and integrating over Ω ,

$$\begin{aligned} 0 &= \langle [L - \Lambda_1]u, -\phi_1 \rangle \\ &= - \int_{\Omega} [(u^-)^{q-2}u - (\Lambda_1 - b)u^+ + \Lambda_1 u^-] \phi_1 dx \geq 0. \end{aligned}$$

Since the condition $b < \Lambda_1$ imply that $-(\Lambda_1 - b)u^+ \leq 0$, $(u^-)^{q-1}u \leq 0$, and $\Lambda_1 u^- \leq 0$ for all real valued function u and $\phi_1(x) > 0$ for all $x \in \Omega$. Therefore the only possibility to hold (1.1) is that $u \equiv 0$. \square

In this section, we suppose $b < 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (1). Here we suppose that f is defined by equation (2).

In the following, we consider the following sequence of subspaces of $L^2(\mathbb{R}^N)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_{Λ} is the eigenspace associated to Λ .

LEMMA 3.2. *The functional I_b satisfies $(P.S.)_{\gamma}^*$ condition, with respect to (H_n) , for all γ .*

Proof. Let (k_n) be any sequence in N with $k_n \rightarrow \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n , $I_b(u_n) \rightarrow \gamma$ and $D(I_b)|_{H_{k_n}}(u_n) \rightarrow 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \rightarrow \infty$ and set $\hat{u}_n = u_n/t_n$. Up to a subsequence $\hat{u}_n \rightharpoonup \hat{u}$ in H for some \hat{u} in H . Moreover

$$\begin{aligned} 0 &< \langle D(I_b)_{H_{k_n}}(u_n), \hat{u}_n \rangle - \frac{2}{t_n} I_b(u_n) \\ &= \frac{2}{t_n} \int_{\Omega} F(u_n) dx - \frac{1}{t_n} \int_{\Omega} f(u_n) u_n dx \\ &= \int_{\Omega} -\frac{p-2}{p} (t_n)^{p-1} [(A\hat{u}_n)^+]^p + \frac{q+2}{q} (t_n)^{q-1} [(\hat{u}_n)^-]^q dx. \end{aligned}$$

Since $t_n \rightarrow \infty$, $(A\hat{u}_n)^+ \rightarrow 0$ and $(A\hat{u}_n)^- \rightarrow 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightarrow u$ for some $u \in H$. We know that

$$D(I_b)_{H_{k_n}}(u_n) = P^+u_n - P^-u_n + b(u_n)^+ - f(u_n).$$

Hence $P^+u_n - P^-u_n$ converges strongly, hence $u_n \rightarrow u$ strongly and $DI_b(u) = 0$. □

4. An application of linking theory

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the main result via a linking argument.

First of all, we introduce a suitable splitting of the space H . Let

$$Z_1 = \bigoplus_{j=i+1}^{\infty} H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \bigoplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+$$

LEMMA 4.1. *There exists R such that $\sup_{v \in Z_1 \oplus Z_2, \|v\|=R} I_b(v) < 0$.*

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2} \int_{\Omega} |[v]^+|^2 dx - \int_{\Omega} F(v) dx.$$

Since

$$\frac{b}{2} \| [Sv]^+ \|^2 - \int_{\Omega} F(Sv) dx = \int_{\Omega} \frac{b}{2} ([Sv]^+)^2 - \frac{1}{p} ([Sv]^+)^p - \frac{1}{q} ([Sv]^-)^q dx,$$

there exists R such that $\frac{b}{2}\|[Sv]^+\|^2 - \int_{\Omega} F(Sv)dx \leq 0$ for all $\|v\| = R$. Hence, for $v \in Z_1 \oplus Z_2$, $\|v\| = R$

$$I_b(v) \leq -\frac{1}{2}\|v\|^2 < 0.$$

□

LEMMA 4.2. *There exists ρ such that $\inf_{u \in Z_2 \oplus Z_3, \|u\|=\rho} I_b(u) > 0$.*

Proof. Let $\sigma \in [0, 1]$. We consider the functional $I_{b,\sigma} : Z_2 \oplus Z_3 \rightarrow R$ defined by

$$I_{b,\sigma}(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2} \int_{\Omega} |[v]^+|^2 dx - \sigma \int_{\Omega} F(u)dx.$$

We claim that there exists a ball $B_{\rho} = \{u \in Z_2 \oplus Z_3 \mid \|u\| < \rho\}$ such that

1. $I_{b,\sigma}$ are continuous with respect to σ ,
2. $I_{b,\sigma}$ satisfies (P.S) condition,
3. 0 is a minimum for $I_{b,0}$ in B_{ρ} ,
4. 0 is the unique critical point of $I_{b,\sigma}$ in B_{ρ} .

Then by a continuation argument of Li-Szulkin's (see[7]), it can be shown that 0 is a local minimum for $I_b|_{Z_2 \oplus Z_3} = I_{b,1}$ and Lemma is proved.

The continuity in σ and the fact that 0 is a local minimum for $I_{b,0}$ are straightforward. To prove (P.S.) condition one can argue as in the previous Lemma, when dealing with I_b .

To prove that 0 is isolated we argue by contradiction and suppose that there exists a sequence (σ_n) in $[0, 1]$ and sequence (u_n) in $Z_2 \oplus Z_3$ such that $DI_{b,\sigma_n}(u_n) = 0$ for all n , $u_n \neq 0$, and $u_n \rightarrow 0$. Set $t_n = \|u_n\|$ and $\hat{u}_n = u_n/t_n$ then $t_n \rightarrow 0$. Let $\hat{v}_n = P^-\hat{u}_n$ and $\hat{w}_n = P^+\hat{u}_n$. Since \hat{v}_n varies in a finite dimensional space, we can suppose that $\hat{v}_n \rightarrow \hat{v}$ for some \hat{v} . We get

$$(2) \quad \frac{1}{t_n} DI_{b,\sigma}(u_n) = \hat{w}_n - \hat{v}_n + \frac{b}{t_n}(u_n)^+ - \frac{\sigma_n}{t_n} f(u_n) = 0.$$

Multiplying by \hat{w}_n yields

$$\|\hat{w}_n\|^2 = \frac{\sigma_n}{t_n} \int_{\Omega} f(u_n) \hat{w}_n dx - \frac{b}{t_n} \int_{\Omega} (u_n)^+ \hat{w}_n dx.$$

We know that

$$\begin{aligned} \int_{\Omega} (u_n)^+ \hat{w}_n dx &= \int_{\Omega} P^+(u_n)^+ \hat{u}_n dx \\ &= \int_{\Omega} P^+(u_n)^+ (\hat{u}_n)^+ dx. \end{aligned}$$

Since $b > 0$, there exists a sequence (ϵ_n) such that $\epsilon_n \rightarrow 0$ and $0 < \epsilon_n < b$ for all n . That is

$$\frac{b}{t_n} \int_{\Omega} (u_n)^+ \hat{w}_n dx \geq \frac{\epsilon_n}{t_n} \int_{\Omega} P^+(u_n)^+ (\hat{u}_n)^+ dx.$$

Then

$$\begin{aligned} \|\hat{w}_n\|^2 &\leq \frac{1}{t_n} \int_{\Omega} f(u_n) \hat{w}_n dx - \frac{\epsilon_n}{t_n} \int_{\Omega} P^+(u_n)^+ (\hat{u}_n)^+ dx \\ &\leq \int_{\Omega} \frac{|f(u_n)|}{t_n} |\hat{w}_n| dx + \epsilon_n \int_{\Omega} |P^+(\hat{u}_n)^+| |(\hat{u}_n)^+| dx. \end{aligned}$$

Hence

$$\begin{aligned} |f(u_n)| &= |\{([t_n \hat{u}_n]^+)^{p-1} - ([t_n \hat{u}_n]^-)^{q-1}\}| \\ &\leq t_n^{p-1} |[\hat{u}_n]^+|^{p-1} + t_n^{q-1} |[\hat{u}_n]^-|^{q-1} \\ &\leq t_n^m (M_1 + t_n^{M-m} M_2) \end{aligned}$$

for some M_1 and M_2 where $m = \min\{p - 1, q - 1\}$ and $M = \max\{p - 1, q - 1\}$. We get that

$$\int_{\Omega} \frac{|f(u_n)|}{t_n} |\hat{w}_n| dx \leq t_n^m (M_1 + t_n^{M-m} M_2) \int_{\Omega} |\hat{w}_n| dx \leq o(1).$$

Hence

$$(3) \quad \|\hat{w}_n\|^2 \leq o(1) + \epsilon_n \int_{\Omega} |P^+(\hat{u}_n)^+| |(\hat{u}_n)^+| dx.$$

Since $\int_{\Omega} |P^+(\hat{u}_n)^+| |(\hat{u}_n)^+| dx$ is bounded and equation (7) holds for every ϵ_n , $\hat{w}_n \rightarrow 0$ and so (\hat{u}_n) converges. Since $|f(u_n)| \leq t_n^m (M_1 + t_n^{M-m} M_2)$, we get

$$\frac{\sigma_n}{t_n} |f(u_n)| \leq \frac{1}{t_n} |f(u_n)| \leq t_n^{m-1} (M_1 + t_n^{M-m} M_2) \leq o(1).$$

Then $\frac{\sigma_n}{t_n} f(u_n) \rightarrow 0$. From equation (6), (\hat{v}_n) converges to zero, but this is impossible if $\|(\hat{u}_n)\| = 1$. □

DEFINITION 4.3. Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:

$$\begin{aligned} B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\ S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\ \Delta_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\ \Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}. \end{aligned}$$

THEOREM 4.1. If $\Lambda_i^- \leq -b$ then problem (1.1) has at least one non-trivial solution.

Proof. Let $e \in Z_2$. By Lemma 4.1 and Lemma 4.2, for a suitable large R and a suitable small ρ , we have the linking inequality

$$(4) \quad \sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)).$$

Moreover $(P.S.)_\gamma^*$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \leq I_b(u) \leq \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$. \square

References

- [1] Q. H. Choi and T. Jung, *Multiplicity of solutions and source terms in a fourth order nonlinear elliptic equation*, Acta Math. Sci. **19** (4) (1999), 361–374.
- [2] Q. H. Choi and T. Jung, *Multiplicity results on nonlinear biharmonic operator*, Rocky Mountain J. Math. **29** (1) (1999), 141–164.
- [3] T. Jung and Q. H. Choi, *Nonlinear biharmonic problem with variable coefficient exponential growth term*, Korean J. Math. **18** (3) (2010), 1–12.
- [4] T. Jung and Q. H. Choi, *Multiplicity results on a nonlinear biharmonic equation*, Nonlinear Anal. **30** (8) (1997), 5083–5092.
- [5] T. Jung and Q. H. Choi, *Nontrivial solution for the biharmonic boundary value problem with some nonlinear term*, Korean J. Math, to be appeared (2013).
- [6] T. Jung and Q. H. Choi, *Fourth order elliptic boundary value problem with nonlinear term decaying at the origin*, J. Inequalities and Applications, **2013** (2013), 1–8.
- [7] S. Li and A. Squilkin, *Periodic solutions of an asymptotically linear wave equation*. Nonlinear Anal. **1** (1993), 211–230.
- [8] J.Q. Liu, *Free vibrations for an asymmetric beam equation*, Nonlinear Anal. **51** (2002), 487–497.
- [9] A. M. Micheletti and A. Pistoia, *Multiplicity results for a fourth-order semilinear elliptic problem*, Nonlinear Anal. TMA, **31** (7) (1998), 895–908.

- [10] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS. Regional conf. Ser. Math., **65**, Amer. Math. Soc., Providence, Rhode Island (1986).
- [11] Tarantello, *A note on a semilinear elliptic problem*, Diff. Integ.Equat. **5** (3) (1992), 561-565.

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