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ABSOLUTE CONTINUITY OF THE REPRESENTING MEASURES OF THE HYPERGEOMETRIC TRANSLATION OPERATORS ATTACHED TO THE ROOT SYSTEM OF TYPE B_2 AND C_2

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Abstract. We prove in this paper the absolute continuity of the representing measures of the hypergeometric translation operators $\mathcal{T}_x$ and $\mathcal{T}_x^W$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type B_2 and C_2 which are studied in [9].

1. Introduction

We consider the differential-difference operators $T_j, j = 1, 2, ...d$ associated with a root system $\mathcal{R}$, a Weyl group W and a multiplicity function k, introduced by I. Cherednik in [2], and called the Cherednik operators in the literature. These operators are helpful for the extension and simplification of the theory of Heckman-Opdam, which is a generalization of the harmonic analysis on the symmetric spaces $G|K$ (see [3, 4, 5, 7]).

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The notion of hypergeometric translation operators introduced in [8] is basic in the harmonic analysis associated to the Cherednik operators and the Heckman-Opdam theory. We have considered in [9] the hypergeometric translation operators $\mathcal{T}_x$, and $\mathcal{T}_x^W$, $x \in \mathbb{R}^2$, associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type B_2 and C_2 we have proved that these operators are integral transforms, more precisely, for all function f in $\mathcal{E}(\mathbb{R}^2)$ (the space of C^∞-functions on $\mathbb{R}^2$) we have

$$\forall\, t \in \mathbb{R}^2, \mathcal{T}_x(f)(t) = \int_{\mathbb{R}^2} f(z) dm_{x,t}(z), \tag{1.1}$$

where $m_{x,t}$ is a positive measure with compact support contained in the set $\{z \in \mathbb{R}^2\,; |\|x\| - \|t\|| \leq \|z\| \leq \|x\| + \|t\|\}$, and of norm equal to 1. From this result we have deduced that for all function f in $\mathcal{E}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{E}(\mathbb{R}^2)$ of W-invariant functions), we have

$$\forall\, t \in \mathbb{R}^2\,, \mathcal{T}_x^W(f)(t) = \int_{\mathbb{R}^2} f(z) dm_{x,t}^W(z), \tag{1.2}$$

where

$$m_{x,t}^W = \frac{1}{|W|^2} \sum_{w,w' \in W} m_{wx,w't}. \tag{1.3}$$

In this paper we prove that for all $x, t \in \mathbb{R}^2_{reg}$ (the regular part of $\mathbb{R}^2$) the measures $m_{x,t}$ and $m_{x,t}^W$ are absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely there exist positive functions $\mathcal{W}(x,t,.)$ and $\mathcal{W}^W(x,t,.)$ such that

$$dm_{x,t}(z) = \mathcal{W}(x,t,z)\mathcal{A}_k(z)dz, \tag{1.4}$$

$$dm_{x,t}^W(z) = \mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz, \tag{1.5}$$

where $\mathcal{A}_k$ is a weight function on $\mathbb{R}^2$ which will be given in the following section (see (2.8)).
The functions $z \to \mathcal{W}(x,t,z)$ and $z \to \mathcal{W}^W(x,t,z)$ have their support contained in the set $\{z \in \mathbb{R}^2; |\|x\| - \|t\|| \leq \|z\| \leq \|x\| + \|t\|\}$ and satisfy

$$\int_{\mathbb{R}^2} \mathcal{W}(x,t,z)\mathcal{A}_k(z)dz = 1, \tag{1.6}$$

and

$$\int_{\mathbb{R}^2} \mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz = 1. \tag{1.7}$$

As applications of the previous results, we prove that for all $\lambda \in \mathbb{C}^2$, the Opdam-Cherednik kernel G_λ and the Heckmann-Opdam hypergeometric function F_λ possess the following product formulas

$$\forall\, x, t \in \mathbb{R}^2_{reg}, G_\lambda(x)G_\lambda(t) = \int_{\mathbb{R}^2} G_\lambda(z)\mathcal{W}(x,t,z)\mathcal{A}_k(z)dz, \tag{1.8}$$

and

$$\forall\, x, t \in \mathbb{R}^2_{reg}, F_\lambda(x)F_\lambda(t) = \int_{\mathbb{R}^2} F_\lambda(z)\mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz. \tag{1.9}$$

2. The Cherednik operators and their eigenfunctions

We consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$ and inner product $\langle ., . \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^2$.

2.1. The root systems of type B_2 and C_2 and the multiplicity functions.

The root system of type B_2 can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\pm e_1, \pm e_2\} \cup \{\pm e_1 \pm e_2\}, \tag{2.1}$$

which can also be written in the form

$$\mathcal{R} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3 \pm \alpha_4\},$$

with

$$\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = (e_1 - e_2), \alpha_4 = (e_1 + e_2). \tag{2.2}$$

We denote by $\mathcal{R}_+$ the set of positive roots

$$\mathcal{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \tag{2.3}$$

and by $\mathcal{R}^o_+$ the set of positive indivisible roots i.e, the roots $\alpha \in \mathcal{R}_+$ such that $\frac{\alpha}{2} \notin \mathcal{R}_+$. Then we have

$$\mathcal{R}^0_+ = \mathcal{R}_+. \tag{2.4}$$

For $\alpha \in \mathcal{R}$, we consider

$$r_\alpha(x) = x - \langle \breve{\alpha}, x\rangle\alpha, \text{ with } \breve{\alpha} = \frac{2\alpha}{\|\alpha\|^2}, \tag{2.5}$$

the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^2$ orthogonal to α. The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(2)$, called the Weyl group associated with $\mathcal{R}$. In this case W is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2,.$

The multiplicity function $k : \mathcal{R} \to]0, +\infty[$ can be written in the form $k = (k_1, k_2)$ where k_1 is the value on the roots α_1, α_2, and k_2 is the value on the roots α_3, α_4.

The positive Weyl chamber denoted by $\mathfrak{a}^+$ is given by

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^2 \,; \forall\, \alpha \in \mathcal{R}_+, \langle \alpha, x\rangle > 0\}, \tag{2.6}$$

it can also be written in the form

$$\mathfrak{a}^+ = \{(x_1, x_2) \in \mathbb{R}^2 \,; x_1 > x_2 > 0\}. \tag{2.7}$$

Let also $\mathbb{R}^2_{reg}$ be the subset of regular elements in $\mathbb{R}^2$, i.e., those elements which belong to no hyperplane $H_\alpha = \{x \in \mathbb{R}^2; \langle \alpha, x\rangle = 0\}, \alpha \in \mathcal{R}$.

Let $\mathcal{A}_k$ denote the weight function

$$\forall\, x \in \mathbb{R}^2, \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |\sinh\langle \frac{\alpha}{2}, x\rangle|^{2k(\alpha)}. \tag{2.8}$$

Remark 2.1. The root system of type C_2 can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\pm 2e_1, \pm 2e_2\} \cup \{\pm e_1 \pm e_2\},$$

which can also be written in the form

$$\mathcal{R} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\},$$

with

$$\alpha_1 = 2e_1, \alpha_2 = 2e_2, \alpha_3 = (e_1 - e_2), \alpha_4 = (e_1 + e_2).$$

The set of positive roots is the following

$$\mathcal{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

If we denote by $W(C_2)$ the Weyl group associated to the root system $\mathcal{R}$ of type C_2, then we have

$$W(C_2) = W(B_2).$$

We denote also by $k = (k_1, k_2)$ the multiplicity function of the root system $\mathcal{R}$ of C_2, where k_1 is the value on the roots α_1, α_2, and k_2 is the value on the roots α_3, α_4.

In the remainder of the paper we shall give the results and their proofs only for the root system of type B_2. It is easy to obtain the analogous of these results in the case of the root system of type C_2.

2.2. The Cherednik operators attached to the root system of type B_2.

The Cherednik operators $T_j, j = 1, 2$, on $\mathbb{R}^2$ associated with the Weyl group W and the multiplicity function k are defined for f of class C^1 on $\mathbb{R}^2$ and $x \in \mathbb{R}_{reg} = \mathbb{R}^2 \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x), \quad (2.9)$$

with

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \qquad \text{and } \alpha^j = \langle \alpha, e_j \rangle. \quad (2.10)$$

These operators can also be written in the following form

$$\begin{aligned} T_1 f(x) &= \frac{\partial}{\partial x_1} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_1} x)\}}{1 - e^{-\langle \alpha_1, x \rangle}} + k_2 \Big[\frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_3, x \rangle}} \\ &\quad + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} \Big] - (\frac{1}{2} k_1 + k_2) f(x). \end{aligned} \quad (2.11)$$

$$\begin{aligned} T_2 f(x) &= \frac{\partial}{\partial x_2} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_2} x)\}}{1 - e^{-\langle \alpha_2, x \rangle}} \\ &\quad + k_2 \Big[- \frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_3, x \rangle}} + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} \Big] - \frac{1}{2} k_1 f(x). \end{aligned} \quad (2.12)$$

2.3. The eigenfunctions of the Cherednik operators attached to the root system of type B_2.

We denote by $G_\lambda, \lambda \in \mathbb{C}^2$, the eigenfunction of the operators $T_j, j = 1, 2$. It is the unique analytic function on $\mathbb{R}^2$ which satisfies the differential difference system

$$\begin{cases} T_j G_\lambda(x) &= -i\lambda_j G_\lambda(x), x \in \mathbb{R}^2, j = 1, 2, \\ G_\lambda(0) &= 1 \end{cases} \quad (2.13)$$

It is called the Opdam-Cherednik kernel.

We consider the function $F_\lambda, \lambda \in \mathbb{C}^2$, defined by

$$\forall\, x \in \mathbb{R}^2, F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \quad (2.14)$$

This function is the unique analytic W-invariant function on $\mathbb{R}^2$, which satisfies the partial differential equation

$$\begin{cases} p(T)F_\lambda(x) & = p(-i\lambda)F_\lambda(x), \quad x \in \mathbb{R}^2, \\ F_\lambda(0) & = 1, \end{cases} \tag{2.15}$$

for all W-invariant polynomials p on $\mathbb{R}^2$ and $p(T) = p(T_1, T_2)$. It is called the Heckman-Opdam hypergeometric function.

The functions G_λ and F_λ possess the following properties

i) For all $x \in \mathbb{R}^2$ the function $\lambda \to G_\lambda(x)$ is entire on $\mathbb{C}^2$.

ii) We have

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{C}^2,\ \ \overline{G_\lambda(x)} = G_{-\overline{\lambda}}(x). \tag{2.16}$$

iii) We have

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{C}^2,\ |G_\lambda(x)| \leq G_{iIm(\lambda)}(x). \tag{2.17}$$

iv) We have

$$\forall\, x \in \mathbb{R}^2\ , \forall\, \lambda \in \mathbb{R}^2, |G_\lambda(x)| \leq 1. \tag{2.18}$$

$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{R}^2, |F_\lambda(x)| \leq 1. \tag{2.19}$$

v) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation

$$\forall\, x \in \mathbb{R}^2, G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda, y\rangle} d\mu_x(y), \tag{2.20}$$

where μ_x is a positive measure on $\mathbb{R}^2$ with support in $\Gamma = \text{conv}\{wx, w \in W\}$ (the convexe hull of the orbit of x under W).

vi) From (2.14), (2.20) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation

$$\forall\, x \in \mathbb{R}^2, F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda, y\rangle} d\mu_x^W(y), \tag{2.21}$$

where μ_x^W is the positive measure with support in Γ given by

$$\mu_x^W = \frac{1}{|W|} \sum_{w \in W} \mu_{wx}. \tag{2.22}$$

3. The hypergeometric translation operator $\mathcal{T}_x$

We consider the hypergeometric translation operator $\mathcal{T}_x, x \in \mathbb{R}^2$, given by the relation (1.1). In the following we give some properties of this operator (see [9]).

i) For all $x \in \mathbb{R}^2$, the operator $\mathcal{T}_x$ is continuous from $\mathcal{E}(\mathbb{R}^2)$ (resp. $\mathcal{D}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$ with compact support) into itself, and for all f in $\mathcal{D}(\mathbb{R}^2)$ with support in the closed ball $\bar{B}(0,a)$ of center 0 and radius $a > 0$, we have

$$supp\mathcal{T}_x(f) \subset \bar{B}(0, a + \|x\|). \tag{3.1}$$

ii) For all f in $\mathcal{E}(\mathbb{R}^2)$ and $x, y \in \mathbb{R}^2$, we have

$$\mathcal{T}_x(f)(0) = f(x), \quad \text{ and } \mathcal{T}_x(f)(y) = \mathcal{T}_y(f)(x). \tag{3.2}$$

iii) For $x \in \mathbb{R}^2, g \in \mathcal{E}(\mathbb{R}^2)$ and f in $\mathcal{D}(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} \mathcal{T}_x(g)(y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^2} g(z) \mathcal{T}_x(\check{f})(-z) \mathcal{A}_k(z) dz, \tag{3.3}$$

where $\check{f}$ is the function given by

$$\forall\, x \in \mathbb{R}^2, \ \check{f}(x) = f(-x).$$

Remark 3.1. The hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^2$, given by the relation (1.2) satisfies the same properties as for the operator $\mathcal{T}_x, x \in \mathbb{R}^2$, by considering the spaces $\mathcal{E}(\mathbb{R}^2)^W$ and $\mathcal{D}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{D}(\mathbb{R}^2)$ of W-invariant functions).

Notation. We denote by $B(c,a)$ the open ball of $\mathbb{R}^2$ of center c in $\mathbb{R}^2$ and radius $a > 0$, and by $\bar{B}(c,a)$ its closure.

Proposition 3.2. Let $y_0 \in \mathbb{R}^2$ and $a > 0$. We consider the sequence $\{f_n\}_{n \in \mathbb{N}\backslash\{0\}}$ of functions in $\mathcal{D}(\mathbb{R}^2)$, positive, increasing such that :

$$\forall\, n \in \mathbb{N}\backslash\{0\}, suppf_n \subset \bar{B}(y_0, a), \forall\, t \in B(y_0, a - \frac{1}{n}), \ f_n(t) = 1,$$

and

$$\forall\, t \in \mathbb{R}^2, \lim_{n \to +\infty} f_n(t) = 1_{B(y_0,a)}(t),$$

where $1_{B(y_0,a)}$ is the characteristic function of the ball $B(y_0,a)$. We have

$$\begin{aligned}\forall\, x,z\in\mathbb{R}^2,\ \lim_{n\to+\infty}\mathcal{T}_x(f_n)(z) &= \lim_{n\to+\infty}\int_{\mathbb{R}^2} f_n(t)dm_{x,z}(t)\\ &= \int_{\mathbb{R}^2} 1_{B(y_0,a)}(t)dm_{x,z}(t).\end{aligned}$$

The function $z\to m_{x,z}(B(y_0,a))=\displaystyle\int_{\mathbb{R}^2}1_{B(y_0,a)}(t)dm_{x,z}(t)$, which can also be denoted by $\mathcal{T}_x(1_{B(y_0,a)})(z)$ is defined almost every where on $\mathbb{R}^2$ (see [1] p. 17), measurable and for all function h in $\mathcal{D}(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} m_{x,z}(B(y_0,a))h(z)\mathcal{A}_k(z)dz=\int_{B(y_0,a)}\mathcal{T}_x(\check{h})(-t)\mathcal{A}_k(t)dt. \tag{3.4}$$

Proof. For all $x\in\mathbb{R}^2$ and $n\in\mathbb{N}\backslash\{0\}$, the function $\mathcal{T}_x(f_n)$ belongs to $\mathcal{D}(\mathbb{R}^2)$. Then we obtain the results of this proposition from the monotonic convergence theorem and the relation (3.3). □

REMARK 3.3. There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$. Then for all $E\in\mathfrak{m}$, the function $z\to m_{x,z}(E)$ is defined almost every where on $\mathbb{R}^2$, measurable and we have the following relation

$$\int_{\mathbb{R}^2} m_{x,z}(E)h(z)\mathcal{A}_k(z)dz=\int_E\mathcal{T}_x(\check{h})(-t)\mathcal{A}_k(t)dt,\quad h\in\mathcal{D}(\mathbb{R}^2). \tag{3.5}$$

In this section we shall prove that for all $x\in\mathbb{R}^2_{reg}$, $t\in\mathbb{R}^2$, the measures $m_{x,t}$ and $m^W_{x,t}$ given by the relations (1.1) and (1.3) are absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

3.1. Absolute continuity of the measure $m_{x,z}$.

Notation. We denote by λ the Lebesgue measure on $\mathbb{R}^2$.

PROPOSITION 3.4. *For $x\in\mathbb{R}^2_{reg}$, $z\in\mathbb{R}^2$, there exists a unique positive function $\ominus(x,z,.)$ integrable on $\mathbb{R}^2$ with respect to the Lebesgue measure λ, and a positive measure $m^s_{x,z}$ on $\mathbb{R}^2$ such that for every Borel set E, we have*

$$m_{x,z}(E)=\int_E\ominus(x,z,t)dt+m^s_{x,z}(E). \tag{3.6}$$

Proof. We deduce (3.6) from (1.1) and Theorem 6.9 of [6] p.129-130, and Theorem 8.6 and its Corollary of [6] p. 166. □

REMARK 3.5.

i) The supports of the function $t \to \ominus(x,z,t)$ and the measure $m^s_{x,z}$ are contained in the set $\{t \in \mathbb{R}^2; |\|x\| - \|z\|| \leq \|t\| \leq \|x\| + \|z\|\}$.

ii) The measures $m^s_{x,z}$ and the Lebesgue mesure λ are mutually singular.

iii) From Theorem 8.6, p.166 and Definition 8.3, p.164, of [6], we have

$$\ominus(x,z,t) = \lim_{a \to 0} \frac{m_{x,z}(B(t,a))}{\lambda(B(t,a))}. \tag{3.7}$$

PROPOSITION 3.6. *We consider $x \in \mathbb{R}^2_{reg}$ and a positive function h in $\mathcal{D}(\mathbb{R}^2)$ with support contained in the ball $\bar{B}(0,R)$, $R > 0$.*

i) *For all Borel set E, we have*

$$\int_E \mathcal{N}^h_x(t)dt = \int_{\bar{B}(0,R)} h(z) m^s_{x,z}(E) \mathcal{A}_k(z)dz, \tag{3.8}$$

where

$$\mathcal{N}^h_x(t) = \mathcal{T}_x(\check{h})(-t)\mathcal{A}_k(t) - \int_{\bar{B}(0,R)} \ominus(x,z,t) h(z)\mathcal{A}_k(z)dz. \tag{3.9}$$

ii) *We have*

$$\forall\, t \in \mathbb{R}^2, \mathcal{N}^h_x(t) \geq 0. \tag{3.10}$$

Proof.

i) By using the relations (3.5), (3.6), we obtain

$$\begin{aligned}\int_E \mathcal{T}_x(\check{h})(-t)\mathcal{A}_k(t)dt &= \int_{\bar{B}(0,R)} m_{x,z}(E)h(z)\mathcal{A}_k(z)dz \\ &= \int_{\bar{B}(0,R)} \Big[\int_E \ominus(x,z,t)dt + m^s_{x,z}(E)\Big] h(z)\mathcal{A}_k(z)dz.\end{aligned}$$

We deduce (3.8) by applying Fubini-Tonelli's theorem to the second member.

ii) From the relation (3.8), the positivity of the measure $m^s_{x,z}$ implies that for all Borel set E, we have

$$\int_E \mathcal{N}^h_x(t)dt \geq 0.$$

Thus

$$\forall\, t \in \mathbb{R}^2, \mathcal{N}^h_x(t) \geq 0.$$

□

Proposition 3.7. *The measure* Λ_x^h *on* $\mathbb{R}^2$ *given for all Borel set* E *by*

$$\Lambda_x^h(E) = \int_E \mathcal{N}_x^h(t)dt, \tag{3.11}$$

is positive and bounded.

Proof.
- The relation (3.10) gives the positivity of the measure Λ_x^h.
- From the relation (3.11) (3.8), for all Borel set E we have

$$\Lambda_x^h(E) \leq \int_{\bar{B}(0,R)} \|m_{x,z}^s\| h(z)\mathcal{A}_k(z)dz. \tag{3.12}$$

On the other hand by using (3.6), we obtain for all $z \in \mathbb{R}^2_{reg}$,

$$m_{x,z}^s(E) \leq m_{x,z}(E),$$

thus

$$\|m_{x,z}^s\| \leq \|m_{x,z}\| = 1.$$

By using this result, the relation (3.12) implies that for all Borel set E, we have

$$\Lambda_x^h(E) \leq M_h,$$

where

$$M_h = \int_{\bar{B}(0,R)} h(z)\mathcal{A}_k(z)dz.$$

Then the measure Λ_x^h is bounded. □

Proposition 3.8. *Let* $x \in \mathbb{R}^2_{reg}$ *and* h *be a positive function in* $\mathcal{D}(\mathbb{R}^2)$ *with support contained in the ball* $\bar{B}(0,R), R > 0$.

i) *For all Borel set* E *we have*

$$\Lambda_x^h(E) = 0 \tag{3.13}$$

ii) *For* $x, t \in \mathbb{R}^2_{reg}$, *we have*

$$\mathcal{T}_x(h)(t) = \int_{\bar{B}(0,R)} h(z)\mathcal{W}(x,t,z)\mathcal{A}_k(z)dz, \tag{3.14}$$

with

$$\mathcal{W}(x,t,z) = \frac{\ominus(x,-z,-t)}{\mathcal{A}_k(t)} \tag{3.15}$$

Proof.

i) From the relations (3.11), (3.8), for all Borel set E the measure Λ_x^h possesses also the following form

$$\Lambda_x^h(E) = \int_{\bar{B}(0,R)} m_{x,z}^s(E) h(z) \mathcal{A}_k(z) dz. \tag{3.16}$$

On the other hand from Proposition 3.7 the measure Λ_x^h is absolute continuous with respect to the Lebesgue measure λ and from Remark 3.5 ii) the measure $m_{x,z}^s$, $z \in \bar{B}(0,R)$ and the Lebesgue measure λ are mutually singular. Then from Proposition 6.8,(f), p. 129, of [6], the measure Λ_x^h and $m_{x,z}^s$, $z \in \bar{B}(0,R)$, are mutually singular. By using the definition of measures mutually singular (see p. 128 of [6]), we deduce (3.13) from (3.16).

ii) By using the i) and (3.11), (3.9), we get

$$\mathcal{T}_x(\check{h})(-t)\mathcal{A}_k(t) = \int_{\bar{B}(0,R)} \ominus(x,z,t) h(z) \mathcal{A}_k(z) dz \tag{3.17}$$

As

$$\mathcal{A}_k(t) \neq 0 \Leftrightarrow t \in \mathbb{R}^2_{reg},$$

then for $t \in \mathbb{R}^2_{reg}$, we deduce (3.14), (3.15) from (3.17).

□

Theorem 3.9. *For all f in $\mathcal{E}(\mathbb{R}^2)$ and $x, t \in \mathbb{R}^2_{reg}$, we have*

$$\mathcal{T}_x(f)(t) = \int_{\mathbb{R}^2} f(z) \mathcal{W}(x,t,z) \mathcal{A}_k(z) dz, \tag{3.18}$$

with

$$\forall\, z \in \mathbb{R}^2, \;\; \mathcal{W}(x,t,z) = \mathcal{W}(t,x,z). \tag{3.19}$$

Proof. We obtain (3.18), (3.19) by writing $f = f^+ - f^-$ and by using Proposition 3.8, and the properties i), ii) of the operator $\mathcal{T}_x$. □

Remark 3.10. Theorem 3.9 shows that for all $x \in \mathbb{R}^2_{reg}$, $t \in \mathbb{R}^2$ the measure $m_{x,t}$ is absolute continuous with respect to the measure $\mathcal{A}_k(z)dz$. More precisely for all $z \in \mathbb{R}^2$, we have

$$dm_{x,t}(z) = \mathcal{W}(x,t,z) \mathcal{A}_k(z) dz. \tag{3.20}$$

Corollary 3.11.

i) *For all $\lambda \in \mathbb{C}^2$ and $x, t \in \mathbb{R}^2_{reg}$, we have*

$$G_\lambda(x) G_\lambda(t) = \int_{\mathbb{R}^2} G_\lambda(z) \mathcal{W}(x,t,z) \mathcal{A}_k(z) dz. \tag{3.21}$$

ii) *For all* $x, t \in \mathbb{R}^2_{reg}$, *we have*

$$\int_{\mathbb{R}^2} \mathcal{W}(x,t,z)\mathcal{A}_k(z)dz = 1. \tag{3.22}$$

iii) *For all* $x, t \in \mathbb{R}^2_{reg}$, *the support of the function* $z \to \mathcal{W}(x,t,z)$ *is contained in the set* $\{z \in \mathbb{R}^d \, ; |\|x\| - \|t\|| \leq \|z\| \leq \|x\| + \|t\|\}$.

Proof. We deduce the results of this Corollary from (1.1), (3.20), Theorem 3.9 and the product formula for the Opdam-Cherednik kernel $G_\lambda, \lambda \in \mathbb{C}^2$, (see [9] p. 24). □

3.2. Absolute continuity of the measure $m^W_{x,t}$.

PROPOSITION 3.12. *For all* $x, t \in \mathbb{R}^2_{reg}$ *the measure* $m^W_{x,t}$ *is absolute continuous with respect to the Lebesgue measure on* $\mathbb{R}^2$. *More precisely for all* $z \in \mathbb{R}^2$, *we have*

$$dm^W_{x,t}(z) = \mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz, \tag{3.23}$$

where $\mathcal{W}^W(x,t,z)$ *is the function given by*

$$\mathcal{W}^W(x,t,z) = \frac{1}{|W|^2} \sum_{w,w' \in W} \mathcal{W}(wx, w't, z). \tag{3.24}$$

Proof. The relation (1.3) and Theorem 3.9 imply (3.23), (3.24). □

COROLLARY 3.13.

i) *For all* $\lambda \in \mathbb{C}^2$ *and* $x, t \in \mathbb{R}^2_{reg}$, *we have*

$$F_\lambda(x)F_\lambda(t) = \int_{\mathbb{R}^2} F_\lambda(z)\mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz. \tag{3.25}$$

ii) *For all* $x, t \in \mathbb{R}^2_{reg}$, *we have*

$$\int_{\mathbb{R}^2} \mathcal{W}^W(x,t,z)\mathcal{A}_k(z)dz = 1. \tag{3.26}$$

iii) *For all* $x, t \in \mathbb{R}^2_{reg}$, *the support of the function* $z \to \mathcal{W}^W(x,t,z)$ *is contained in the set* $\{z \in \mathbb{R}^2; |\|x\| - \|t\|| \leq \|z\| \leq \|x\| + \|t\|\}$.

Proof. We obtain the results of this Corollary from the relation (1.2), Proposition 3.12, and the product formula for the Heckman-Opdam hypergeometric function $F_\lambda, \lambda \in \mathbb{C}^2$, (see [9] p. 27). □

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