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ON NILPOTENT POWER SERIES WITH NILPOTENT COEFFICIENTS

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Abstract. Antoine studied conditions which are connected to the question of Amitsur of whether or not a polynomial ring over a nil ring is nil, introducing the notion of nil-Armendariz rings. Hizem extended the nil-Armendariz property for polynomial rings onto power-series rings, say *nil power-serieswise rings.* In this paper, we introduce the notion of power-serieswise CN rings that is a generalization of nil power-serieswise Armendariz rings. Finally, we study the nil-Armendariz property for Ore extensions and skew power series rings.

1. Introduction

Throughout this note every ring is associative with identity unless otherwise stated. The polynomial ring and the (formal) power series ring with an indeterminate x over a ring R (possibly without identity) are denoted by $R[x]$ and $R[[x]]$, respectively. Let $N^*(R)$ and $N(R)$ denote the upper nilradical (i.e., sum of nil ideals), and the set of all nilpotent elements in R, respectively. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial or a power series $f(x)$. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Z}_n$ denotes the ring of integers modulo n. Denote the n by n full

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(resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

A ring is called *reduced* if it has no nonzero nilpotent elements. Armendariz [3, Lemma 1] proved that for a reduced ring R,

$$ab = 0 \text{ for all } a \in C_{f(x)} \text{ and } b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0$$

where $f(x)$, $g(x)$ are in $R[x]$. From this result, Rege and Chhawchharia [19] called a ring (not necessarily reduced) *Armendariz* if it satisfies this condition. So reduced rings are clearly Armendariz. Armendariz rings are Abelian (i.e., its idempotents are central) by the proof of [1, Theorem 6].

Brewer et al. [5, Lemma 1], and Gilmer et al. [6, Proposition 3.5] extended the study of zero divisors in polynomial rings to power series rings when base rings are commutative reduced. While, Anderson and Camillo discussed the same argument on noncommutative reduced rings in the statements before [1, Example 10]. Kim et al. [12] called a ring R *power-serieswise Armendariz* (we will use *ps-Armendariz* for simplicity) if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two power series $f(x), g(x)$ in $R[[x]]$ satisfy $f(x)g(x) = 0$. Ps-Armendariz rings are Armendariz by the definition, but the converse need not hold by [12, Example 2.1]. Reduced rings are ps-Armendariz by [12, Lemma 2.3], but there exist many kinds of non-reduced ps-Armendariz rings as we see in [12, Section 3]. It is obvious that the class of (ps-)Armendariz rings is closed under subrings and direct products. Due to Bell [4], a ring R is called *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is obtained through simple computations that reduced rings are IFP and ps-Armendariz rings are IFP by [12, Lemma 2.3(2)].

On the other hand, Antoine [2] called a ring R *nil-Armendariz* if $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in N(R)[x]$. Armendariz rings are nil-Armendariz, but the converse need not hold [2, Proposition 2.7 and Example 4.9]. Marks [17] called a ring R *NI* if $N^*(R) = N(R)$. Note that a ring R is NI if and only if $N(R)$ forms a two-sided ideal if and only if $R/N^*(R)$ is reduced. NI rings are nil-Armendariz but not conversely by [2, Proposition 2.1, and Example 4.8].

Recently, the concept of the nil-Armendariz ring property for polynomial rings is extended to power series rings by S. Hizem. A ring R is called *nil power-serieswise Armendariz* (we will use *nil-ps-Armendariz*)

[8, Definition 3] if for any $f(x), g(x) \in R[[x]]$ such that $f(x)g(x) \in N(R)[[x]]$ then $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Note that a ring R is nil-ps-Armendariz if and only if R is NI [8, Theorem 1], and hence nil-ps-Armendariz rings are nil-Armendariz. All IFP rings are both NI and Abelian by simple computations. Since ps-Armendariz rings are IFP by [12, Lemma 2.3], ps-Armendariz rings are nil-ps-Armendariz. However, the concepts of Armendariz rings and nil-ps-Armendariz rings are independent of each other by [2, Example 4.8] and [11, Proposition 4.1].

Antoine showed that if R is an Armendariz ring then $N(R)[x] \subseteq N(R[x])$, in [2, Lemma 2.6]. So one may conjecture that if R is an Armendariz ring then $N(R)[[x]] \subseteq N(R[[x]])$. But the following example erases the possibility.

Example 1.1. Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_{2^{i+1}}$ be the direct product of $\mathbb{Z}_{2^j}$'s for $j \geq 2$. Every $\mathbb{Z}_{2^{i+1}}$ is ps-Armendariz by [12, Proposition 3.2], and so R is also ps-Armendariz. Consider $f(x) = \sum_{i=0}^{\infty}(a(i)_j)x^i \in R[[x]]$ where $a(i)_j = 2$ for $j = i+1$ and $a(i)_j = 0$ for $j \neq i+1$. Then for any $k \geq 1$,

$$\begin{aligned} f(x)^k &= (a(k-1)_j)^k x^{k(k-1)} + (a(k)_j)^k x^{k^2} + \cdots \\ &= (0, \ldots, 0, 2^k, 0, \ldots)x^{k(k-1)} + \cdots \neq 0 \end{aligned}$$

This implies $f(x) \notin N(R[[x]])$, but $(a(i)_j)^{i+2} = 0$ for all $i \in \{0, 1, 2, \ldots\}$.

In the following example, the ring R, in [2, Example 4.8], is (nil-)Armendariz but not NI and so R is not nil-ps-Armendariz. However, we here prefer a computation to be concerned with the product of power series.

Example 1.2. ([2, Example 4.8]) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be an ideal of A generated by a^2, and $R = A/I$. Identify a, b with their images in R for simplicity. Then R is (nil-)Armendariz.

Consider two power series $f(x) = a - abx$ and $g(x) = \sum_{i=0}^{\infty} b^i x^i ab$ in $R[[x]]$. Then

$$f(x)g(x) = a(1 - bx)(1 + bx + b^2x^2 + \cdots + b^n x^n + \cdots)ab = a^2 b = 0,$$

but $abab$ is non-nilpotent, entailing that R is not NI.

For a ring R and $n \geq 2$, let $D_n(R)$ be the ring of all matrices in $U_n(R)$ whose diagonal entries are all equal, and $V_n(R)$ be the ring of all matrices

(a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$.

Note that for a ring R and $n \geq 2$, R is nil ps-Armendariz if and only if $D_n(R)$ is a nil-ps-Armendariz ring if and only if $V_n(R)$ is a nil-ps-Armendariz ring, with help of [11, Proposition 2.4(2) and Proposition 4.1(1)]. Moreover, R is a nil-ps-Armendariz ring if and only if the factor ring $R[x]/\langle x^n \rangle$ is nil-ps-Armendariz, where $\langle x^n \rangle$ is a two-sided ideal of $R[x]$ generated by x^n, since $V_n(R) \cong R[x]/\langle x^n \rangle$ by [15]. However $Mat_n(R)$ $(n \geq 2)$ is not a nil-ps-Armendariz ring over any ring R. Here observe the following product of power series: For the matrix units e_{ij}',

$$e_{11}(1-(e_{11}+e_{12})x)(1+(e_{11}+e_{12})x+(e_{11}+e_{12})^2x^2+\cdots)(e_{21}+e_{22})$$
$$= e_{11}(e_{21}+e_{22}) = 0.$$

But $e_{11}(e_{11}+e_{12})(e_{21}+e_{22}) = e_{11}+e_{12}$ is non-nilpotent, showing that $Mat_n(R)$ is not nil-ps-Armendariz for $n \geq 2$.

2. Properties of power-serieswise CN rings

According to [14, Definition 2], a nilpotent polynomial $f(x)$ over a ring R is called *CN* if every coefficient of $f(x)$ is nilpotent. A ring R is called *CN* if $N(R[x]) \subseteq N(R)[x]$ (i.e., every nilpotent polynomial over R is CN). Every nil Armendariz ring is CN, but not conversely by [14, Example 1]. Similarly, if a ring R is nil-ps-Armendariz then $N(R[[x]]) \subseteq N(R)[[x]]$ by [8, Lemma 2], but the converse does not hold by [8, Remark 3]. Hence, we define a new class of rings.

Definition 2.1. A ring R is called *power-serieswise CN* (simply, *ps-CN*) if $N(R[[x]]) \subseteq N(R)[[x]]$, equivalently, whenever any power series $f(x) \in N(R[[x]])$, $a \in N(R)$ for all $a \in C_{f(x)}$.

The following diagram shows all implications among the concepts (with no other implications holding, except by transitivity):

$$\begin{array}{ccccc}
\text{IFP} & \Longrightarrow & \text{NI} & & \\
\Uparrow & & \Updownarrow & & \\
\text{Ps-Armendariz} & \Longrightarrow & \text{Nil-ps-Armendariz} & \Longrightarrow & \text{Ps-CN} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Armendariz} & \Longrightarrow & \text{Nil-Armendariz} & \Longrightarrow & \text{CN}
\end{array}$$

Remark 2.2. (1) The class of ps-CN rings is closed under subrings obviously.

(2) For any ring R and $n \geq 2$, $Mat_n(R)$ is not a ps-CN ring with help of [14, Example 12].

(3) The class of ps-CN rings is not closed under homomorphic images. For example, $R/qR \cong Mat_2(\mathbb{Z}_q)$ by the argument in [7, Exercise 2A], where R is the ring of quaternions with integer coefficients and q is any odd prime integer.

A ring R is said to be *of bounded index* (*of nilpotency*) if there exists a positive integer n such that $a^n = 0$ for all $a \in N(R)$.

Lemma 2.3. (1) *For any family $\{R_\gamma \mid \gamma \in \Gamma\}$ of rings, suppose that the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ (resp., $\oplus_{\gamma\in\Gamma} R_\gamma$) is of bounded index. Then R_γ is a ps-CN ring for all $\gamma \in \Gamma$ if and only if R is ps-CN.*

(2) *For a central idempotent e in a ring R of bounded index, a ring R is ps-CN if and only if eR and $(1-e)R$ are ps-CN rings.*

(3) *If R is a ps-CN ring and I is a nilpotent ideal of R, then R/I is a ps-CN ring.*

Proof. (1) We show the necessity for the direct product. Let k be the bounded index of R. Then R_γ is also of bounded index $\leq k$ for each $\gamma \in \Gamma$. Assume that every R_γ is ps-CN. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in N(R[[x]])$ where $a_i = (a_{i_\gamma})_{\gamma\in\Gamma} \in R$. Let $f_\gamma(x) = \sum_{i=0}^{\infty} a_{i_\gamma} x^i \in R_\gamma[[x]]$ for each $\gamma \in \Gamma$. Then $f_\gamma(x) \in N(R_\gamma[[x]])$ from $f(x) \in N(R[[x]])$. Since every R_γ is ps-CN and has the bounded index $\leq k$, we obtain $a_{i_\gamma}^k = 0$ for all i and so $a_i^k = 0$, entailing $a_i \in N(R)$. Therefore R is ps-CN.

The converse comes from Remark 2.2(1).

(2) It comes from Remark 2.2(1) and (1), since $R \cong eR \oplus (1-e)R$.

(3) We apply the method in the proof of [14, Proposition 8(1)]. Let $\bar{R} = R/I$ and $f(x) \in R[[x]]$. Suppose that R is ps-CN. If $\bar{f}(x) \in N(\bar{R}[[x]]]$, then $f(x)^n \in I[[x]]$ for some $n \geq 2$, and so $f(x) \in N(R[[x]])$ since I is nilpotent. Since R is ps-CN, $a \in N(R)$ for any $a \in C_{f(x)}$, and hence $\bar{a} \in N(\bar{R})$ for any $\bar{a} \in C_{\bar{f}(x)}$, proving that R/I is ps-CN. □ □

The n by n lower triangular matrix ring over R is denoted by $L_n(R)$. For any set M of matrices over a ring R, M^T denotes the set of all transposes of matrices in M.

Theorem 2.4. *For a ring R of bounded index and $n \geq 2$, the following are equivalent:*

(1) *R is a ps-CN ring;*
(2) *$U_n(R)$ is a ps-CN ring;*
(3) *$D_n(R)$ is a ps-CN ring;*
(4) *$V_n(R)$ is a ps-CN ring;*
(5) *$R[x]/\langle x^n\rangle$ is a ps-CN ring;*
(6) *$L_n(R)$ is a ps-CN ring;*
(7) *$D_n(R)^T$ is a ps-CN ring;*
(8) *$V_n(R)^T$ is a ps-CN ring.*

Proof. (1)⇔(2) Assume (1) and let $n \geq 2$. For a nilpotent ideal $I = \{A \in U_n(R) \mid \text{each diagonal entry of } A \text{ is zero}\}$ of $U_n(R)$, $U_n(R)/I \cong \oplus_{i=1}^n R_i$ where $R_i = R$ is ps-CN by Lemma 2.3(1). Hence $U_n(R)$ is also a ps-CN ring by Lemma 2.3(3). The converse follows from Remark 2.2(1).

(2)⇒(3)⇒(4)⇒(1) are obvious as subrings.

(4)⇔(5) follows from $V_n(R) \cong R[x]/\langle x^n\rangle$ by [15].

Similarly, (1)⇔(6)⇒(7)⇒(8)⇒(1) can be shown. □

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 2.5. *Let R be a ring of bounded index. Then R is ps-CN if and only if the trivial extension $T(R, R)$ is ps-CN.*

By the same arguments as in Theorem 2.4, we have the following.

Proposition 2.6. *Let R and S be rings of bounded indexes. For a bimodule ${}_RM_S$* (resp., ${}_SM_R$), $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ (resp., $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$) *is a ps-CN ring if and only if R and S are ps-CN.*

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Note that NI rings (i.e., nil-ps-Armendariz rings) are directly finite by [11, Proposition 2.7(1)], and moreover ps-CN rings are directly finite, since CN rings are directly finite by [14, Proposition 15].

3. Extensions of power-serieswise CN rings

THEOREM 3.1. *Assume that $N(R)[[x]] \subseteq N(R[[x]])$ for a ring R. A ring R is ps-CN if and only if $R[x]$ is a ps-CN ring.*

Proof. It is enough to show the necessity. Note that $N(R)[[x]] \subseteq N(R[[x]])$ implies that R is of bounded index by [8, Proposition 2]. Assume that R is ps-CN. Let $f(t) = \sum_{i=0}^{\infty} f_i t^i \in R[x][[t]]$ where $f_i \in R[x]$. Suppose that $f(t) \in N(R[x][[t]])$ and let $k_n = \deg f_0 + \cdots + \deg f_n + 1$, where the degree is considered as polynomials in x and the degree of the zero polynomial is taken to be zero. Let $g(x) = f_0 + f_1 x^{k_1} + f_2 x^{k_2} + \cdots$. Since $f(t) \in N(R[x][[t]])$, we have $g(x) \in N(R[[x]])$, and moreover, the set of coefficients of f_i's equals to the set of coefficients of $g(x)$. By assumption, $N(R[[x]]) = N(R)[[x]]$. Thus all coefficients of f_i's are in $N(R)$, leading to $f_i \in N(R)[x] \subseteq N(R)[[x]] = N(R[[x]])$ for all i. This implies $f_i \in N(R[x])$ and therefore $f(t) \in N(R[x])[[t]]$, completing the proof. □

COROLLARY 3.2. (1) *If a ring R with $N(R)[[x]] = N(R[[x]])$, then both R and $R[[x]]$ are ps-CN rings.*

(2) *If R is a nil-ps-Armendariz ring (i.e., an NI ring) with $N(R)[[x]] \subseteq N(R[[x]])$, then both R and $R[[x]]$ are ps-CN rings.*

(3) *Let $N(R[[x]])$ be a subring of $R[[x]]$ for a ring R. Then R is ps-CN if and only if $R[[x]]$ is a ps-CN ring.*

Proof. (1) follows immediately from Theorem 3.1.

(2) If R is a nil-ps-Armendariz ring with $N(R)[[x]] \subseteq N(R[[x]])$, then actually $N(R[[x]]) = N(R)[[x]]$ by [8, Lemma 2].

(3) Let $N(R[[x]])$ be a subring of $R[[x]]$. For any $a \in N(R)$ and nonnegative integer t, ax^t is nilpotent. Thus $ax^t \in N(R[[x]])$, and so $N(R)[[x]] \subseteq N(R[[x]])$. Thus, R is ps-CN if and only if $R[[x]]$ is ps-CN by Theorem 3.1. □

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* is defined similarly, and *regular* means both left and right regular (hence not a zero divisor).

PROPOSITION 3.3. *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is a ps-CN ring if and only if $\Delta^{-1}R$ is a ps-CN ring.*

Proof. It is enough to show the necessity. Assume that R is ps-CN. Let $F(x) \in N((\Delta^{-1}R)[[x]])$ for $F(x) = \sum_{i=0}^{\infty} u^{-1}a_i x^i \in (\Delta^{-1}R)[[x]]$ where $a_i \in R$ with regular $u \in R$. Then there exists a positive integer k such that $(F(x))^k = 0$. Since Δ is contained in the center of R, we have $0 = (F(x))^k = (a_0 + a_1 x + \cdots)^k (u^k)^{-1}$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then $f(x) \in N(R[[x]])$. Since R is ps-CN, $a \in N(R)$ for any $a \in C_{f(x)}$ and hence $au^{-1} \in N(\Delta^{-1}R)$, proving that $\Delta^{-1}R$ is ps-CN. $\square$

The ring of *Laurent series* in x, coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{\infty} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are integers. This ring is usually written by $R[[x; x^{-1}]]$.

Corollary 3.4. *For a ring R, $R[[x]]$ is ps-CN if and only if $R[[x; x^{-1}]]$ is ps-CN.*

Proof. It follows directly from Proposition 3.3.
For, let $\Delta = \{1, x, x^2, \ldots\}$, then clearly Δ is a multiplicatively closed subset of $R[[x]]$ and $R[[x; x^{-1}]] = \Delta^{-1}R[[x]]$. $\square$

Notice that ps-CN rings need not be Abelian as can be seen by the ps-CN ring $U_2(R)$ where R is a ps-CN ring of bounded index by Theorem 2.4. There exists an Abelian ring which is not ps-CN by [14, Example 18].

Recall that R is a *weakly IFP* ring [16, Definition 2.1] if $ab = 0$ for any $a, b \in R$ implies $aRb \subseteq N(R)$. Clearly every IFP ring is weakly IFP, but not conversely by [16, Example 2.1]. NI rings are weakly IFP by [8, Lemma 3]. Similarly, $U_2(D)$ over a division ring D is weakly IFP by [16, Example 2.2], but it is neither IFP nor Abelian, while the following example shows that Abelian rings need not be weakly IFP.

Example 3.5. We consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_2(\mathbb{Z}) \mid a \equiv d, b \equiv 0 \text{ and } c \equiv 0 \pmod 2 \right\}.$$

Then the only idempotents of R are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so R is an Abelian ring.

Note that R is not weakly IFP. In fact,

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \notin N(R)),$$

entailing that R is not weakly IFP.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$.

Observe that the ring R of Example 3.5 is not regular. Indeed, for a nonzero element $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of R, we have $\begin{pmatrix} 2a & 0 \\ 2c & 0 \end{pmatrix} \in uR$. Thus uR cannot be generated by an idempotent element, showing that R is not regular. However, we have the following.

Theorem 3.6. *Given a regular ring R, the following conditions are equivalent:*

(1) *R is reduced;*
(2) *R is Armendariz;*
(3) *R is nil-Armendariz;*
(4) *R is CN;*
(5) *R is Abelian;*
(6) *R is NI;*
(7) *R is IFP;*
(8) *R is weakly IFP;*
(9) *R is ps-Armendariz;*
(10) *R is nil-ps-Armendariz; and*
(11) *R is ps-CN.*

Proof. The equivalence from (1) to (6) as seen in [14, Theorem 19]. (1)⇒(9) By [12, Lemma 2.3]. The implications (9)⇒(10)⇒(11)⇒(4), (1)⇒(7)⇒(6) and (7)⇒(8) are obvious.

(8)⇒(5) Let R be weakly IFP. Assume on the contrary that there exist $r, e = e^2 \in R$ with $re \neq er$. Let $a = ere - re$. Then $ea = 0$. Since R is regular there exists $b \in R$ such that $ae = aebae$. From $ea = 0$, we have $eba \in N(R)$ since R is weakly IFP. Thus $(eba)^n = 0$ for some positive integer n. But we obtain $eba = 0$ by the facts $ae = a$ and $ea = 0$. Hence $0 = eba = aebae = ae$, entailing $ere = re$. Now let $c = ere - er$, then $ce = 0$. Since R is regular, there exists $d \in R$ such that $ec = ecdec$. From $ce = 0$, we have $cde = 0$ by the similar argument as above, using $ec = c$. Then $0 = cde = ecdec = ec$ and so $ere = er$. Consequently, $re = ere = er$, a contradiction. □

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are clearly π-regular. However the preceding results need not hold on π-regular rings. $U_n(D)$ ($n \geq 2$ and D is a division ring) is ps-CN by Theorem 2.4 and π-regular through a simple computation, but it is neither regular nor Abelian.

Finally, we study the nil-Armendariz property of the Ore extension type and the skew power series ring type.

Recall that for an endomorphism σ of a ring R, the additive map $\delta : R \to R$ is called a *σ-derivation* if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b) \text{ for any } a, b \in R.$$

For a ring R with an endomorphism σ of R and a σ-derivation δ, the *Ore extension* $R[x; \sigma, \delta]$ of R is the ring obtained by giving the polynomial ring over R with new multiplication

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$. If $\delta = 0$, we write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and it is called an *Ore extension of endomorphism type*. The ring $R[[x; \sigma]]$ is called a *skew power series ring*.

According to Krempa [13], an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. Hong et al. [9] called R a *σ-rigid ring* if there exists a rigid endomorphism σ of R. Clearly, the endomorphism σ of a σ-rigid ring is a monomorphism.

For a σ-ideal I (i.e., $\sigma(I) \subseteq I$) of a ring R, I is called a *σ-rigid ideal* of R if $a\sigma(a) \in I$ for $a \in R$ implies $a \in I$ [10]. Obviously, R is a σ-rigid ring if and only if the zero ideal of R is a σ-rigid ideal. If R is a σ-rigid ring, then $N^*(R)$ is clearly a σ-rigid ideal, but the converse does not hold in [10].

Following [18], for integers i, j with $0 \leq i \leq j$, let $f_i^j \in End(R, +)$ be the map which is the sum of all possible words in σ, δ built with i letters σ and $j - i$ letters δ. For example, $f_0^0 = 1, f_j^j = \sigma^i, f_0^j = \delta^j$ and

$$f_{j-1}^j = \sigma^{j-1}\delta + \sigma^{j-2}\delta\sigma + \cdots + \delta\sigma^{j-1}.$$

Lemma 3.7. *For a ring R, suppose that $N^*(R)$ is a σ-rigid δ-ideal of R. Then we get the following.*

(1) *R is NI.*

(2) *$ab \in N(R)$ implies $af_i^j(b) \in N(R)$ for all $j \geq i \geq 0$ and $a, b \in R$.*

Proof. (1) and (2) come from [10, Corollary 2.3(2) and Proposition 2.4], respectively. □

A (σ, δ)-ideal means that it is both a σ-ideal and a δ-ideal. For a (σ, δ)-ideal I of a ring R, we will call I a *(σ, δ)-rigid* ideal of R if $af_i^1(a) \in I$ for $1 \geq i \geq 0$ and $a \in R$ implies $a \in I$. Clearly, a (σ, δ)-rigid ideal is a σ-rigid δ-ideal.

Let $C_{p(x)}$ also denote the set of all coefficients of $p(x)$ for $p(x) \in R[x; \sigma, \delta]$ (or $R[[x; \sigma]]$). We call a ring R *nil-Armendariz of the Ore extension type* (resp., *nil-Armendariz of the skew power series ring type*) if $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x; \sigma, \delta]$ (resp., $R[[x; \sigma]]$) satisfy $f(x)g(x) \in N(R)[x; \sigma, \delta]$ (resp., $N(R)[[x; \sigma]]$).

Proposition 3.8. *For a ring R, we have the following.*

(1) *Assume that $N^*(R)$ is a (σ, δ)-ideal of R. Then $N^*(R)$ is a (σ, δ)-rigid ideal of R if and only if R is NI and R is nil-Armendariz of the Ore extension type.*

(2) *Assume that $N^*(R)$ is a σ-ideal of R. Then $N^*(R)$ is a σ-rigid ideal of R if and only if R is NI and R is nil-Armendariz of the skew power series ring type.*

Proof. (1) If $N^*(R)$ is a σ-rigid ideal, then we directly have that R is NI and R is nil-Armendariz of the Ore extension type by [10, Corollary 2.3 and Theorem 2.5]. Conversely, assume that R is NI and R is nil-Armendariz of the Ore extension type. Then we get $ab \in N(R)$ for any $a \in C_{p(x)}$ and $b \in C_{q(x)}$ whenever $p(x)q(x) \in N(R)[x; \sigma, \delta]$ for $p(x), q(x) \in R[x; \sigma, \delta]$ and $N^*(R) = N(R)$. Let $af_i^1(a) \in N^*(R)$ for $1 \geq i \geq 0$ and $a \in R$. For $p(x) = ax, q(x) = a \in R[x; \sigma, \delta]$, we have $p(x)q(x) = af_1^1(a)x + af_0^1(a) \in N(R)[x; \sigma, \delta]$. By assumption, $a^2 \in N(R)$ and so $a \in N(R)$. Thus $N(R) = N^*(R)$ is a (σ, δ)-rigid ideal.

(2) is the similar argument to the proof of (1), with help of [10, Proposition 2.7]. □

Example 3.9. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring (and so an NI ring) and $N^*(R) = 0 = N(R)$. Define $\sigma : R \to$

R by $\sigma(a,b) = (b,a)$. Note that $N^*(R)$ is not a σ-rigid ideal, since $(1,0)\sigma(1,0) \in N(R)$ but $(1,0) \notin N(R)$. Moreover, for $p(x) = (1,0)x$ and $q(x) = (1,0) \in R[x;\sigma]$, we have $p(x)q(x) \in N(R)[x;\sigma]$ but $(1,0) \notin N(R)$.

Corollary 3.10. (1) *Assume that a ring R is nil-Armendariz of the Ore extension type and $N^*(R)$ is a (σ,δ)-ideal with $N^*(R)[x;\sigma,\delta] \subseteq N^*(R[x;\sigma,\delta])$. Then R is NI if and only if $R[x;\sigma,\delta]$ is NI.*

(2) *Assume that a ring R is nil-Armendariz of the skew power series ring type and $N^*(R)$ is a σ-ideal with $N^*(R)[[x;\sigma]] \subseteq N^*(R[[x;\sigma]])$. Then R is NI if and only if $R[[x;\sigma]]$ is NI.*

Proof. Note that the class of NI rings is closed under subrings.

(1) Let R be NI. Then $N^*(R)$ is a (σ,δ)-rigid ideal by Proposition 3.8(1). If $p(x) \in N(R[x;\sigma,\delta])$, then $p(x) \in N(R)[x;\sigma,\delta] = N^*(R)[x;\sigma,\delta]$ by [10, Proposition 3.8]. Hence $p(x) \in N^*(R[x;\sigma,\delta])$ by assumption, completing the proof.

(2) is the similar to the proof of (1), combining Proposition 3.8(1) and [10, Proposition 3.14]. $\square$

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