

THE PRIMITIVE BASES OF THE SIGNED CYCLIC GRAPHS

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ABSTRACT. The base $l(S)$ of a signed digraph S is the maximum number k such that for any vertices u, v of S , there is a pair of walks of length k from u to v with different signs. A graph can be regarded as a digraph if we consider its edges as two-sided arcs. A signed cyclic graph \widetilde{C}_n is a signed digraph obtained from the cycle C_n by giving signs to all arcs. In this paper, we compute the base of a signed cyclic graph \widetilde{C}_n when \widetilde{C}_n is neither symmetric nor anti-symmetric. Combining with previous results, the base of all signed cyclic graphs are obtained.

1. Introduction

A *sign pattern matrix* A of order n is the $n \times n$ matrix with entries 1, 0 and -1 . When we compute the entries of the powers of A , we use the operation rule that continues to hold the sign of the usual addition and multiplication, that is for any $a \in \{1, 0, -1\}$

$$1+1 = 1; (-1)+(-1) = -1; 1+0 = 0+1 = 1; (-1)+0 = 0+(-1) = -1$$

$$0 \cdot a = a \cdot 0 = 0; 1 \cdot 1 = (-1) \cdot (-1) = 1; 1 \cdot (-1) = (-1) \cdot 1 = -1.$$

Received January 2, 2013. Revised February 13, 2013. Accepted February 15, 2013.

2010 Mathematics Subject Classification: 15B35, 05C20, 05C22.

Key words and phrases: base, sign pattern matrix, directed cycle.

This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

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In this case we contact the ambiguous situations $1 + (-1)$ and $(-1) + 1$, which we will use the notation " $\#$ " as in [3]. Define the addition and multiplication involving the symbol $\#$ as follows:

$$\begin{aligned} (-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \text{ for any } a \in \{1, -1, \#, 0\} \\ 0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \text{ for any } b \in \{1, -1, \#\}. \end{aligned}$$

A *generalized sign pattern matrix* A of order n is the $n \times n$ matrix with entries $1, 0, -1$ and the ambiguous sign $\#$. A least positive integer l such that there is a positive integer p satisfying $A^l = A^{l+p}$ is called the *base* of A , and denoted by $l(A)$. And the least such positive integer p is called to be the *period* of A , and denoted by $p(A)$. A generalized sign pattern matrix A is called *powerful* if there appears no $\#$ entry in any power of A . And A is *non-powerful* if it is not powerful. If a sign pattern matrix A is non-powerful and there is a number l such that every entry of A^l is $\#$, then the least such integer l is the base of A .

In [3], Li, Hall and Stuart showed that if the sign pattern matrix A is powerful, then $l(A) = l(|A|)$ where $|A|$ is the matrix by assigning each non-zero entry of A to 1. If A is non-powerful, then the $\#$ entry appears and we have a different situation. We introduce a graph theoretic method to study the powers of a sign pattern matrix.

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1 . The *sign* of the walk W in S , denoted by $\text{sgn}(W)$, is defined to be the product of signs of all arcs in W . If two walks W_1 and W_2 have the same initial points, the same terminal points, the same lengths and different signs, then we say W_1 and W_2 a *pair of SSSD walks*. A signed digraph S is *powerful* if S contains no pair of SSSD walks. So every non-powerful primitive signed digraph contains a pair of SSSD walks. Let $A = A(S) = (a_{ij})$ be the adjacency matrix of a signed digraph S , that is $\text{sgn}(i, j) = \alpha$ if and only if $a_{ij} = \alpha$ where $\alpha = 1$, or -1 for an arc (i, j) of S . In this case A is a sign pattern matrix which satisfies that (i, j) -entry of $A^k = 0$, if and only if there is no walk of length k from i to j . Moreover (i, j) -entry of A^k is 1 (or -1), if and only if all walks of length k from i to j are of sign 1 (or, -1). Also (i, j) -entry of A^k is $\#$ if and only if there is a pair of SSSD walks of length k from i to j . Thus we see from the above relations between matrices and graphs that each power of a signed digraph S contains no pair of SSSD walks if and only if the adjacency sign pattern matrix $A(S)$ is powerful. A signed digraph S is also said to be powerful or non-powerful if its adjacency

sign pattern matrix is powerful or non-powerful respectively. There is an important characterization for powerful irreducible sign pattern matrices given in [2] which will be the starting point of our study on the bases of non-powerful irreducible sign pattern matrices. Let S be a strongly connected signed digraph and h be the index of imprimitivity of S (i.e., h is the greatest common divisor of the lengths of all the cycles of S). Then S is powerful if and only if S satisfies the following two conditions:

(A1) All cycles in S with lengths even multiples of h (if any) are positive.

(A2) All cycles in S with lengths odd multiples of h have the same sign.

From now on we assume that S is a primitive non-powerful signed digraph of order n . For each pair of vertices v_i, v_j of S , we define the *local base* $l_S(v_i, v_j)$ from v_i to v_j to be the smallest integer l such that for each $k \geq l$, there is a pair of SSSD walks of length k in S from v_i to v_j . The *base* $l(S)$ of S is defined to be $\max\{l_S(v_i, v_j) | v_i, v_j \in V(S)\}$. It follows directly from the definitions that $l(S) = l(A)$ where A is the adjacency matrix of S . You et al. [7] found upper bounds for the bases of primitive nonpowerful sign pattern matrices and completely characterized extremal cases. Gao, Huang and Shao [1], Shao and Gao [6] and Li and Liu [4] studied the base and local base of the primitive non-powerful signed symmetric digraphs with loops. Liang, Liu and Lai [5] gave the bounds on the k -th multiple generalized base index for a class of non-powerful generalized sign pattern matrices. They also characterized the extremal graphs for the (generalized) base for primitive anti-symmetric sign pattern matrices.

Let us assume that \widetilde{C}_n is a signed digraph of order n which is the cyclic graph C_n on n vertices by assigning signs to each arc such that it becomes a signed digraph. Liang, Liu and Lai [5] proved that the base of anti-symmetric signed cyclic graph \widetilde{C}_n on n vertices is $2n - 1$. In this paper we find the base of \widetilde{C}_n when \widetilde{C}_n is neither symmetric nor anti-symmetric.

Let Q be the canonical cycle in C_n . We then can summarize the main contributions of the present paper as follows:

(C1) If the cycle Q and its inverse cycle $-Q$ have the same sign, then the base of \widetilde{C}_n is $n + 1$.

(C2) If the cycle Q and its inverse cycle $-Q$ have distinct sign, then the base of \widetilde{C}_n is n .

Consequently the base of all signed cyclic graphs are obtained.

2. Main theorem

In this section we assume that n is an odd positive integer, $C_n = (V, E)$ where $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{\{v_i, v_j\} | j \equiv i+1 \pmod{n}\}$. Thus C_n is a cyclic graph of odd order. If $A = \{(v_i, v_j) | \{v_i, v_j\} \in E\}$ and $f : A \rightarrow \{\pm 1\}$, then $\widetilde{C}_n = (V, A, f)$ is a signed digraph. If $a = (v, w) \in A$, then $a^{-1} = (w, v)$ is the inverse of a and $e = \{v, w\}$ is the underlying edge of a . If $W = w_0w_1 \cdots w_k$ where $w_0, w_1, \dots, w_k \in V$ is a walk of length k in C_n , then $-W = w_kw_{k-1} \cdots w_1w_0$ is the inverse of W . If $W_1 = v_0v_1 \cdots v_n$ and $W_2 = v_nv_{n+1} \cdots v_m$ are two walks in a graph, then we use $W_1 + W_2$ to be the walk $v_0v_1 \cdots v_m$. We also use the notation $kW = W + W + \cdots + W$ (k -times) for a circuit W .

The *sign* $f(W)$ of W is $f(w_0w_1)f(w_1w_2) \cdots f(w_{k-1}w_k)$. If $e = \{v, w\}$, then the *sign* of e is $f(vw)f(wv)$. Note that \widetilde{C}_n is symmetric when the sign of every edge is 1, and anti-symmetric when the sign of every edge is -1 . If $W = w_0w_1 \cdots w_k$ is a cycle of length k , then $W' = w_iw_{i+1} \cdots w_kw_0w_1 \cdots w_i$ is a *rotation* of W for $0 \leq i \leq k$.

LEMMA 1. *If $W = w_0w_1 \cdots w_k$ is a walk of length k in an odd cycle C_n with $w_0 = w_k$, then for each $e \in E$, the number of i such that $\{w_i, w_{i+1}\} = e$ is congruent to k modulo 2.*

Proof. Since $C_n - e$ is isomorphic to the path P_n , which is bipartite, there are $V_0, V_1 \subset V$ such that $V_0 \cup V_1 = V$ and $V_0 \cap V_1 = \phi$ and every edge except e joins a vertex of V_0 and a vertex of V_1 . We may assume that the two vertices incident to e belong to V_0 . For each $i = 0, 1, \dots, k-1$, the membership of w_i and w_{i+1} among V_0 and V_1 is changed if and only if $\{w_i, w_{i+1}\} \neq e$. Since $w_0 = w_k$, the number of i such that $\{w_i, w_{i+1}\} \neq e$ is even. So the number of i such that $\{w_i, w_{i+1}\} = e$ is congruent to k modulo 2. \square

LEMMA 2. *Every even cycle in an odd cycle C_n is a 2-cycle.*

Proof. Let $Z = w_0w_1 \cdots w_k$ be an even cycle. If $e = \{w_0, w_1\}$, then by Lemma 1 the number of i such that $\{w_i, w_{i+1}\} = e$ is even. Hence there is a t such that $t \geq 1$ and $\{w_t, w_{t+1}\} = e$. Since Z is a cycle, $w_i \neq w_j$ for $i \neq j$ except $i = 0, j = k$ or $i = k, j = 0$. Hence $t = 1$ and $w_2 = w_0$. Hence we have $k = 2$ and $Z = w_0w_1w_0$. \square

Let Q be the canonical n -cycle $v_0v_1 \cdots v_{n-1}v_0$ in C_n . Then $-Q = v_0v_{n-1}v_{n-2} \cdots v_0$.

LEMMA 3. *Let C_n be a cyclic graph of odd order. Then there are exactly two odd cycles Q and $-Q$ up to a rotation in C_n .*

Proof. If $Z = w_0w_1 \cdots w_k$ is an odd cycle, then by Lemma 1, for each edge e , the number of i such that $\{w_i, w_{i+1}\} = e$ is odd. Since Z doesn't visit the same vertex twice, except $w_0 = w_k$, for all edge e of C_n , there is exactly one i such that $\{w_i, w_{i+1}\} = e$. Thus $k = n$ and $\{w_0, w_1, \cdots, w_{n-1}\} = V$. We may assume that $w_0 = v_0$. Since v_1 and v_{n-1} are only two vertices adjacent to v_0 , w_1 are v_1 or v_{n-1} . If $w_1 = v_1$, then w_2 is v_0 or v_2 . Since Z is a cycle, $w_2 \neq v_0$. Hence $w_2 = v_2$. Similarly we have $w_i = v_i$ for any $3 \leq i \leq n-1$ and $w_n = v_0$. Therefore $Z = Q$. If $w_1 = v_{n-1}$, then by the same method we have $Z = -Q$. \square

PROPOSITION 1. *If a signed odd cyclic graph \widetilde{C}_n is symmetric, then \widetilde{C}_n is powerful.*

Proof. If Z is an even cycle, then by Lemma 2 Z is a 2-cycle. Hence $Z = w_0w_1w_0$ for some $w_0, w_1 \in V$. Thus $f(Z) = f(w_0w_1)f(w_1w_0)$ is the same with the sign of edge $\{v_0, v_1\}$. Since \widetilde{C}_n is symmetric, $f(Z) = 1$. So there is no even cycle of sign -1 . By Lemma 3 the odd cycles of C_n are Q and $-Q$ up to translation. Since \widetilde{C}_n is symmetric, we have $f(-Q) = f(w_0w_{n-1})f(w_{n-1}w_{n-2}) \cdots f(w_1w_0) = f(w_0w_1)f(w_1w_2) \cdots f(w_{n-1}w_0) = f(Q)$. Thus all odd cycles in \widetilde{C}_n have the same signs. Hence every even cycle in \widetilde{C}_n has sign 1 and every odd cycles, Q and $-Q$, have the same signs. By the characterization of powerful signed digraph provided in introduction, \widetilde{C}_n is powerful. \square

It is known [3] that the base of a primitive powerful signed digraph S is equal to the exponent of S . Hence we have the following Corollary.

COROLLARY 1. *If a signed odd cyclic graph \widetilde{C}_n is symmetric, then the base of \widetilde{C}_n is $n-1$.*

The following Proposition is due to Liang, Liu and Lai [5].

PROPOSITION 2. *If a signed odd cyclic graph \widetilde{C}_n is anti-symmetric, then $l(\widetilde{C}_n) = 2n-1$.*

LEMMA 4. *There is only one walk of length $n-1$ from v_0 to v_{n-1} in an odd cycle C_n .*

Proof. If $W = w_0w_1 \cdots w_k$ is a walk of length $n - 1$ from v_0 to v_{n-1} in C_n , then since $|E| = n$, there is $e \in E$ such that $\{w_i, w_{i+1}\} \neq e$ for all $i = 0, 1, \dots, n - 2$. If $e \neq \{w_0, w_{n-1}\}$, then since $C_n - e$ is bipartite, there is no walk of even length from v_0 to v_{n-1} . This contradicts to the fact that W is a walk of even length $n - 1$ from v_0 to v_{n-1} . Thus $e = \{v_{n-1}, v_0\}$. Since the distance from v_0 to v_{n-1} in $C_n - \{v_0, v_{n-1}\}$ is $n - 1$, we have $W = v_0v_1 \cdots v_{n-1}$ \square

LEMMA 5. *There are exactly two walks Q and $-Q$ of length n from v_0 to v_0 in an odd cycle C_n .*

Proof. If $W = w_0w_1 \cdots w_n$ is a walk of length n from v_0 to v_0 in C_n , then w_{n-1} is v_{n-1} or v_1 . If $w_{n-1} = v_{n-1}$, then by Lemma 4 $w_0w_1 \cdots w_{n-1} = v_0v_1 \cdots v_{n-1}$. Hence $W = v_0v_1 \cdots v_{n-1} = Q$. By the same method, if $w_{n-1} = v_1$, then we have $W = -Q$. \square

PROPOSITION 3. *Assume that an odd cycle \widetilde{C}_n is neither symmetric nor anti-symmetric. Then $l(\widetilde{C}_n) = n + 1$ if $f(Q) = f(-Q)$, and $l(\widetilde{C}_n) = n$ if $f(Q) = -f(-Q)$.*

Proof. Let $v, w \in V$. We may assume that $v = v_0$ and $w = v_t$ for $0 \leq t \leq n - 1$. Let $\alpha = n + 1$ if $f(Q) = f(-Q)$, and $\alpha = n$ if $f(Q) = -f(-Q)$. Let $e_i = \{v_i, v_{i+1}\}$ for all $i = 0, 1, \dots, n - 2$ and $e_{n-1} = \{v_{n-1}, v_0\}$. Since \widetilde{C}_n is neither symmetric nor anti-symmetric, there is s such that $0 \leq s \leq n - 2$ and $f(v_s v_{s+1})f(v_{s+1} v_s) = -f(v_{n-1} v_0)f(v_0 v_{n-1})$. Let $Z = v_0 v_{n-1} v_0$, $Z_1 = v_s v_{s+1} v_s$ and $Z_2 = v_{s+1} v_s v_{s+1}$. Therefore $f(Z) = -f(Z_1) = -f(Z_2)$. Since n is odd, $\alpha \equiv t \pmod{2}$ or $\alpha \equiv n - t \pmod{2}$. We may assume that $\alpha \equiv t \pmod{2}$.

If $t \geq 1$ and $0 \leq s \leq t$, then since $\alpha - t - 2$ is even and $\alpha - t - 2 \geq n - (n - 1) - 2 = -1$, $\alpha - t - 2 = 2k$ for all $k \geq 0$. Let $W_1 = v_0 v_1 \cdots v_s$ and $W_2 = v_s v_{s+1} \cdots v_t$. Then $(k + 1)Z + W_1 + W_2$ and $kZ + W_1 + Z_1 + W_2$ are SSSD walks of length α from v_0 to v_t .

If $t \geq 1$ and $t \leq s \leq n - 2$, then since $n - t - 1 = (n - s - 1) + (s - t)$, $s - t \leq \frac{n-t-1}{2}$ or $n - s - 1 \leq \frac{n-t-1}{2}$. Let $X_1 = v_0 v_1 \cdots v_s$, $X_2 = v_t v_{t+1} \cdots v_s$ and $X_3 = v_0 v_{n-1} v_{n-2} \cdots v_{s+1}$. If $s - t \leq \frac{n-t-1}{2}$, since $\alpha - 2s + t - 2$ is even and $\alpha - 2s + t - 2 \geq n - 2s + (2s + 1 - n) - 2 = -1$, $\alpha - 2s + t - 2 = 2k$ for some $k \geq 0$. Then $(k + 1)Z + X_1 + X_2 - X_2$ and $kZ + X_1 + X_2 + Z_1 - X_2$ are SSSD walks of length α from v_0 to v_t . If $n - s - 1 \leq \frac{n-t-1}{2}$, by the similar method with $\alpha = 2k + 2(n - s) + t$, we can show that $(k + 1)Z + X_3 - X_3 + X_1$ and $kZ + X_3 + Z_2 - X_3 + X_1$ are SSSD walks of length α from v_0 to v_t .

If $t = 0$ and $f(Q) = -f(-Q)$, then Q and $-Q$ are SSSD walks of length n from v_0 to v_t . So $l(\widetilde{C}_n) \leq n = \alpha$. If $t = 0$ and $f(Q) = f(-Q)$, then $s \leq \frac{n-1}{2}$ or $n - s - 1 \leq \frac{n-1}{2}$. Let $Y_1 = v_0v_1 \cdots v_s$ and $Y_2 = v_0v_{n-1}v_{n-2} \cdots v_{s+1}$. Since $\alpha = n + 1$ is even, $\alpha - 2s - 2$ is even. If $s \leq \frac{n-1}{2}$, then since $\alpha - 2s - 2 \geq n + 1 - (n - 1) - 2 = 0$, we have $\alpha - 2s - 2 = 2k$ for some $k \geq 0$. Hence $(k + 1)Z + Y_1 - Y_1$ and $kZ + Y_1 + Z_1 - Y_1$ are SSSD walks of length $n + 1$ from v_0 to v_0 . Similarly $\alpha - 2n - 2s = 2l$ for some $l \geq 0$. If $n - s - 1 \leq \frac{n-1}{2}$, then $(l + 1)Z + Y_2 - Y_2$ and $lZ + Y_2 + Z_2 - Y_2$ are SSSD walks of length $n + 1$ from v_0 to v_0 . So $l(\widetilde{C}_n) \leq n + 1 = \alpha$.

If $f(Q) = -f(-Q)$, then by Lemma 4 $l(\widetilde{C}_n) \geq n$. So $l(\widetilde{C}_n) = n = \alpha$. If $f(Q) = f(-Q)$, then by Lemma 5 Q and $-Q$ are only 2 walks of length n from v_0 to v_0 . Since $f(Q) = f(-Q)$, there is no walk of length n from v_0 to v_0 with sign $-f(Q)$. Thus $l(\widetilde{C}_n) \leq n + 1$. As a consequence we have $l(\widetilde{C}_n) = n + 1 = \alpha$. \square

From Propositions 1, 2 and 3 we conclude the following.

THEOREM 1. *Let \widetilde{C}_n be a signed odd cyclic graph of order n . Then*

$$l(\widetilde{C}_n) = \begin{cases} n - 1, & \text{if } \widetilde{C}_n \text{ is symmetric;} \\ 2n - 1, & \text{if } \widetilde{C}_n \text{ is anti-symmetric;} \\ n + 1, & \text{if } \widetilde{C}_n \text{ is neither anti-symmetric nor symmetric,} \\ & \text{and } f(Q) = f(-Q); \\ n, & \text{if } \widetilde{C}_n \text{ is neither anti-symmetric nor symmetric,} \\ & \text{and } f(Q) \neq f(-Q). \end{cases}$$

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