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A STUDY ON QUADRATIC CURVES AND GENERALIZED ECCENTRICITY IN POLAR TAXICAB GEOMETRY

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Abstract. Over the years, there has been much research conducted on quadratic curves and the set of points with the generalized notion of eccentricity in a plane with metrics such as taxicab distance or Chinese-checker distance. On the other hand, polar taxicab distance has been newly proposed on the polar coordinate system, a type of curvilinear coordinate system, to overcome the limitation of pre-existing metrics in terms of describing curved routes. Previous study has looked into the fundamental properties of this metric. From this point of view, we study the quadratic curves and the set of points with the generalized notion of eccentricity in a plane with polar taxicab distance.

1. Introduction

Euclidean Geometry is widely used since it is easy to understand intuitively and is appropriate for applying to various theories. Nevertheless, in modern cities, it is sometimes impossible to follow the path defined

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by Euclidean distance in order to move from one place to another. Consequently, the idea of how a taxi travels in modern cities was developed into a practical distance notion, taxicab distance[6]. Since not all motions consist of horizontal and vertical movements, taxicab distance has been generalized to alpha-distance, which includes Chinese-checker distance as well[9]. In addition, generalized absolute value metric was proposed as a generalization of the distance functions mentioned above[2]. Meanwhile, unlike metrics such as taxicab distance, alpha-distance, and generalized absolute value metric, which are defined on the Cartesian coordinate system, a new metric called polar taxicab distance was defined on the polar coordinate system, a type of curvilinear system[5]. Distance functions on the Cartesian coordinate system have been studied steadily[1, 3, 7, 8], and especially, research on quadratic curves has advanced significantly using those distance functions.

Some authors even studied the set of points with the generalization of eccentricity, $C_{AB}(k)$, in Euclidean geometry[3]. Contrary to the other distance functions which have been studied a lot, only basic research has been conducted about polar taxicab distance. Considering this, this study looks at the geometric properties and classifications of $C_{AB}(k)$ in a plane with polar taxicab distance.

2. Polar Taxicab Distance and Quadratic Curves

In [5], polar taxicab distance, a metric in the polar coordinate system, was introduced. From now on, all the coordinates are defined on polar coordinate system. Polar taxicab distance between two points in $\mathbb{R}^2$ is defined as follows.

Definition 1. [5] The polar taxicab distance between two points $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ with $r_1 \geq 0$, $r_2 \geq 0$, $0 \leq \theta_1 < 2\pi$, and $0 \leq \theta_2 < 2\pi$ on the plane with polar coordinates is defined as follows.

$$d_{PT}(A, B) = \begin{cases} \min\{r_1, r_2\} \times |\theta_2 - \theta_1| + |r_2 - r_1| & (0 \leq |\theta_2 - \theta_1| \leq 2), \\ r_1 + r_2 & (2 < |\theta_2 - \theta_1| \leq \pi). \end{cases}$$

In [5], the following lemma is also proved.

Lemma 2. *The rotation around the origin preserves the polar taxicab distance between the two points.*

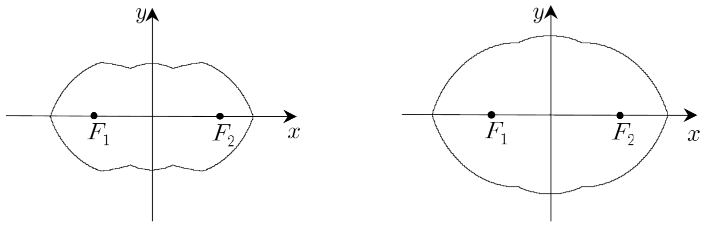


FIGURE 1.

Now quadratic curves in a plane with polar taxicab distance will be examined. An ellipse in a plane with polar taxicab distance is defined analogously as it is defined in Euclidean geometry. It is defined as follows.

DEFINITION 3. An ellipse in polar taxicab geometry is the set of all points in a plane, the sum of whose polar taxicab distances from two fixed points is a given positive constant.

If the two foci are symmetric about the origin, then the shape of the ellipse in a polar taxicab plane is determined by the ratio of the length of the major axis and the polar taxicab distance between the origin and the focus. This can be summarized as the following theorem.

THEOREM 4. *In polar taxicab geometry, an ellipse can have only two shapes as seen in Figure 1 if the two foci F_1 and F_2 are symmetric about the origin.*

Proof. Since Lemma 2 guarantees that the rotation around the origin preserves the shape of a figure, we can rotate the ellipse to place F_1 and F_2 on the axis. Without loss of generality, let the coordinates of two foci be $F_1(f, 0)$ and $F_2(f, \pi)$. Let $P(r, \theta)$ be an arbitrary point on the ellipse. Then, $d_{PT}(P, F_1) + d_{PT}(P, F_2) = 2a$ holds, where a is a given constant. Since the figure is an ellipse, we have $a \geq f$ by using the definition of the distance function.

Without loss of generality, we only consider $P(r, \theta)$ with $0 \leq \theta \leq \frac{\pi}{2}$. Since the polar taxicab distance between two points depends on the difference of their arguments, we should divide it into the following cases: 1) $0 \leq \theta \leq \pi - 2$ and 2) $\pi - 2 < \theta < \frac{\pi}{2}$.

Case 1) $0 \leq \theta \leq \pi - 2$

If $r \leq f$, then we obtain $r\theta = 2(a - f)$ by the definition of the ellipse in polar taxicab geometry. $r \leq f$ leads to $2(a - f) \leq f(\pi - 2)$. Hence, $a \leq \frac{\pi}{2}$. This implies that if $a \leq \frac{\pi}{2}f$, we cannot find such ellipse. If $r > f$, then we obtain $r = \frac{2a-f\theta}{2}$. The inequality $r > f$ implies $\theta < \frac{2a}{f} - 2$. The fact that $a \geq f$ leads us to find P on $r > f$ and $\theta < \frac{2a}{f} - 2$.

Case 2) $\pi - 2 < \theta < \frac{\pi}{2}$

If $r \leq f$, then $a \leq \frac{\pi}{2}f$ is similarly obtained. If $r > f$, then we get $a > \frac{\pi}{2}f$ in the same way.

From our discussion, we conclude that only a and f can change the shape of the figure. We also conclude that the ellipse in polar taxicab geometry can have only two shapes. Namely, the shape is Figure 1(a) if $f \leq a \leq \frac{\pi}{2}f$, and Figure 1(b) if $\frac{\pi}{2}f < a$. □

If a is smaller than $\frac{\pi}{2}f$, then the shape of the ellipse is Figure 1(a). As a increases, the shape gradually changes from Figure 1(a) to Figure 1(b). If a grows larger than $\frac{\pi}{2}f$, then the shape of the ellipse is Figure 1(b), and the scale becomes larger.

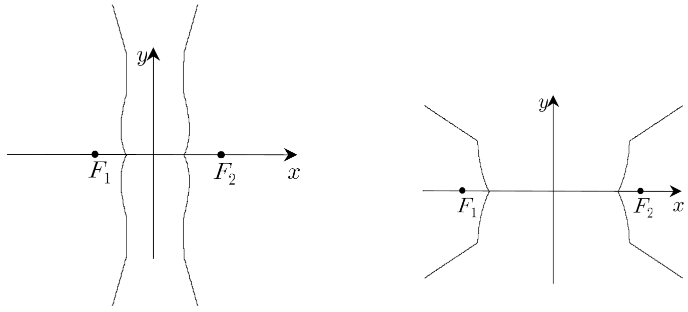
Next, hyperbola will be examined. A hyperbola in a plane with polar taxicab distance is also defined analogously as it is defined in Euclidean geometry. It is defined as follows.

Definition 5. A hyperbola in polar taxicab geometry is the set of all points in a plane, the difference of whose polar taxicab distances from the two fixed points is a given positive constant.

If the two foci are symmetric about the origin, then the shape of the hyperbola in a polar taxicab plane is determined by the ratio of the length of the major axis and the polar taxicab distance between the origin and the focus. This can be summarized as the following theorem.

Theorem 6. *In polar taxicab geometry, a hyperbola can have only two shapes as seen in Figure 2 if the two foci F_1 and F_2 are symmetric about the origin.*

Proof. Using the discussion in Theorem 4, we can assume that the coordinates of F_1 and F_2 are $(f, 0)$ and (f, π), respectively. Let $P(r, \theta)$ be a point and $|d_{PT}(P, F_1) - d_{PT}(P, F_2)| = 2a$, where a is a constant. Since the figure is a hyperbola, we have $a \leq f$ by using the definition of the distance function. We divide it into the following cases as in Theorem 4 : 1) $0 \leq \theta \leq \pi - 2$ and 2) $\pi - 2 < \theta < \frac{\pi}{2}$.



(a) (b)

FIGURE 2.

Case 1) $0 \leq \theta \leq \pi - 2$

If $r \leq f$, then we obtain $r(2-\theta) = 2a$ by the definition of the hyperbola in polar taxicab geometry. $r \leq f$ leads to $\theta \leq 2 - \frac{2a}{f}$. Therefore, we can find P on $r \leq f$ and $\theta \leq 2 - \frac{2a}{f}$. If $r > f$, then we conclude that $\left(2 - \frac{\pi}{2}\right) f \leq a \leq f$ in the same way.

Case 2) $\pi - 2 < \theta < \frac{\pi}{2}$

If $r \leq f$, then we obtain $r|\pi - 2\theta| = 2a$. Because of the fact that $2a \leq f|\pi - 2\theta|$ and $\pi - 2 < \theta \leq \frac{\pi}{2}$, we cannot find such hyperbola if $a \geq \left(2 - \frac{\pi}{2}\right) f$. If $r > f$, then we obtain $\theta = \frac{\pi}{2} \pm \frac{a}{f}$. Since $\pi - 2 < \theta \leq \frac{\pi}{2}$, we conclude that $a < \left(2 - \frac{\pi}{2}\right) f$.

From our discussion, we conclude that only a and f can change the shape of the figure. We also conclude that the hyperbola in polar taxicab geometry can have only two shapes. Namely, the shape is Figure 2(a) if $a < \left(2 - \frac{\pi}{2}\right) f$, and Figure 2(b) if $\left(2 - \frac{\pi}{2}\right) f \leq a \leq f$. □

If a is smaller than $\left(2 - \frac{\pi}{2}\right) f$, then the shape of the hyperbola is Figure 2(a). As a increases, the shape gradually changes from Figure 2(a) to Figure 2(b). If a grows larger than $\left(2 - \frac{\pi}{2}\right) f$, then the shape of the hyperbola is Figure 2(b), and the scale becomes larger.

Next, Apollonius circle will be examined. An Apollonius circle in a plane with polar taxicab distance is also defined analogously as it is defined in Euclidean geometry. It is defined as follows.

DEFINITION 7. An Apollonius circle in polar taxicab geometry is the set of all points in a plane, the ratio of whose polar taxicab distances from two fixed points is a given positive constant.

If the two foci are symmetric about the origin, then the shape of the an Apollonius circle in a polar taxicab plane is determined by the constant. This can be summarized as the following theorem.

THEOREM 8. *In polar taxicab geometry, an Apollonius circle can have only two shapes as seen in Figure 3 if the two foci F_1 and F_2 are symmetric about the origin.*

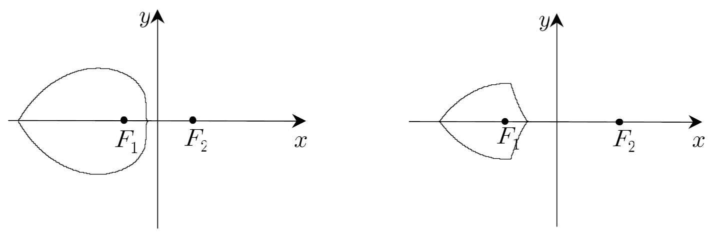


FIGURE 3.

Proof. Using the discussion in Theorem 4, we can assume that the coordinates of F_1 and F_2 are $(f, 0)$ and (f, π), respectively. Let $P(r, \theta)$ be a point and $d_{PT}(P, F_1) = kd_{PT}(P, F_2)$, where k is a constant. If $0 < k < 1$, then $k' d_{PT}(P, F_1) = d_{PT}(P, F_2)$, and $k' > 1$ holds for $k' = \frac{1}{k}$. Therefore, we assume $k > 1$ without loss of generality. Also, we only consider $P(r, \theta)$ with $0 \leq \theta \leq \pi$ without loss of generality. Since the polar taxicab distance between two points depends on the difference of their arguments, we should divide it into the following cases: 1) $0 \leq \theta \leq \pi - 2$, 2) $\pi - 2 < \theta < 2$, and 3) $2 \leq \theta \leq \pi$.

Case 1) $0 \leq \theta \leq \pi - 2$

If $0 \leq r \leq f$, then we obtain $r\theta = (k-1)f + (k+1)r$ by the definition of the Apollonius circle in polar taxicab geometry. $0 \leq \theta \leq \pi - 2$ leads to two inequalities, $(1-k)f \leq (k+1)r$ and $(k-1)f \leq r(\pi - 3 - k)$. Since $0 < (k-1)f$ and $r(\pi - 3 - k) < r(\pi - 4) < 0$ lead us to a contradiction, we cannot find such part of Apollonius circle. If $r > f$, then it is also proved similarly that we cannot find such part of Apollonius circle.

Case 2) $\pi - 2 < \theta < 2$

If $0 \leq r \leq f$, then we obtain $(k-1)f = r\{(k+1)\theta - 1 - k(\pi - 1)\}$. Since $k > 1$, θ satisfies $\theta > \frac{k\pi - k + 1}{k+1}$. Meanwhile, $\pi - 2 < \theta$ leads to $k < \frac{1}{\pi - 3}$. The inequality $0 < (k+1)\theta - 1 - k(\pi - 1) < 1 - (\pi - 3)k$ holds as $\theta < 2$.

Hence, $\frac{k-1}{1-(\pi-3)k}f < r$. At the time, $r \leq f$ leads to $k < \frac{2}{\pi-2}$. If $r > f$, then we obtain $f\theta(k+1) = (k-1)r + f(k\pi - k + 1)$. The inequality $\pi-2 < \theta < 2$ is equivalent to $f(-k+\pi-3) < (k-1)r < f(-k\pi+3k+1)$, and $k > 1$ guarantees $f(-k+\pi-3) < (k-1)r$. Hence, we can find P on $r < \frac{(3-\pi)k+1}{k-1}f$. At the time, $r > f$ leads to $k < \frac{2}{\pi-2}$.

Case 3) $2 \leq \theta \leq \pi$

If $0 \leq r \leq f$, then we obtain $r\{1 - k(\pi - \theta - 1)\} = (k-1)f$. Since $k > 1$, we can find P on $\theta > \pi - 1 - \frac{1}{k}$. Using the inequality $0 \leq r \leq f$, we conclude that P can be found on $0 \leq r \leq f$ and $\theta \geq \pi - \frac{2}{k}$. If $r > f$, then we obtain $(k-1)r = \{1 - k(\pi - \theta - 1)\}f$. Using the inequalities $k > 1$ and $r > f$, we conclude that $\theta > \pi - 1 - \frac{1}{k}$ and $\theta > \pi - \frac{2}{k}$.

From our discussion, we conclude that only k can change the shape of the figure. We also conclude that Apollonius circle in polar taxicab geometry can have only two shapes. Namely, the shape is Figure 3(a) if $k < \frac{2}{\pi-2}$, and Figure 3(b) if $\frac{2}{\pi-2} \leq k$. □

If k is smaller than $\frac{2}{\pi-2}$, then the shape of the Apollonius circle is Figure 3(a). As k increases, the shape gradually changes from Figure 3(a) to Figure 3(b). If k grows larger than $\frac{2}{\pi-2}$, then the shape of the Apollonius circle is Figure 3(b), and the scale becomes larger.

For the final subject of this section, lemniscate will be examined. A lemniscate in a polar taxicab plane is also defined analogously as it is defined in Euclidean geometry. It is defined as follows.

Definition 9. A lemniscate in polar taxicab geometry is a set of all points in the plane, the product of whose polar taxicab distances from two fixed points is a given positive constant.

If the two foci are symmetric about the origin, then the shape of the lemniscate in a polar taxicab plane is determined by the ratio of the constant and the square of the polar taxicab distance between the origin and the focus. This can be summarized as the following theorem.

Theorem 10. *In polar taxicab geometry, a lemniscate can have only four shapes as seen in Figure 4 if the two foci F_1 and F_2 are symmetric about the origin.*

Proof. Using the discussion in Theorem 4, we can assume that the coordinates of F_1 and F_2 are $(f, 0)$ and (f, π), respectively. Let $P(r, \theta)$ be a point and $d_{PT}(P, F_1) \times d_{PT}(P, F_2) = k$, where k is a given constant.

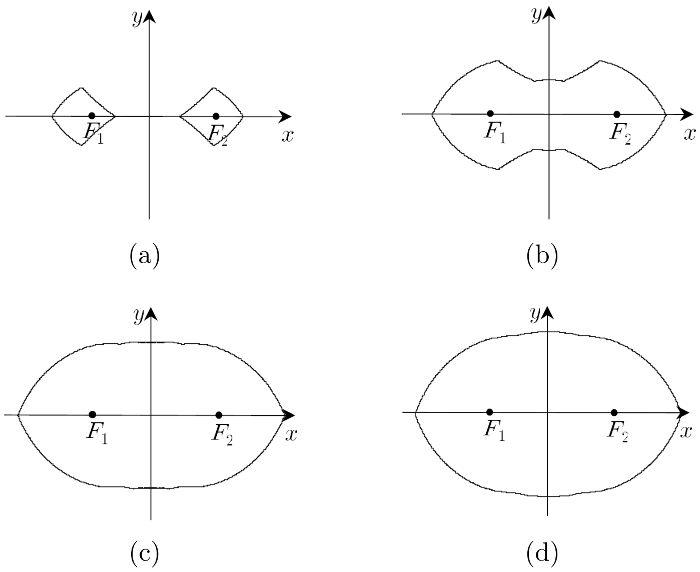


FIGURE 4.

We also divide the cases as we did in Theorem 4: 1) $0 \leq \theta \leq \pi - 2$ and 2) $\pi - 2 < \theta < \frac{\pi}{2}$.

Case 1) $0 \leq \theta \leq \pi - 2$

If $0 \leq r \leq f$, then we obtain $r(r+f)\theta = k - f^2 + r^2$ by the definition of the lemniscate in polar taxicab geometry. $0 \leq \theta \leq \pi - 2$ implies two inequalities $f^2 - k \leq r^2$ and $(\pi - 3)r^2 + f(\pi - 2)r + f^2 - k \geq 0$. Since $\sqrt{f^2 - k} \leq r \leq f$, we conclude that $k \leq (2\pi - 4)f^2$ by using the properties of quadratic functions. If $r > f$, we can similarly conclude that $k \leq (2\pi - 4)f^2$.

Case 2) $\pi - 2 < \theta < \frac{\pi}{2}$

If $0 \leq r \leq f$, let $\phi = \frac{\pi}{2} - \theta$, where $0 \leq \phi \leq 2 - \frac{\pi}{2}$. The definition of ϕ and $0 \leq \phi \leq 2 - \frac{\pi}{2}$ lead to $\left\{r\left(\frac{\pi}{2} - 1\right) + f\right\}^2 - k = \phi^2 r^2$, and $(r + f)\{(\pi - 3)r + f\} \leq k \leq \left\{r\left(\frac{\pi}{2} - 1\right) + f\right\}^2$, respectively. In the same way as we did, we get $f^2 \leq k \leq \frac{\pi^2}{4} f^2$. If $r > f$, let $d = r - f$ and $\phi = \frac{\pi}{2} - \theta$, where $d \geq 0$ and $0 \leq \phi \leq 2 - \frac{\pi}{2}$. The definition of ϕ and $0 \leq \phi \leq 2 - \frac{\pi}{2}$ lead to $d^2 + \pi f d + \frac{\pi^2}{4} f^2 - k = \phi^2 f^2$, and $d^2 + \pi f d + (-k - 4f^2 + 2\pi f^2) \leq 0 \leq d^2 + \pi f d + \frac{\pi^2}{4} f^2 - k$, respectively. In the same way as we did, we get $(2\pi - 4)f^2 \leq k$.

From our discussion, we conclude that only k and f can change the shape of the figure. We also conclude that the lemniscate in polar taxicab geometry can have only four shapes. Namely, the shape is Figure 4(a) if $0 \leq k \leq f^2$, Figure 4(b) if $f^2 \leq k \leq (2\pi - 4)^2$, Figure 4(c) if $(2\pi - 4)f^2 \leq k \leq \frac{\pi^2 f^2}{4}$, and Figure 4(d) if $\frac{\pi^2 f^2}{4} \leq k$. □

If k is smaller than f^2, then the shape of the lemniscate is Figure 4(a). As k increases, the shape gradually changes from Figure 4(a) to Figure 4(b), then Figure 4(c), and eventually Figure 4(d). If k grows larger than $\frac{\pi^2 f^2}{4}$, then the shape of the lemniscate is Figure 4(d), and the scale becomes larger.

3. Generalized Eccentricity in Polar Taxicab Geometry

Let A and B denote a point, a line, or a circle, respectively, in the plane. For a positive constant k, $C_{AB}(k)$ is a locus of points P whose distances from A and B are, respectively, in a constant ratio k. In [4], equivalent conditions for conic sections in Euclidean geometry were studied. Now, we consider $C_{AB}(k)$ on polar taxicab plane.

Let us define polar taxicab distance between a point and a circle whose center is the origin. The circumstances in which the polar taxicab distance from a point to the points on the circle becomes minimal are stated in the following theorem.

Theorem 11. *For a fixed point $X(x, \alpha)$ and a point $Q(R, \phi)$ on the circle whose center is the origin, $d_{PT}(X, Q)$ has the minimum value at $\phi = \alpha$.*

Proof. It is clear that $d_{PT}(X, Q) \geq |x - R|$, and equality holds when $\phi = \alpha$. Therefore, $d_{PT}(X, Q)$ has the minimum value when $\phi = \alpha$. □

Referring to Theorem 11, we can find out the polar taxicab distance between a point and a circle centered at the origin.

Corollary 12. *For a point $P(r, \theta)$ and a circle Γ whose center is the origin and radius is R, the polar taxicab distance between P and Γ is $d_{PT}(\Gamma, P) = |R - r|$.*

Now, $C_{AB}(k)$ will be examined, where A is a point, and B is a circle whose center is the origin.

Theorem 13. *Let a point and a circle whose center is the origin are given. The locus of the point whose ratio of polar taxicab distances to the given point and the given circle, respectively, is constant can have only four shapes as seen in Figure 5.*

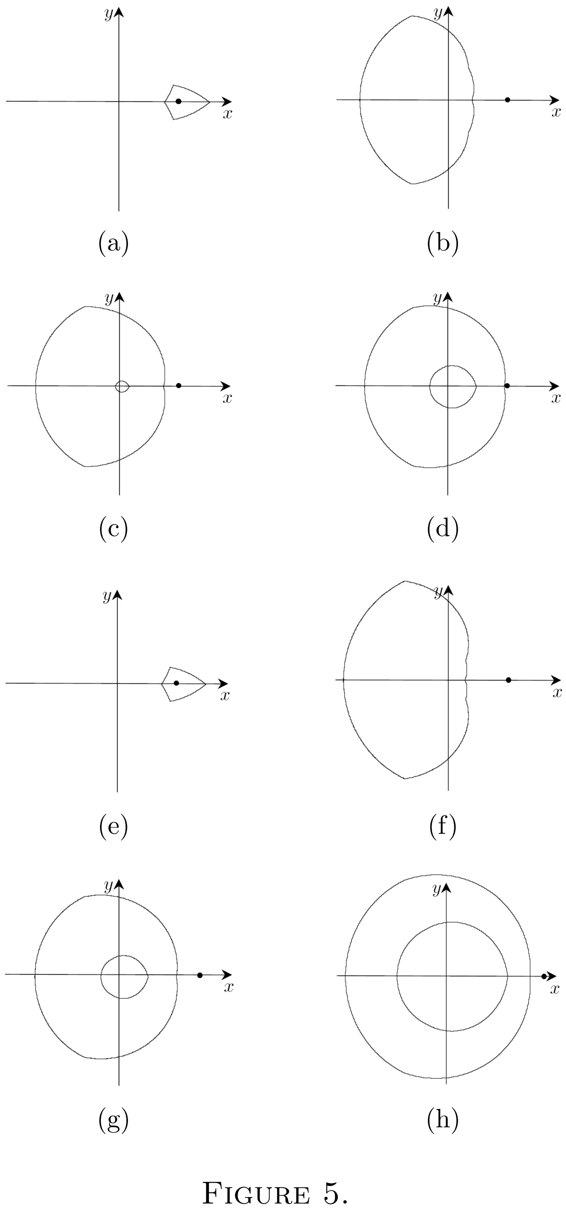


Figure 5.

Proof. Let A and B denote a given point and a circle, respectively. Let $P(r,\theta)$ be a point on $C_{AB}(k) = \{P \mid d_{PT}(A,P) = kd_{PT}(B,P)\}$. Using the discussion in Theorem 4, we can assume that the coordinate of A is $(f,0)$. Let the radius of B be R, where $f > R$. Since the polar taxicab distance between two points depends on the difference of their arguments, we should divide it into the following cases: 1) $0 \le \theta \le 2$ and 2) $2 \le \theta \le \pi$.

Case 1) $0 \le \theta \le 2$

If $r \le R$, then we obtain $r(\theta + k = 1) = kR - f$ by the definition of $C_{AB}(k)$. If $kR - f < 0$, then we can find P on $\theta < 1 - k$. Hence, $k < 1$. In this case, $r \le R$ leads to $\theta \le 1 - \frac{f}{R}$, which is a contradiction since $\theta \ge 0$. Therefore, we cannot find such part of $C_{AB}(k)$. If $kR - f \ge 0$, then $k \ge \frac{f}{R} \ge 1$. If $R < r < f$, then we obtain $r(k + 1 - \theta) = kR + f$, and we can find P on $\theta < k + 1$. It is easily shown that P is found on $R < r < f$ and $1 - \frac{f}{R} < \theta < \left(1 - \frac{R}{f}\right) k$. If $f \le r$, then we obtain $r(k-1) - f\theta = kR - f$. If $k < 1$, it is easily shown that P is found on $f \le r$ and $\theta \le k\left(1 - \frac{R}{f}\right)$. If $k \ge 1$, then $f \le r$ leads to $k\left(1 - \frac{R}{f}\right) \le \theta$. Hence, $k \le \frac{2f}{f-R}$.

Case 2) $2 \le \theta \le \pi$

If $r \le R$, then we obtain $r = \frac{kR-f}{k+1}$, and $0 \le r \le R$ implies $k \ge \frac{f}{R}$ and $0 \le f + R$. If $R < r$, then we obtain $(k-1)r = kR + f$, so that $k > 1$. Then, we can find P on $2 \le \theta \le \pi$.

From our discussion, we conclude that only k, f and R can change the shape of the figure. We also conclude that the figure can have only four shapes. Suppose that $f \le 3R$. Then the shape is Figure 5(a) if $k < 1$, Figure 5(b) if $1 \le k < \frac{f}{R}$, Figure 5(c) if $\frac{f}{R} \le k \le \frac{2f}{f-R}$, and Figure 5(d) if $\frac{2f}{f-R} < k$. On the contrary, suppose that $f > 3R$. Then the shape is Figure 5(e) if $k < 1$, Figure 5(f) if $1 \le k < \frac{2f}{f-R}$, Figure 5(g) if $\frac{2f}{f-R} \le k \le \frac{f}{R}$, and Figure 5(h) if $\frac{f}{R} < k$. □

Suppose that $f \le 3R$. If k is smaller than 1, then the shape of $C_{AB}(k)$ is Figure 5(a). As k increases, the shape gradually changes from Figure 5(a) to Figure 5(b), then Figure 5(c), and eventually Figure 5(d). If k grows larger than $\frac{2f}{f-R}$, then the shape is Figure 5(d), and the scale becomes larger. On the other hand, suppose that $f > 3R$. If k is smaller than 1, then the shape is Figure 5(e). As k increases, the shape gradually changes from Figure 5(e) to Figure 5(f), then Figure 5(g), and eventually Figure 5(h). If k grows larger than $\frac{f}{R}$, then the shape is Figure 5(h), and the scale becomes larger.

Next, the case when A is a point and B is the line $\theta = \frac{\pi}{2}$ will be examined. Let us define polar taxicab distance between a point and the line B. The circumstances that the polar taxicab distance from the point to the points on the line become minimal are stated in the following theorem.

THEOREM 14. *For a fixed point $X(x, \alpha)$ $\left(x > 0\,,\ 0 \leq \alpha \leq \frac{\pi}{2}\right)$ and a point $Q\left(t, \frac{\pi}{2}\right)$ on the line $\theta = \frac{\pi}{2}$, $d_{PT}(X < Q)$ has the minimum value at $t = 0$ if $0 \leq \alpha \leq \frac{\pi}{2}$, and $t = x$ if $\frac{\pi}{2} - 1 \leq \alpha \leq \frac{\pi}{2}$.*

Proof. Let $Y\left(y, \frac{\pi}{2}\right)$ and $Y'\left(-y, \frac{\pi}{2}\right)$ be points on the line $\theta = \frac{\pi}{2}$, where y is a positive number. $d_{PT}(X, Y') = x + y \geq y = d_{PT}(X, O)$ if $\alpha \geq 2 - \frac{\pi}{2}$, and $d_{PT}(X, Y') = |x - y| + \min(x, y)\left|\alpha + \frac{\pi}{2}\right| \geq |x - y| + \min(x, y)\left|\frac{\pi}{2} - \phi\right| = d_{PT}(X, Y)$ if $\alpha < 2 - \frac{\pi}{2}$. Thus, Y' is not the point where the polar taxicab distance has the minimum value. If $0 \leq \phi \leq \frac{\pi}{2} - 1$, we obtain $d_{PT}(X, Q) = x - t + t\left(\frac{\pi}{2} - \phi\right) \geq x = d_{PT}(X, O)$ if $t \leq x$, and $d_{PT}(X, Q) = t - x + x\left(\frac{\pi}{2} - \phi\right) > x = d_{PT}(X, O)$ if $t > x$. Equality holds when $t = 0$. If $\frac{\pi}{2} - 1 \leq \phi \leq \frac{\pi}{2}$, then we obtain $d_{PT}(X, Q) = x - t + t\left(\frac{\pi}{2} - \phi\right) \geq x\left(\frac{\pi}{2} - \phi\right)$ if $t \leq x$ and $d_{PT}(X, Q) = t - x + x\left(\frac{\pi}{2} - \phi\right) \geq x\left(\frac{\pi}{2} - \phi\right)$ if $t > x$. Equality holds when $t = x$. Thus, $t = 0$ when $0 \leq \alpha \leq \frac{\pi}{2} - 1$ and $t = x$ when $\frac{\pi}{2} - 1 \leq \alpha \leq \frac{\pi}{2}$ in order for $d_{PT}(X, Q)$ to be minimum. □

Referring to Theorem 14, we can find out the polar taxicab distance between a point and the line $\theta = \frac{\pi}{2}$.

COROLLARY 15. *For a point $P(r, \theta)$ and the line l: $\theta = \frac{\pi}{2}$, polar taxicab distance between P and l is*

$$d_{PT}(l, P) = \begin{cases} r, & \left(0 \leq \theta \leq \frac{\pi}{2} - 1\right), \\ r\left(\frac{\pi}{2} - \theta\right) & \left(\frac{\pi}{2} - 1 < \theta \leq \frac{\pi}{2}\right). \end{cases}$$

Now, we study about the case when A is a point and B is the line $\theta = \frac{\pi}{2}$.

THEOREM 16. *Let the line $\theta = \frac{\pi}{2}$ and a point on the line $\theta = 0$ be given. The locus of the point whose ratio of polar taxicab distances to the given point and the given line, respectively, is constant can have only four shapes as seen in Figure 6.*

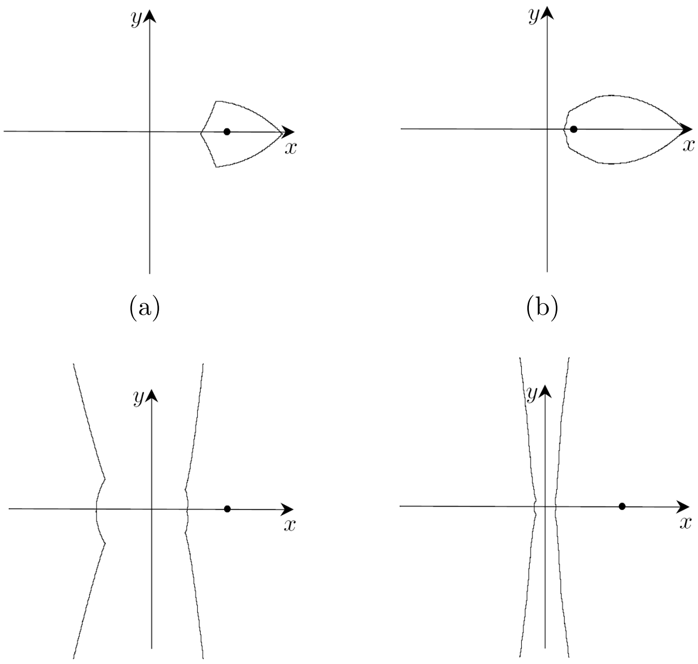


FIGURE 6.

Proof. Let A and B denote the given point and the line $\theta = \frac{\pi}{2}$. Let $P(r, \theta)$ be a point on $C_{AB}(k) = \{P \mid d_{PT}(A, P) = kd_{PT}(B, P)\}$. Using the discussion in Theorem 4, we can assume that the coordinate of A is $(R, 0)$. Since the polar taxicab distance from a point to another point or the line $\theta = \frac{\pi}{2}$ depends on the arguments of a point or the points, we should divide it into the following cases: 1) $0 \leq \theta \leq \frac{\pi}{2} - 1$, 2) $\frac{\pi}{2} - 1 < \theta < 2$, 3) $2 \leq \theta \leq \frac{\pi}{2} + 1$, and 4) $\frac{\pi}{2} + 1 \leq \theta \leq \pi$.

Case 1) $0 \leq \theta \leq \frac{\pi}{2} - 1$

If $r \leq R$, then we obtain $r(k+1-\theta) = R$ by the definition of $C_{AB}(k)$. Thus, we can find P on $\theta \leq k+1$. Since $r \leq R$ implies $\theta \leq k$, we can find P on $r \leq R$ and $\theta \leq k$. If $R < r$, then we obtain $r(1-k) + R\theta = R$. If $k < 1$, then we can find P on $R < r$ and $\theta < k$. If $k \geq 1$, then $R < r$ implies $k < \theta$, and $1 \leq k < \theta$ leads us to a contradiction since $\theta \leq \frac{\pi}{2} - 1$. Therefore, we cannot find such part of $C_{AB}(k)$.

Case 2) $\frac{\pi}{2} - 1 < \theta < 2$

If $r \leq R$, then we obtain $r\left\{\frac{\pi}{2}k+1-(k+1)\theta\right\} = R$. We can find P on $\theta < \frac{\pi k+2}{2(k+1)}$. Since $\frac{\pi}{2}-2 < k$, there always exists θ such that $\theta < \frac{\pi k+2}{2(k+1)}$. Meanwhile, $r \leq R$ leads to $\theta \leq \frac{\pi}{2(k+1)}$. Since $\frac{\pi}{2} = 1 < \theta$, we obtain $\frac{\pi}{2}-1 < k$. If $R < r$, then we obtain $r\left(1-\frac{\pi}{2}k+k\theta\right)+R\theta = R$. When $\theta \leq \frac{\pi}{2}-\frac{1}{k}$, we can say that $1 < k$. The inequality $R < r$ leads to $\frac{\pi k}{2(k-1)} < \theta$. Hence, $\frac{4}{4-\pi} < k$. When $\theta \geq \frac{\pi}{2}-\frac{1}{k}$, the inequality $R < r$ leads to $\theta < \frac{\pi k}{2(k+1)}$. It is proved similarly that $\frac{\pi}{2}-1 < k$.

Case 3) $2 \leq \theta \leq \frac{\pi}{2}+1$

We obtain $r\left(k\theta-\frac{\pi}{2}k-1\right) = R$, and P is found on $\theta > \frac{\pi}{2}+\frac{1}{k}$. Therefore, $k > 1$.

Case 4) $\frac{\pi}{2}+1 \leq \theta \leq \pi$

We obtain $(k-1)r = R$. Therefore, $k > 1$.

From our discussion, we conclude that only k can change the shape of the figure. We also conclude that the figure can have only four shapes. Namely, the shape is Figure 6(a) if $k \leq \frac{\pi}{2}-1$, Figure 6(b) if $\frac{\pi}{2}-1 < k \leq 1$, Figure 6(c) if $1 < k < \frac{4}{4-\pi}$, and Figure 6(d) if $\frac{4}{4-\pi} < k$. □

If k is smaller than $\frac{\pi}{2}-1$, then the shape of the ellipse is Figure 6(a). As k increases, the shape gradually changes from Figure 6(a) to Figure 6(b), then Figure 6(c), and eventually Figure 6(d). If k grows larger than $\frac{4}{4-\pi}$, then the shape of the ellipse is Figure 6(d), and the scale becomes larger.

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