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CONVERGENCE THEOREMS FOR THE CHOQUET-PETTIS INTEGRAL

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Abstract. In this paper, we introduce the concept of Choquet-Pettis integral of Banach-valued functions using the Choquet integral of real-valued functions and investigate convergence theorems for the Choquet-Pettis integral.

1. Introduction

The fuzzy measure was introduced by Sugeno [9] and the Choquet integral of real-valued functions with respect to a fuzzy measure was introduced by Murofushi and Sugeno [5]. The Choquet integral is a generalization of the Lebesgue integral, since they coincide when μ is a classical $\sigma-$additive measure. The Choquet integral is a basic tool for the subjective evaluation and decision analysis. The convergence theorems are very important in classical integral theory and also Choquet integral theory. Narukawa, Murofushi and Sugeno [8] introduced the regular fuzzy measure on a locally compact Hausdorff space and showed the usefulness in the point of representation of some functional.

In this paper, we introduce the concept of Choquet-Pettis integral of Banach-valued functions using the Choquet integral of real-valued functions. The Choquet-Pettis integral is an extension of the Choquet

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integral for Banach-valued functions and this integral is also a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when μ is a classical $\sigma-$additive measure. We also investigate convergence theorems for this integral.

2. Preliminaries

Throughout this paper, X denotes a real Banach space and X^* its dual. Let Ω be a nonempty classical set, Σ a σ-algebra formed by the subsets of Ω and (Ω, Σ) a measurable space.

DEFINITION 2.1.[7,9]. A *fuzzy measure* on a measurable space (Ω, Σ) is an extended real-valued set function $\mu : \Sigma \to [0, \infty]$ satisfying
(i) $\mu(\emptyset) = 0$,
(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B,\ A, B \in \Sigma$.
When $\mu(\Omega) < \infty$, we say that μ is *finite.* When μ is finite, we define *the conjugate* μ^c of μ by

$$\mu^c(A) = \mu(\Omega) - \mu(A^c),$$

where A^c is the complement of $A \in \Sigma$.
A fuzzy measure μ is said to be *lower semi-continuous* if it satisfies

$$A_1 \subset A_2 \subset \cdots \text{ implies } \mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

A fuzzy measure μ is said to be *upper semi-continuous* if it satisfies

$$A_1 \supset A_2 \supset \cdots \text{ and } \mu(A_1) < \infty \text{ implies } \mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

A fuzzy measure μ is said to be *continuous* if it is both lower and upper semi-continuous.

If a fuzzy measure μ is lower(resp., upper) semi-continuous, then μ^c is upper(resp., lower) semi-continuous.

The class of real-valued measurable functions is denoted by M and the class of nonnegative real-valued measurable functions is denoted by M^+. The class of non-negative upper semi-continuous real-valued functions with compact support is denoted by $USCC^+$ and the class of non-negative lower semi-continuous real-valued functions is denoted by LSC^+.

Definition 2.2.[1,5]. (1) *The Choquet integral* of $f \in M^+$ with respect to a fuzzy measure μ on $A \in \Sigma$ is defined by

$$(C)\int_A f d\mu = \int_0^\infty \mu((f \geq r) \cap A)dr,$$

where the right-hand side integral is the Lebesgue integral and $(f \geq r) = \{\omega \in \Omega \mid f(\omega) \geq r\}$ for all $r \geq 0$.

If $(C)\int_A f d\mu < \infty$, then we say that f is *Choquet integrable* on A with respect to μ. Instead of $(C)\int_\Omega f d\mu$, we will write $(C)\int f d\mu$.

(2) Suppose $\mu(\Omega) < \infty$. The Choquet integral of $f \in M$ with respect to a fuzzy measure μ on $A \in \Sigma$ is defined by

$$(C)\int_A f d\mu = (C)\int_A f^+ d\mu - (C)\int_A f^- d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. When the right-hand side is $\infty - \infty$, the Choquet integral is not defined. If $(C)\int_A f d\mu$ is finite, then we say that f is *Choquet integrable* on A with respect to μ.

$L_1^+(\mu)$ denotes the class of nonnegative Choquet integrable functions. That is,

$$L_1^+(\mu) := \left\{ f \mid f \in M^+, (C)\int f d\mu < \infty \right\}.$$

The Choquet integral is a generalization of the Lebesgue integral, since they coincide when μ is a classical σ-additive measure. For each $f \in M^+$, we also have

$$(C)\int_A f d\mu = \int_0^\infty \mu((f > r) \cap A)dr, \ \forall A \in \Sigma,$$

where $(f > r) = \{\omega \in \Omega \mid f(\omega) > r\}$ for all $r \geq 0$.

Definition 2.3.[2]. Let $f, g \in M$. We say that f and g are *comonotonic* if $f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega')$ for $\omega, \omega' \in \Omega$. We denote $f \sim g$ when f and g are comonotonic.

Definition 2.4.[3]. A sequence (f_n) of real-valued measurable functions is said to *converge* to f *in distribution,* in symbols $f_n \underset{\rightarrow}{D} f$, if

$$\lim_{n\to\infty} \mu((f_n \geq r)) = \mu(f \geq r)) \quad \text{e.c.},$$

where "e.c." stands "except at most countably many values of r".

3. Results

We introduce the concept of Choquet-Pettis integral of Banach-valued functions. The concept of Pettis integral and its properties may be found in [4].

Definition 3.1. A function $f : \Omega \to X$ is called *Choquet-Pettis integrable* if for each $x^* \in X^*$ the function x^*f is Choquet integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. The vector x_A is called *the Choquet-Pettis integral* of f on A and is denoted by $(CP)\int_A f d\mu$.

The Choquet-Pettis integral is a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when μ is a classical σ-additive measure.

Definition 3.2. (1) Let $f : \Omega \to X$ and $g : \Omega \to X$ be weakly measurable. f and g are said to be *weakly comonotonic* if for each $x^* \in X^*$ x^*f and x^*g are comonotonic. We denote $f \sim_w g$ when f and g are weakly comonotonic.

(2) A sequence (f_n) of X-valued weakly measurable functions is said to *converge weakly* to f *in distribution* on Ω, in symbols $f_n \underrightarrow{wD} f$, if for each $x^* \in X^*$ (x^*f_n) converges to x^*f in distribution.

A set $N \in \Sigma$ is called *a null set* with respect to μ if $\mu(A \cup N) = \mu(A)$ for all $A \in \Sigma$ [6]. "$P(\omega)$ μ-a.e. on A" means that there exists a null set N such that $P(\omega)$ is true for all $\omega \in A - N$, where $P(\omega)$ is a proposition concerning the point of A.

Theorem 3.3. *Let $f : \Omega \to X$ and $g : \Omega \to X$ be Choquet-Pettis integrable. Then*

(1) *af is Choquet-Pettis integrable and*

$$(CP)\int_A af d\mu = a(CP)\int_A f d\mu$$

for all $A \in \Sigma$ and $a \geq 0$;

(2) *if $f \sim_w g$, then $f + g$ is Choquet-Pettis integrable and*

$$(CP)\int_A (f+g) d\mu = (CP)\int_A f d\mu + (CP)\int_A g d\mu$$

for all $A \in \Sigma$;

(3) *if* $f = g$ μ*-a.e. and* μ^c*-a.e. on* Ω*, then*

$$(CP)\int_A f d\mu = (CP)\int_A g d\mu$$

for all $A \in \Sigma$

Proof. (1) Since $f : \Omega \to X$ is Choquet-Pettis integrable, for each $x^* \in X^*$ x^*f is Choquet integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Hence for each $x^* \in X^*$ $x^*(af)$ is Choquet integrable and for every $A \in \Sigma$ $x^*(ax_A) = (C)\int_A x^*(af)d\mu$ for all $x^* \in X^*$. Thus af is Choquet-Pettis integrable and $(CP)\int_A af d\mu = ax_A = a(CP)\int_A f d\mu$ for all $A \in \Sigma$ and $a \geq 0$.

(2) Since $f : \Omega \to X$ and $g : \Omega \to X$ are Choquet-Pettis integrable, for each $x^* \in X^*$ x^*f and x^*g are Choquet integrable and for every $A \in \Sigma$ there exist $x_A, y_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ and $x^*(y_A) = (C)\int_A x^* g d\mu$ for all $x^* \in X^*$. Since $f \sim_w g$, for each $x^* \in X^*$ $x^*(f+g)$ is Choquet integrable and for every $A \in \Sigma$ $x^*(x_A + y_A) = (C)\int_A x^*(f+g)d\mu$ for all $x^* \in X^*$. Thus $f+g$ is Choquet-Pettis integrable and $(CP)\int_A (f+g)d\mu = x_A + y_A = (CP)\int_A f d\mu + (CP)\int_A g d\mu$ for all $A \in \Sigma$.

(3) Since $f : \Omega \to X$ and $g : \Omega \to X$ are Choquet-Pettis integrable, for each $x^* \in X^*$ x^*f and x^*g are Choquet integrable and for every $A \in \Sigma$ there exist $x_A, y_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ and $x^*(y_A) = (C)\int_A x^* g d\mu$ for all $x^* \in X^*$. Since $f = g$ μ-a.e. and μ^c-a.e. on Ω, $x^*f = x^*g$ μ-a.e. and μ^c-a.e. on Ω for all $x^* \in X^*$. Hence for every $A \in \Sigma$ $(C)\int_A x^* f d\mu = (C)\int_A x^* g d\mu$ i.e., $x^*(x_A) = x^*(y_A)$ for all $x^* \in X^*$. Hence $x_A = y_A$ i.e., $(CP)\int_A f d\mu = (CP)\int_A g d\mu$.

$\square$

THEOREM 3.4. *Let* X *be a reflexive Banach space and let* (f_n) *be a sequence of Choquet-Pettis integrable* X*-valued functions on* Ω*. If* (f_n) *converges weakly to* f *in distribution on* Ω *and if* g *and* h *are Choquet-Pettis integrable* X*-valued functions on* Ω *such that* $\mu((x^*h \geq r)) \leq \mu((x^*f_n \geq r)) \leq \mu((x^*g \geq r))$ *e.c. for* $n = 1, 2, \cdots$ *and* $x^* \in X^*$*, then* f *is Choquet-Pettis integrable and* $(CP)\int f_n d\mu \to (CP)\int f d\mu$ *weakly.*

Proof. Since g and h are Choquet-Pettis integrable, for each $x^* \in X$ x^*g and x^*h are Choquet integrable. Since (f_n) converges weakly to f in distribution, for each $x^* \in X$ (x^*f_n) converges to x^*f in distribution.

By hypothesis, $\mu((x^*h \geq r)) \leq \mu((x^*f_n \geq r)) \leq \mu((x^*g \geq r))$ e.c. for $n = 1, 2, \cdots$ and $x^* \in X^*$. By [3, Theorem 8.9] x^*f is Choquet integrable and $\lim_{n\to\infty}(C)\int_A x^*f_n d\mu = (C)\int_A x^*f d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$. Since f_n is Choquet-Pettis integrable for $n = 1, 2, \cdots$, for each $A \in \Sigma$ there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (C)\int_A x^*f_n d\mu$ for all $x^* \in X^*$, i.e., $x_{n,A} = (CP)\int_A f_n d\mu$. Thus $(x_{n,A})$ is a weak Cauchy sequence in X. Since X is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $\lim_{n\to\infty} x^*(x_{n,A}) = x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (C)\int_A x^*f d\mu$ for all $x^* \in X^*$. Thus f is Choquet-Pettis integrable and $x_A = (CP)\int_A f d\mu$ for each $A \in \Sigma$. In particular, $(CP)\int f_n d\mu \to (CP)\int f d\mu$ weakly.

□

THEOREM 3.5. (1) *Let μ be a finite and lower semi-continuous fuzzy measure and let (f_n) be a sequence of real-valued measurable functions. If $f_n \uparrow f$ μ-a.e. and μ^c-a.e. and there exists a Choquet integrable function g such that $f_1^- \leq g$ on Ω, then f is Choquet integrable and $(C)\int f_n d\mu \uparrow (C)\int f d\mu$.*

(2) *Let μ be a finite and upper semi-continuous fuzzy measure and let (f_n) be a sequence of real-valued measurable functions. If $f_n \downarrow f$ μ-a.e. and μ^c-a.e. and there exists a Choquet integrable function g such that $f_1^+ \leq g$ on Ω, then f is Choquet integrable and $(C)\int f_n d\mu \downarrow (C)\int f d\mu$.*

Proof. (1) Since $f_n \uparrow f$ μ-a.e. and μ^c-a.e., $f_n^+ \uparrow f^+$ μ-a.e. and $f_n^- \downarrow f^-$ μ^c-a.e. Since μ is lower semi-continuous, by [11, Theorem 2.4] f^+ is Choquet integrable with respect to μ and $(C)\int f_n^+ d\mu \uparrow (C)\int f^+ d\mu$. Since μ is lower semi-continuous, μ^c is upper semi-continuous. Since there exists a Choquet integrable function g such that $f_1^- \leq g$ on Ω, by [11, Theorem 2.4] f^- is Choquet integrable with respect to μ^c and $(C)\int f_n^- d\mu^c \downarrow (C)\int f^- d\mu^c$. Hence f is Choquet integrable and $(C)\int f_n d\mu \uparrow (C)\int f d\mu$.

(2) The proof is similar to (1).

□

THEOREM 3.6. *Let μ be a finite and continuous fuzzy measure and let X be a reflexive Banach space and let (f_n) be a sequence of Choquet-Pettis integrable X-valued functions on Ω.*

(1) *If $f_n \uparrow f$ weakly μ-a.e. and μ^c-a.e. and there exists a Choquet integrable function g such that $(x^*f_1)^- \leq g$ on Ω for all $x^* \in X^*$,*

then f is Choquet-Pettis integrable and $(CP)\int f_n d\mu \uparrow (CP)\int f d\mu$ *weakly.*

(2) *If $f_n \downarrow f$ weakly μ-a.e. and μ^c-a.e. and there exists a Choquet integrable function g such that $(x^* f_1)^+ \leq g$ on Ω for all $x^* \in X^*$, then f is Choquet-Pettis integrable and* $(CP)\int f_n d\mu \downarrow (CP)\int f d\mu$ *weakly.*

Proof. (1) Let $A \in \Sigma$. Since $f_n \uparrow f$ weakly μ-a.e. and μ^c-a.e. and there exists a Choquet integrable function g such that $(x^* f_1)^- \leq g$ on Ω for all $x^* \in X^*$, by Theorem 3.5 $x^* f$ is Choquet integrable and $(C)\int_A x^* f_n d\mu \uparrow (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Since f_n is Choquet-Pettis integrable for $n = 1, 2, \cdots$, there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (C)\int_A x^* f_n d\mu$ for all $x^* \in X^*$ i.e., $x_{n,A} = (CP)\int_A f_n d\mu$. Thus $(x_{n,A})$ is a weak Cauchy sequence in X. Since X is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $x^*(x_{n,A}) \uparrow x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Thus f is Choquet-Pettis integrable and $x_A = (CP)\int_A f d\mu$. In particular, $(CP)\int f_n d\mu \uparrow (CP)\int f d\mu$ weakly.

(2) The proof is similar to (1).

□

THEOREM 3.7. *Let μ be a finite and continuous fuzzy measure and let X be a reflexive Banach space and let (f_n) be a sequence of Choquet-Pettis integrable X-valued functions on Ω. If $f_n \to f$ weakly μ-a.e. and μ^c-a.e. and there exist Choquet integrable functions g and h such that $h \leq x^* f_n \leq g$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, then f is Choquet-Pettis integrable and* $(CP)\int f_n d\mu \to (CP)\int f d\mu$ *weakly.*

Proof. Let $A \in \Sigma$. Since $f_n \to f$ weakly μ-a.e., $(x^* f_n)^+ \to (x^* f)^+$ μ-a.e. for all $x^* \in X^*$. Since $x^* f_n \leq g$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, $(x^* f_n)^+ \leq g^+$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$. By [11, Theorem 2.7] $(x^* f)^+$ is Choquet integrable with respect to μ and $\lim_{n\to\infty}(C)\int_A (x^* f_n)^+ d\mu = (C)\int_A (x^* f)^+ d\mu$ for all $x^* \in X^*$. Since $f_n \to f$ weakly μ^c-a.e., $(x^* f_n)^- \to (x^* f)^-$ μ^c-a.e. for all $x^* \in X^*$. Since $h \leq x^* f_n$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, $(x^* f_n)^- \leq h^-$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$. By [11, Theorem 2.7] $(x^* f)^-$ is Choquet integrable with respect to μ^c and $\lim_{n\to\infty}(C)\int_A (x^* f_n)^- d\mu^c = (C)\int_A (x^* f)^- d\mu^c$ for all $x^* \in X^*$. Hence $x^* f$ is Choquet integrable with respect to μ and

$$\begin{aligned}\lim_{n\to\infty}(C)\int_A x^* f_n d\mu &= \lim_{n\to\infty}\left((C)\int_A (x^* f_n)^+ d\mu - (C)\int_A (x^* f_n)^- d\mu^c\right)\\ &= (C)\int_A (x^* f)^+ d\mu - (C)\int_A (x^* f)^- d\mu^c\\ &= (C)\int_A x^* f d\mu\end{aligned}$$

for all $x^* \in X^*$. Since f_n is Choquet-Pettis integrable for $n = 1, 2, \cdots$, there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (C)\int_A x^* f_n d\mu$ for all $x^* \in X^*$ i.e., $x_{n,A} = (CP)\int_A f_n d\mu$. Since $\lim_{n\to\infty}(C)\int_A x^* f_n d\mu = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$, $(x_{n,A})$ is a weak Cauchy sequence in X. Since X is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $\lim_{n\to\infty} x^*(x_{n,A}) = x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Thus f is Choquet-Pettis integrable and $x_A = (CP)\int_A f d\mu$. In particular, $(CP)\int f_n d\mu \to (CP)\int f d\mu$ weakly.

□

In the sequel, we assume that Ω is a locally compact Hausdorff space, $\mathcal{B}$ is the class of Borel subsets of Ω, $\mathcal{C}$ is the class of compact subsets of Ω and $\mathcal{O}$ is the class of open subsets of Ω.

Definition 3.8.[8]. Let μ be a fuzzy measure on the measurable space $(\Omega, \mathcal{B})$. μ is said to be *outer regular* if

$$\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$$

for all $B \in \mathcal{B}$.

The outer regular fuzzy measure μ is said to be *regular* if

$$\mu(O) = \inf\{\mu(C) | C \in \mathcal{C}, C \subset O\}$$

for all $O \in \mathcal{O}$.

The next theorem follows immediately from [8, Proposition 3.3].

Theorem 3.9. *Let μ be a regular fuzzy measure.*

(1) *If $f_n \in LSC^+$ for $n = 1, 2, \cdots$ and $f_n \uparrow f$ on Ω, then f is Choquet integrable and*

$$\lim_{n\to\infty}(C)\int f_n d\mu = (C)\int f d\mu.$$

(2) *If $f_n \in USCC^+$ for $n = 1, 2, \cdots$ and $f_n \downarrow f$ on Ω, then f is Choquet integrable and*

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu.$$

THEOREM 3.10. *Let μ be a finite and regular fuzzy measure. If (f_n) is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on Ω, then f is Choquet integrable and*

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu.$$

Proof. Since (f_n) is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on Ω, $f_n^+ \in LSC^+$ for $n = 1, 2, \cdots$ and $f_n^+ \uparrow f^+$. By Theorem 3.9,

$$\lim_{n\to\infty} (C) \int f_n^+ d\mu = (C) \int f^+ d\mu.$$

Since (f_n) is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on Ω, $f_n^- \in USCC^+$ for $n = 1, 2, \cdots$ and $f_n^- \downarrow f^-$. By Theorem 3.9,

$$\lim_{n\to\infty} (C) \int f_n^- d\mu = (C) \int f^- d\mu.$$

Hence f is Choquet integrable and

$$\begin{aligned}
\lim_{n\to\infty} (C) \int f_n d\mu &= \lim_{n\to\infty} \left[(C) \int f_n^+ d\mu - (C) \int f_n^- d\mu \right] \\
&= \lim_{n\to\infty} (C) \int f_n^+ d\mu - \lim_{n\to\infty} (C) \int f_n^- d\mu \\
&= (C) \int f^+ d\mu - (C) \int f^- d\mu \\
&= (C) \int f d\mu.
\end{aligned}$$

□

THEOREM 3.11. *Let μ be a finite and regular fuzzy measure and let X be a reflexive Banach space. If (f_n) is a sequence of continuous Choquet-Pettis integrable X-valued functions with compact support and $f_n \uparrow f$*

weakly on Ω, then f is Choquet-Pettis integrable and

$$\lim_{n\to\infty}(CP)\int f_n d\mu = (CP)\int f d\mu \text{ weakly.}$$

Proof. Let $A \in \Sigma$. Since (f_n) is a sequence of continuous X-valued functions with compact support, $(x^* f_n)$ is a sequence of continuous real-valued functions with compact support for all $x^* \in X^*$. Since $f_n \uparrow f$ weakly on Ω, $x^* f_n \uparrow x^* f$ on Ω for all $x^* \in X^*$. By Theorem 3.10, $x^* f$ is Choquet integrable for all $x^* \in X^*$ and $\lim_{n\to\infty}(C)\int_A x^* f_n d\mu = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Since f_n is Choquet-Pettis integrable for $n = 1, 2, \cdots$, there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (C)\int_A x^* f_n d\mu$ for all $x^* \in X^*$ and $n = 1, 2, \cdots$. That is, $x_{n,A} = (CP)\int_A f_n d\mu$ for $n = 1, 2, \cdots$. Thus $(x_{n,A})$ is a weak Cauchy sequence in X. Since X is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $\lim_{n\to\infty} x^*(x_{n,A}) = x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. Thus f is Choquet-Pettis integrable and $x_A = (CP)\int_A f d\mu$ for each $A \in \Sigma$. In particular,

$$\lim_{n\to\infty}(CP)\int f_n d\mu = (CP)\int f d\mu \text{ weakly.}$$

□

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