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HAMILTONIAN SYSTEM WITH THE SUPERQUADRATIC NONLINEARITY AND THE LIMIT RELATIVE CATEGORY THEORY

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Abstract. We investigate the number of the weak periodic solutions for the bifurcation problem of the Hamiltonian system with the superquadratic nonlinearity. We get one theorem which shows the existence of at least two weak periodic solutions for this system. We obtain this result by using variational method, critical point theory induced from the limit relative category theory.

1. Introduction

Let $G(t, z(t))$ be a C^2 function defined on $R^1 \times R^{2n}$ which is $2\pi-$periodic with respect to the first variable t and $\lambda \in R$. In this paper we investigate the multiplicity of the 2π-periodic solutions for the bifurcation problem of the following Hamiltonian system

$$\dot{p(t)} = -\lambda q(t) - G_q(t, p(t), q(t)), \tag{1.1}$$

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$$\dot{q(t)} = \lambda p(t) + G_p(t, p(t), q(t)),$$

where p, $q \in R^n$. Let $z = (p, q)$ and J be the standard symplectic structure on R^{2n}, i. e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Then (1.1) can be rewritten as

$$-J\dot{z} = \lambda z + G_z(t, z(t)), \tag{1.2}$$

where $\dot{z} = \frac{dz}{dt}$ and G_z is the gradient of G. We assume that $G \in C^2(R^1 \times R^{2n}, R^1)$ satisfies the following conditions:
$(G1)$ $G \in C^2(R^1 \times R^{2n}, R)$, $G(0, \theta) = 0$, where $\theta = (0, \cdots, 0)$.
$(G2)$ There exist $1 < p_1 \leq p_2 < 2p_1 + 1$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 \geq 0$ such that

$$\alpha_1 \|z(t)\|_{R^{2n}}{}^{p_1+1} - \beta_1 \leq G(t, z(t)) \leq \alpha_2 \|z(t)\|_{R^{2n}}{}^{p_2+1}, \qquad \text{for every } z \in R^{2n}.$$

$(G3)$ G is $2\pi-$periodic function with respect to t.

Several authors ([1], [5], [6], [7] etc.) studied the nonlinear Hamiltonian system. Jung and Choi ([5], [6], [7]) considered (1.1) with nonsingular potential nonlinearity or jumping nonlinearity crossing one eigenvalue, or two eigenvalues, or several eigenvalues. Chang ([1]) proved that (1.1) has at least two nontrivial $2\pi-$periodic weak solutions under some asymptotic nonlinearity. Jung and Choi ([5]) proved that (1.1) has at least m weak solutions, which are geometrically distinct and nonconstant under some jumping nonlinearity.

We are looking for the weak solutions of (1.1) under the conditions $(G1)$-$(G3)$. The 2π-periodic weak solution $z = (p, q) \in E$ of (1.1) satisfies

$$\int_0^{2\pi} (\dot{z} - \lambda z(t) - JG_z(t, z(t))) \cdot Jw dt = 0 \qquad \text{for all } w \in E,$$

i.e.,

$$\int_0^{2\pi} [(\dot{p} + \lambda q(t) + G_q(t, z(t))) \cdot \psi - (\dot{q} - \lambda p(t) - G_p(t, z(t))) \cdot \phi] dt = 0$$
$$\text{for all } \zeta = (\phi, \psi) \in E,$$

where E is introduced in section 2.

Our main result is as follows:

THEOREM 1.1. *Assume that G satisfies the conditions $(G1)-(G3)$, and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then there exists a small number $\delta > 0$ such that for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$, system (1.1) has at least two nontrivial 2π-periodic solutions.*

The outline of the proof of Theorem 1.1 is as follows: In section 2, we introduce the perturbed operator $A_\epsilon = \epsilon I + A$ of the operator A $(A(z(t)) = -J\dot{z}(t))$ because of the lack of the compactness of the operator A^{-1}, approach the variational method, obtain some results on the corresponding functional of the perturbed problem $A_\epsilon(z) = \lambda z + \epsilon z + G_z(t, z(t))$ of (1.1) and recall the critical point theory induced from the limit relative category, which plays a crucial role to prove the multiplicity result. In section 3, we prove Theorem 1.1 by the critical point theory induced from the limit relative category.

2. Variational Formulation

Let $L^2([0, 2\pi], R^{2n})$ denote the set of $2n$-tuples of the square integrable $2\pi-$periodic functions and choose $z \in L^2([0, 2\pi], R^{2n})$. Then it has a Fourier expansion $z(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{ikt}$, with $a_k = \frac{1}{2\pi}\int_0^{2\pi} z(t)e^{-ikt}dt \in C^{2n}$, $a_{-k} = \bar{a_k}$ and $\sum_{k\in Z} |a_k|^2 < \infty$. Let

$$A : z(t) \mapsto -J\dot{z}(t)$$

with domain

$$\begin{aligned} D(A) &= \{z(t) \in H^1([0, 2\pi], R^{2n})|\ z(0) = z(2\pi)\} \\ &= \{z(t) \in L^2([0, 2\pi], R^{2n})|\ \sum_{k\in Z}(\epsilon + |k|)^2|a_k|^2 < +\infty\}, \end{aligned}$$

where ϵ is a positive small number. Then A is self-adjoint operator. Let $\{M_\lambda\}$ be the spectral resolution of A, and let τ be a positive number such that $\tau \notin \sigma(A)$ and $[-\tau, \tau]$ contains only one element 0 of $\sigma(A)$. Let

$$P_0 = \int_{-\tau}^{\tau} dM_\lambda, \qquad P_+ = \int_{\tau}^{+\infty} dM_\lambda, \qquad P_- = \int_{-\infty}^{-\tau} dM_\lambda,$$

$$P_{(-\infty, j_1-1]} = \int_{-\infty}^{j_1-1} dM_\lambda, \qquad P_{[j_1, j_0]} = \int_{j_1}^{j_0} dM_\lambda.$$

Let

$$L_0 = P_0L^2([0, 2\pi], R^{2n}), L_+ = P_+L^2([0, 2\pi], R^{2n}), L_- = P_-L^2([0, 2\pi], R^{2n}),$$

$L_{(-\infty,j_1-1]} = P_{(-\infty,j_1-1]}L^2([0,2\pi],R^{2n}), \quad L_{[j_1,j_0]} = P_{[j_1,j_0]}L^2([0,2\pi],R^{2n}).$
For each $u \in L^2([0,2\pi],R^{2n})$, we have the composition

$$u = u_0 + u_+ + u_-,$$

where $u_0 \in L_0$, $u_+ \in L_+$, $u_- \in L_-$. According to A, there exists a small number $\epsilon > 0$ such that $-\epsilon \notin \sigma(A)$. Let us define the space E as follows:

$$E = D(|A|^{\frac{1}{2}}) = \{z \in L^2([0,2\pi],R^{2n})| \sum_{k\in Z}(\epsilon + |k|)|a_k|^2 < \infty\}$$

with the scalar product

$$(z,w)_E = \epsilon(z,w)_{L^2} + (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_{L^2}$$

and the norm

$$\|z\| = (z,z)_E^{\frac{1}{2}} = (\sum_{k\in Z}(\epsilon + |k|)|a_k|^2)^{\frac{1}{2}}.$$

The space E endowed with this norm is a real Hilbert space continuously embedded in $L^2([0,2\pi],R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-\frac{1}{2},2}([0,2\pi],R^{2n})$. We note that the operator $(\epsilon + |A|)^{-1}$ is a compact linear operator from $L^2([0,2\pi],R^{2n})$ to E such that

$$((\epsilon + |A|)^{-1}w, z)_E = \int_0^{2\pi}(w(t),z(t))dt.$$

Let

$$A_\epsilon = \epsilon I + A.$$

Let

$$E_0 = |A_\epsilon|^{-\frac{1}{2}}L_0, \quad E_+ = |A_\epsilon|^{-\frac{1}{2}}L_+, \quad E_- = |A_\epsilon|^{-\frac{1}{2}}L_-,$$

$$E_{(-\infty,j_1-1]} = |A_\epsilon|^{-\frac{1}{2}}L_{(-\infty,j_1-1]}, \qquad E_{[j_1,j_0]} = |A_\epsilon|^{-\frac{1}{2}}L_{[j_1,j_0]}.$$

Then $E = E_0 \oplus E_+ \oplus E_-$, and for $z \in E$, z has the decomposition $z = z_0 + z_+ + z_- \in E$, where

$$z_0 = |A_\epsilon|^{-\frac{1}{2}}u_0, \qquad z_+ = |A_\epsilon|^{-\frac{1}{2}}u_+, \qquad z_- = |A_\epsilon|^{-\frac{1}{2}}u_-. \tag{2.1}$$

Thus we have

$$\|z_0\|_{E_0} = \|u_0\|_{L_0}, \qquad \|z_+\|_{E_+} = \|u_+\|_{L_+}, \qquad \|z_-\|_{E_-} = \|u_-\|_{L_-}$$

and that E_0, E_+, E_- are isomorphic to L_0, L_+, L_-, respectively. Moreover $E = E_{(-\infty,j_1-1]} \oplus E_{[j_1,j_2]} \oplus E_+$. Let us define a functional

$$f(u) = \frac{1}{2}(\|u_+\|_{L^2}^2 + \|M_+u_0\|_{L^2}^2 - \|M_-u_0\|_{L^2}^2 - \|u_-\|_{L^2}^2) - \psi_\epsilon(z),$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2}\|z\|_{L^2}^2$, $\psi(z) = \int_0^{2\pi}[\frac{\lambda}{2}z(t)^2 + G(t,z(t))]dt$, $M_+ = \int_0^\infty dM_\lambda$, $M_- = \int_{-\infty}^0 dM_\lambda$. By $G \in C^2$, $\psi(z) = \int_0^{2\pi}[\frac{\lambda}{2}z(t)^2 + G(t,z(t))] \in C^2$. Let

$$F(z) = \lambda z(t) + G_z(t,z(t)), \qquad F_\epsilon(z) = \epsilon z + F(z) = \epsilon z + \lambda z(t) + G_z(t,z(t)).$$

Then (1.2) can be rewritten as

$$A_\epsilon(z) = F_\epsilon(z). \tag{2.2}$$

The Euler equation of the functional $f(u)$ is the system

$$u_+ = |A_\epsilon|^{-\frac{1}{2}}P_+F_\epsilon(z), \tag{2.3}$$

$$u_- = -|A_\epsilon|^{-\frac{1}{2}}P_-F_\epsilon(z), \tag{2.4}$$

$$M_+u_0 = |A_\epsilon|^{-\frac{1}{2}}M_+P_0F_\epsilon(z) \qquad M_-u_0 = -|A_\epsilon|^{-\frac{1}{2}}M_-P_0F_\epsilon(z). \tag{2.5}$$

The system (2.3)-(2.5) is reduced to

$$A_\epsilon z_+ = P_+F_\epsilon(z_0+z_++z_-) \quad \text{or} \quad z_+ = (A_\epsilon)^{-1}P_+F_\epsilon(z_0+z_++z_-), \tag{2.6}$$

$$A_\epsilon z_- = P_-F_\epsilon(z_0+z_++z_-) \quad \text{or} \quad z_- = (A_\epsilon)^{-1}P_-F_\epsilon(z_0+z_++z_-), \tag{2.7}$$

$$A_\epsilon M_+z_0 = M_+P_0F_\epsilon(z_0+z_++z_-),\ A_\epsilon M_-z_0 = M_-P_0F_\epsilon(z_0+z_++z_-), \tag{2.8}$$

It follows from (2.3)-(2.8) that $z = z_0 + z_+ + z_-$ is a solution of (1.2) if and only if $u = u_0 + u_+ + u_-$ is a critical point of f. By (2.1), we define a functional

$$I(z) = f(u(z)).$$

The functional $I(z)$ is of the form

$$I(z) = \frac{1}{2}(\||A_\epsilon|^{\frac{1}{2}}z_+\|_{L^2}^2 + \||A_\epsilon|^{\frac{1}{2}}M_+z_0\|_{L^2}^2 - \|(-|A_\epsilon|)^{\frac{1}{2}}z_-\|_{L^2}^2 \\ -\|(-|A_\epsilon|)^{\frac{1}{2}}M_-z_0\|_{L^2}^2) - \psi_\epsilon(z).$$

Thus it suffices to find the critical points of the functional I to find the critical points of the functional f. By the following Lemma 2.1, the weak solutions of (2.2) coincide with the critical points of the functional $I(z)$.

LEMMA 2.1. *Assume that G satisfies the conditions $(G1)-(G3)$ and $\lambda \notin \sigma(A)$, $\lambda < 0$. Then $I(z)$ is continuous and Fréchet differentiable in E with Fréchet derivative*

$$DI(z)w = \int_0^{2\pi} (JA_\epsilon z(t) - (\lambda+\epsilon)Jz(t) - JG_z(t, z(t))) \cdot Jw dt \text{ for all } w \in E, \tag{2.9}$$

Moreover $DI \in C$. That is, $I \in C^1$.

Proof. First we prove that $I(z) = \int_0^{2\pi} [\frac{1}{2}A_\epsilon z - \frac{\epsilon+\lambda}{2}z^2 - G(t, z(t))]dt$ is continuous in E. For $z, w \in E$,

$$\begin{aligned}
& |I(z+w) - I(z)| \\
= \; & |\int_0^{2\pi} \frac{1}{2}A_\epsilon(z+w)\cdot(z+w) - \int_0^{2\pi} [G(t, z+w) + \frac{\epsilon+\lambda}{2}(z+w)^2] \\
& - \int_0^{2\pi} \frac{1}{2}A_\epsilon(z)\cdot z + \int_0^{2\pi} [G(t,z) + \frac{\epsilon+\lambda}{2}z^2] \\
= \; & |\int_0^{2\pi} \frac{1}{2}[A_\epsilon(z)\cdot w + A_\epsilon(w)\cdot z + A_\epsilon(w)\cdot w] \\
& - \int_0^{2\pi} [G(t, z+w) - G(t,z) + \frac{\epsilon+\lambda}{2}(2z\cdot w + w^2))]|.
\end{aligned}$$

We note that

$$|\int_0^{2\pi} \frac{1}{2}[A_\epsilon(z)\cdot w + A_\epsilon(w)\cdot z + A_\epsilon(w)\cdot w| = O(\|w\|_{R^{2n}})$$

and

$$\begin{aligned}
|\int_0^{2\pi} [G(t, z+w) - G(t,z)]dt| &\le |\int_0^{2\pi} [G_z(t, z(t))\cdot w + O(\|w\|_{R^{2n}})]dt| \\
&= O(\|w\|_{R^{2n}}).
\end{aligned} \tag{2.10}$$

Thus we have

$$|I(z+w) - I(z)| = O(\|w\|_{R^{2n}}).$$

Next we shall prove that $I(z)$ is *Fréchet* differentiable in E. For $z, w \in E$,

$$\begin{aligned}
& |I(z+w) - I(z) - DI(z)w| \\
= \; & |\int_0^{2\pi} \frac{1}{2} A_\epsilon(z+w)\cdot(z+w) - \int_0^{2\pi} [G(t,z+w) + \frac{\epsilon+\lambda}{2}(z+w)^2] \\
& - \int_0^{2\pi} \frac{1}{2} A_\epsilon(z)\cdot z + \int_0^{2\pi} [G(t,z) + \frac{\epsilon+\lambda}{2} z^2] \\
& - \int_0^{2\pi} JA_\epsilon(z)\cdot Jw + \int_0^{2\pi} [JG_z(t,z)\cdot Jw + (\epsilon+\lambda) Jz\cdot Jw]| \\
= \; & |\int_0^{2\pi} \frac{1}{2}[A_\epsilon(w)\cdot z + A_\epsilon(w)\cdot w] \\
& - \int_0^{2\pi} [G(t,z+w) - G(t,z) - G_z(t,z)\cdot w + \frac{\epsilon+\lambda}{2} w^2)]|.
\end{aligned}$$

By (2.10), we have

$$\int_0^{2\pi} [G(t,z+w) - G(t,z) - G_z(t,z)]dt = O(\|w\|_{R^{2n}}).$$

Thus

$$|I(z+w) - I(z) - DI(z)w| = O(\|w\|_{R^{2n}}).$$

□

Now, we recall the critical point theory on the manifold with boundary. Let E be a Hilbert space and M be the closure of an open subset of E such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of the critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)_c^*$ condition and the limit relative category (see [4]).

Definition 2.1. If $z \in M$, the lower gradient of f on M at z is defined by

$$grad_M^- f(z) = \begin{cases} Df(z) & \text{if } z \in int(M), \\ Df(z) + [< Df(z), \nu(z) >]^- \nu(z) & \text{if } z \in \partial M, \end{cases} \tag{2.11}$$

where we denote by $\nu(z)$ the unit normal vector to ∂M at the point z, pointing outwards. We say that z is a lower critical for f on M, if

$grad_M^- f(z) = 0$.
Since the functional $I(z)$ is strongly indefinite, the notion of the $(P.S.)_c^*$ condition and the limit relative category is a very useful tool for the proof of the main theorems.
Let $(E_n)_n$ be a sequence of closed subspaces of E with the conditions:

$$E_n = E_{n-} \oplus E_0 \oplus E_{n+}, \text{ where } E_{n+} \subset E_+, \ E_{n-} \subset E_- \text{ for all } n, \quad (2.12)$$

$$(E_{n+} \text{ and } E_{n-} \text{ are subspaces of } E),$$

$$\dim E_n < +\infty, E_n \subset E_{n+1}, \cup_{n \in N} E_n \text{ is dense in } E.$$

Let P_{E_n} be the orthogonal projections from E onto E_n. $M_n = M \cap E_n$, for any n, be the closure of an open subset of E_n and has the structure of a C^2 manifold with boundary in E_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset E$, we will write $B_n = B \cap E_n$.

Definition 2.2. Let $c \in R$. We say that f satisfies the $(P.S.)_c^*$ condition with respect to $(M_n)_n$, on the manifold M with boundary, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $f(u_n) \to c$, $grad_{M_{k_n}}^- f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $grad_M^- f(u) = 0$.

Let Y be a closed subspace of M.

Definition 2.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets U_0, $U_1, \ldots$, U_h with the following properties:
$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{array}{rcll} F(x,0) & = & x & \forall x \in U_0, \\ F(x,t) & \in & Y & \forall x \in Y, \ \forall t \in [0,1], \\ F(x,1) & \in & Y & \forall x \in U_0. \end{array}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Definition 2.4. Let (X, Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X, Y), with respect to $(X_n)_n$, by

$$cat^*_{(X,Y)}(B) = \limsup_{n\to\infty} cat_{(X_n,Y_n)}(B_n). \tag{2.13}$$

Now we consider a theorem which gives an estimate of the number of critical points of a functional, in terms of the limit relative category of its sublevels. The theorem is proved repeating the classical arguments, using the nonsmooth version of the classical Deformation Lemma for functions on manifolds with boundary.

Let Y be a fixed subset of M. We set

$$\mathcal{B}_{\mathrm{i}} = \{\mathrm{B} \subset \mathrm{M} |\ \mathrm{cat}^*_{(\mathrm{M},\mathrm{Y})}(\mathrm{B}) \geq \mathrm{i}\}, \tag{2.14}$$

$$c_i = \inf_{B\in\mathcal{B}_{\mathrm{i}}} \sup_{x\in B} f(x). \tag{2.15}$$

We have the following multiplicity theorem.

Theorem 2.1. *Let $i \in N$ and assume that*
(1) $c_i < +\infty$,
(2) $\sup_{x\in Y} f(x) < c_i$,
(3) *the $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.*
Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad^-_M f(x) = 0\}) \geq k.$$

Proof. Let $c = c_i$; using the $(P.S.)^*_c$ condition, with respect to $(M_n)_n$, one can prove that, for any neighborhood N of

$$K_c = \{x|\ f(x) = c,\ \ grad^-_M f(x) = 0\},$$

there exist n_0 in N and $\delta > 0$ such that $\|grad^-_M\| \geq \delta$ for all $n \geq n_0$ and all $x \in E_n \backslash N$ with $c - \delta \leq f(x) \leq c + \delta$. Moreover it is not difficult to see that, for all n, the function $\tilde{f}_n : E_n \to R \cup \{+\infty\}$ defined by $\tilde{f}_n = f(x)$, if $x \in M_n$, $\tilde{f}_n(x) = +\infty$, otherwise, is ϕ-convex of order two, according to the definitions of [7]. Then the conclusion follows using the same arguments of [1, 8] and the nonsmooth version of the classical Deformation Lemma. □

Now we state the following multiplicity result (for the proof see Theorem 4.6 of [8]) which will be used in the proof of our main theorem.

THEOREM 2.2. *Let E be a Hilbert space and let $E = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of E with X_1, X_2 of finite dimension. For a given subspace X of E, let P_X be the orthogonal projection from E onto X. Set*

$$C = \{x \in E |\ \|P_{X_2}x\| \geq 1\}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $R_1 > 0$. We define

$$\begin{aligned}
\Delta &= \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, 1 \leq \|x_2\| \leq R\}, \\
\Sigma &= \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = 1\} \\
&\quad \cup \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = R\} \\
&\quad \cup \{x_1 + x_2 |\ x_1 \in X_1, x_2 \in X_2, \|x_1\| = R_1, 1 \leq \|x_2\| \leq R\}, \\
S &= \{x \in X_2 \oplus X_3 |\ \|x\| = \rho\}, \\
B &= \{x \in X_2 \oplus X_3 |\ \|x\| \leq \rho\}.
\end{aligned}$$

(a) Assume that

$$\sup f(\Sigma) < \inf f(S)$$

and (b) that the $(P.S.)_c$ condition holds for f on C, with respect to the sequence $(C_n)_n$, $\forall c \in [\alpha, \beta]$, where

$$\alpha = \inf f(S), \qquad \beta = \sup f(\Delta).$$

(c) Moreover we assume $\beta < +\infty$ and $f|_{X_1 \oplus X_3}$ has no critical points z in $X_1 \oplus X_3$ with $\alpha \leq f(z) \leq \beta$.
Then there exist two lower critical points z_1, z_2 for f on C such that $\alpha \leq f(z_i) \leq \beta$, $i = 1.2$.

3. Proof of Theorem 1.1

Throughout this section we assume that G satisfies the conditions $(G1) - (G3)$, $\lambda \notin Z$ and $\lambda < 0$. Let $I(z)$ be the functional defined in section 2, i.e.,

$$\begin{aligned}
I(z) = \frac{1}{2}(\||A_\epsilon|^{\frac{1}{2}} z_+\|_{L^2}^2 + \||A_\epsilon|^{\frac{1}{2}} M_+ z_0\|_{L^2}^2 - \|(-|A_\epsilon|)^{\frac{1}{2}} z_-\|_{L^2}^2 \\
- \|(-|A_\epsilon|)^{\frac{1}{2}} M_- z_0\|_{L^2}^2) - \psi_\epsilon(z),
\end{aligned}$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2}\|z\|^2_{L^2}$, $\psi(z) = \int_0^{2\pi}[\frac{\lambda}{2}z(t)^2 + G(t, z(t))]dt$, $M_+ = \int_0^\infty dM_\lambda$, $M_- = \int_{-\infty}^0 dM_\lambda$.

We shall show that the functional $I(z)$ satisfies the geometric assumptions of Theorem 2.2.

Lemma 3.1. *((P.S.)* condition)*
Assume that G satisfies the conditions $(G1) - (G3)$ and $\lambda \notin Z$. Then $I(z)$ satisfies the $(P.S.)^*_\gamma$ condition with respect to $(E_n)_n$, for any $\gamma \in R$.

Proof. Let $(k_n)_n$ and $(z_n)_n$ be two sequences such that $k_n \to +\infty$, and for any sequence $(z_n)_n$ with $z_n \in E_{k_n}$,

$$I(z_n) \to \gamma$$

and

$$DI_{k_n}(z_n) \to \theta,$$

where I_{k_n} is a restriction of I on E_{k_n} and $\theta = (0, \cdots, 0)$. It follows from $DI_{k_n}(z_n) \to 0$ that

$$P_{E_{k_n}} z_n - P_{E_{k_n}} A_\epsilon^{-1}((\lambda + \epsilon)z_n + G_z(t, z_n(t))) \longrightarrow \theta,$$

where A_ϵ^{-1} is a compact operator. We claim that $(z_n)_n$ is bounded. By contradiction, we suppose that $\|z_n\| \to \infty$. If $w_n = \frac{z_n}{\|z_n\|}$, we can suppose that $w_n \rightharpoonup w_0$ weakly for some $w_0 \in E$. We have

$$0 \longleftarrow \langle P_{E_{k_n}} w_n - A_\epsilon^{-1} P_{E_{k_n}}((\lambda + \epsilon)w_n + \frac{G_z(t, z_n(t))}{\|z_n\|}), w_n\rangle$$

$$= P_{E_{k_n}}\langle w_n, w_n\rangle - \langle A_\epsilon^{-1} P_{E_{k_n}}((\lambda + \epsilon)w_n + \frac{G_z(t, z_n(t))}{\|z_n\|}), w_n\rangle.$$

Since A_ϵ^{-1} is a compact operator, $(\lambda+\epsilon)w_n$ is bounded and $\frac{G_z(t,z_n(t))}{\|z_n\|} \to 0$, $A_\epsilon^{-1}(P_{E_{k_n}}((\lambda + \epsilon)w_n + \frac{G_z(t,z_n(t))}{\|z_n\|}))$ converges to $A_\epsilon^{-1}((\lambda + \epsilon)w_0)$ and we have

$$0 = \langle w_0, w_0\rangle - \langle A_\epsilon^{-1}((\lambda + \epsilon)w_0), w_0\rangle = \|w_0\|^2 - \langle A_\epsilon^{-1}(\lambda + \epsilon))w_0, w_0\rangle,$$

from which w_0 is a solution of the equation

$$A_\epsilon w = (\lambda + \epsilon)w.$$

Since $\lambda \notin \sigma(A)$, $w_0 = 0$, which is a contradiction to the fact that $\|w_0\| = 1$. Thus $(z_n)_n$ is bounded. We can suppose that $z_n \rightharpoonup z_0$ weakly in E,

for some z_0 in E. We have

$$\langle P_{E_{k_n}} DI(z_n), P_+ z_n + P_- z_n\rangle = \|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2$$
$$- P_{E_{k_n}} \int_0^{2\pi} (\lambda z_n(t) + \epsilon z_n(t) + G_z(t, z_n))(P_+ z_n + P_- z_n) \to 0.$$

By $(G1)$ and the boundedness of $G_z(t, z_n) \cdot (P_+ z_n + P_- z_n)$,

$$\lim_{n\to\infty} \|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2 = \int_0^{2\pi} (\lambda z(t) + \epsilon z(t) + G_z(t, z))z,$$

i.e., $\|P_{E_{k_n}} P_+ z_n\|^2 - \|P_{E_{k_n}} P_- z_n\|^2$ converges strongly, which implies that, up to a subsequence, $P_{E_{k_n}} z_n$ converges strongly to z, and we prove the lemma and have

$$DI(z) = \lim_{n\to\infty} P_{E_{k_n}} DI(z_n) = 0,$$

so z is the critical point of I. □

Let us set

$$X_1 = E_{(-\infty, j_1-1]}, \qquad X_2 = E_{[j_1, j_0]}, \qquad X_3 = E_+.$$

Then E is the topological direct sum of the subspaces X_1, X_2 and X_3. Let P_X be the orthogonal projection from E onto X. Let us set

$$C = \{z \in E |\ \|P_{X_2} z\| \geq 1\} \tag{3.1}$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap E_n$. Let us define a functional $\Psi : E \setminus \{X_1 \oplus X_3\} \to E$ by

$$\Psi(z) = z - \frac{P_{X_2} z}{\|P_{X_2} z\|} = P_{X_1 \oplus X_3} z + (1 - \frac{1}{\|P_{X_2} z\|}) P_{X_2} z. \tag{3.2}$$

We have

$$\nabla\Psi(z) \cdot w = w - \frac{1}{\|P_{X_2} z\|}(P_{X_2} w - \langle \frac{P_{X_2} z}{\|P_{X_2} z\|}, w\rangle \frac{P_{X_2} z}{\|P_{X_2} z\|}). \tag{3.3}$$

Let us define the constrained functional $\tilde{I} : C \to R$ by

$$\tilde{I} = I \circ \Psi. \tag{3.4}$$

Then $\tilde{I} \in C^{1,1}_{loc}$. It turns out that

$$\mathrm{grad}^-_C \tilde{I}(\tilde{z}) = \begin{cases} P_{X_1 \oplus X_3} DI(z) + (1 - \frac{1}{\|P_{X_2}\tilde{z}\|_E}) P_{X_2} DI(z) & \text{if } z \in \mathrm{int}(C), \\ P_{X_1 \oplus X_3} DI(z) - \langle DI(z), \frac{P_{X_2}\tilde{z}}{\|P_{X_2}\tilde{z}\|_E}\rangle^+ \frac{P_{X_2}\tilde{z}}{\|P_{X_2}\tilde{z}\|_E} & \text{if } z \in \partial C. \end{cases} \tag{3.5}$$

We note that if $\tilde{z}$ is the critical point of $\tilde{I}$ and lies in the interior of C, then $z = \Psi(\tilde{z})$ is the critical point of I. Thus it suffices to find the critical points, which lies in the interior of C, for $\tilde{I}$. We also note that

$$\|grad_C^- \tilde{I}(\tilde{z})\|_E \geq \|P_{X_1 \oplus X_3} DI(\Psi(\tilde{z}))\|_E \qquad \forall \tilde{z} \in \partial C. \tag{3.6}$$

Let us set

$$\begin{aligned}
S_{23}(\rho) &= \{z \in X_2 \oplus X_3 |\ \|z\|_E = \rho\},\ \rho > 0,\\
\tilde{S_{23}}(\rho) &= \Psi^{-1}(S_{23}(\rho)),\\
\Delta_{12}(R, R_1) &= \{z_1 + z_2 |\ z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, 1 \leq \|z_2\|_E \leq R\},\\
\Delta_{12}(\tilde{R}, R_1) &= \Psi^{-1}(\Delta_{12}(R, R_1))\\
\Sigma_{12}(R, R_1) &= \{z_1 + z_2 |\ z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, \|z_2\|_E = 1\}\\
&\quad \cup \{z_1 + z_2 |\ z_1 \in X_1, z_2 \in X_2, \|z_1\|_E \leq R_1, \|z_2\|_E = R\}\\
&\quad \cup \{z_1 + z_2 |\ z_1 \in X_1, z_2 \in X_2, \|z_1\|_E = R_1, 1 \leq \|z_2\|_E \leq R\},\\
\Sigma_{12}(\tilde{R}, R_1) &= \Psi^{-1}(\Sigma_{12}(R, R_1)).
\end{aligned}$$

We will prove the multiplicity result by using Theorem 2.2 for $\tilde{I}$, C, $\tilde{S_{23}}(\rho)$, $\Delta_{12}(\tilde{R}, R_1)$ and $\Sigma_{12}(\tilde{R}, R_1)$. Now we have the following linking geometry for $\tilde{I}$.

Lemma 3.2. *Assume that G satisfies the conditions $(G1) - (G3)$, and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then there exist a small number $\delta_1 > 0$, $R > \rho > 0$, $R_1 > 0$, $R > 1$ and $\rho > 0$ with $R > \rho$ such that for any λ with $j_1 - \delta_1 < \lambda < j_1 < j_0 < 0$,*

$$\sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < 0 < \inf_{\tilde{w} \in \tilde{S_{23}}(\rho)} \tilde{I}(\tilde{w}). \tag{3.7}$$

Moreover

$$-\infty < \inf_{\tilde{w} \in \tilde{B_{23}}(\rho)} \tilde{I}(\tilde{w}), \qquad \sup_{\tilde{z} \in \Delta_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < \infty.$$

Proof. It suffices to show that there exist $R > \rho > 0$, $R_1 > 0$ and $R > 1$ such that for $z = \psi(\tilde{z})$, $w = \psi(\tilde{w})$,

$$\sup_{z \in \Sigma_{12}(R, R_1)} I(z) < \inf_{w \in S_{23}(\rho)} I(w).$$

because

$$\sup_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} \tilde{I}(\tilde{z}) = \sup_{z\in\Sigma_{12}(R,R_1)} I(z), \qquad \inf_{\tilde{w}\in S_{23}(\rho)} \tilde{I}(\tilde{w}) = \inf_{w\in S_{23}(\rho)} I(w). \tag{3.8}$$

Let $z = z_1 + z_2 \in X_1 \oplus X_2$. By $(G2)$ we have

$$\begin{aligned} I(z) &= \frac{1}{2}(\||A_\epsilon|^{\frac{1}{2}} z_+\|_{L^2}^2 + \||A_\epsilon|^{\frac{1}{2}} M_+ z_0\|^2 - \|(-|A_\epsilon|)^{\frac{1}{2}} z_-\|_{L^2}^2 \\ &\quad -\|(-|A_\epsilon|)^{\frac{1}{2}} M_- z_0\|^2) - \int_0^{2\pi} [\frac{1}{2}(\lambda+\epsilon)z^2 + G(t,z(t))]dt \\ &\le \frac{j_1+\epsilon-\lambda-\epsilon}{2}\|z_2\|_{L^2}^2 + \frac{j_0+\epsilon-\lambda-\epsilon}{2}\|z_2\|_{L^2}^2 \\ &\quad - \int_0^{2\pi} [\alpha_1|z|^{p_1+1} - \beta_1]dt. \\ &\le \frac{\delta_1}{2}\|z_2\|_{L^2}^2 + \frac{j_0+\epsilon-\lambda-\epsilon}{2}\|z_2\|_{L^2}^2 - \int_0^{2\pi} [\alpha_1|z|^{p_1+1} - \beta_1]dt. \end{aligned}$$

Since $j_0-\lambda > 0$ and $p_1+1 > 2$, there exist a small number $\delta_1 > 0$, $R > 0$, $R_1 > 0$ and $R > 1$ such that for $z \in X_1 \oplus X_2$, $\sup_{z\in\Sigma_{12}(R,R_1)} I(z) < 0$. If $z \in \Delta_{12}(R,R_1)$, then $\sup_{z\in\Delta_{12}(R,R_1)} I(z) < \frac{\delta_1}{2}\|z_2\|_{L^2}^2 + \frac{j_0+\epsilon-\lambda-\epsilon}{2}\|z_2\|_{L^2}^2 + 2\pi\beta_1 < \infty$. On the other hand, if $z \in X_2 \oplus X_3$, then

$$\begin{aligned} I(z) &= \frac{1}{2}(\||A_\epsilon|^{\frac{1}{2}} z_+\|_{L^2}^2 + \||A_\epsilon|^{\frac{1}{2}} M_+ z_0\|^2 - \|(-|A_\epsilon|)^{\frac{1}{2}} z_-\|_{L^2}^2 \\ &\quad -\|(-|A_\epsilon|)^{\frac{1}{2}} M_- z_0\|^2) - \int_0^{2\pi} [\frac{1}{2}(\lambda+\epsilon)z^2 + G(t,z(t))]dt \\ &\ge \frac{j_1+\epsilon-\lambda-\epsilon}{2}\|z\|_{L^2}^2 - \int_0^{2\pi} \alpha_2|z|^{p_2+1}dt. \end{aligned}$$

Since $j_1 - \lambda > 0$ and $p_2 + 1 > 2$, there exists a small number $\rho > 0$ with $R > \rho > 0$ such that for $z \in X_2 \oplus X_3$, $inf_{z\in S_{23}(\rho)} I(z) > 0$. If $z \in B_{23}(\rho)$, then $inf_{z\in B_{23}(\rho)} I(z) > -\infty$. Thus we prove the lemma. □

Lemma 3.3. *Assume that G satisfies the conditions $(G1)-(G3)$, and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Then for any λ with $j_1 - 1 \le \lambda < 0$, $\tilde{I}$ has no critical point $\tilde{z}$ such that $\tilde{I}(\tilde{z}) = c$ and $\tilde{z} \in \partial C$, where $\inf_{\tilde{z}\in B_{23}(\rho)} \tilde{I}(\tilde{z}) \le c \le \sup_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} \tilde{I}(\tilde{z}) < 0$.*

Proof. It suffices to prove that I has no critical point $z = \psi(\tilde{z})$ such that $I(z) = c$ and $z \in X_1 \oplus X_3$. We notice that from Lemma 3.2, for

fixed $z_1 \in X_1$, *the functional* $z_3 \mapsto I(z_1 + z_3)$ *is weakly convex in* X_3, *while, for fixed* $z_3 \in X_3$, *the functional* $z_1 \mapsto I(z_1+z_3)$ *is strictly concave in* X_1. *Moreover* $(0,0)$ *is a critical point in* $X_1 \oplus X_3$ *with* $I(0,0) = 0$. *So if* $z = z_1 + z_3$ *is another critical point for* $I|_{X_1 \oplus X_3}$, *then we have*

$$0 = I(0,0) \leq I(z_3) \leq I(z_1 + z_3) \leq I(z_1) \leq I(0,0) = 0.$$

So $I(z_1 + z_3) = I(0,0) = 0$. □

LEMMA 3.4. *Assume that* G *satisfies the conditions* $(G1) - (G3)$, *and that* j_0, j_1 *are negative integers with* $j_1 < j_0 < 0$. *Then there exists a constant* $\delta_2 > 0$ *such that for any* λ *with* $j_1 - 1 - \delta_2 \leq \lambda < j_1 < j_0 < 0$, $\tilde{I}$ *has no critical point* $\tilde{z}$ *such that* $\inf_{\tilde{z} \in B_{23}(\rho)} \tilde{I}(\tilde{z}) \leq \tilde{I}(\tilde{z}) \leq \sup_{\tilde{z} \in \Sigma_{12}(\tilde{R}, R_1)} \tilde{I}(\tilde{z}) < 0$ *and* $\tilde{z} \in \partial C$.

Proof. It suffices to show that $I(z)$ has no critical point z such that $\inf_{z \in B_{23}(\rho)} I(z) \leq I(z) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0$ and $z \in X_1 \oplus X_3$. By contradiction we suppose that we can find two sequences $(\lambda_n)_n$ in R with $\lambda_n \in (j_1 - 1 - \delta_2, j_1)$ and $(z_n)_n$ in $X_1 \oplus X_3$ such that $\lambda_n \to \lambda \in [j_1 - 1, j_1)$, $\inf_{z \in B_{23}(\rho)} I(z) \leq I(z) \leq \sup_{z \in \Sigma_{12}(R, R_1)} I(z) < 0$ and $DI|_{X_1 \oplus X_3}(z_n) = 0$. We claim that $(z_n)_n$ is bounded. If not we can suppose that $\|z_n\| \to +\infty$ and set $w_n = \frac{z_n}{\|z_n\|}$. Since w_n is bounded, up to a subsequence w_n converges weakly to w_0, for some $w_0 \in X_1 \oplus X_3$. Furthermore since $P_{X_1} z_n \in E_-$, $\|P_+ P_{X_1} z_n\| = 0$ and we have

$$\begin{aligned} \langle DI(z_n), P_{X_1} z_n \rangle &= \|P_+ P_{X_1} z_n\|^2 - \|P_- P_{X_1} z_n\|^2 \\ &\quad - \langle (\lambda + \epsilon) z_n + G_z(t, z_n), P_{X_1} z_n \rangle \\ &= -\|P_- P_{X_1} z_n\|^2 - \langle (\lambda + \epsilon) z_n + G_z(t, z_n), P_{X_1} z_n \rangle \\ &\longrightarrow 0. \end{aligned} \tag{3.9}$$

Moreover since $P_{X_3} z_n \in E_+$, $\|P_- P_{X_3} z_n\| = 0$ and we have

$$\langle DI(z_n), P_{X_3} z_n \rangle = \|P_+ P_{X_3} z_n\|^2 - \langle (\lambda + \epsilon) z_n + G_z(t, z_n), P_{X_3} z_n \rangle \longrightarrow 0. \tag{3.10}$$

Adding (3.9) to (3.10), we have

$$\lim_{n \to \infty} (\|P_+ P_{X_3} z_n\|^2 - \|P_- P_{X_1} z_n\|^2) = \lim_{n \to \infty} \langle (\lambda + \epsilon) z_n + G_z(t, z_n), P_{X_1 \oplus X_3} z_n \rangle. \tag{3.11}$$

Dividing (3.11) by $\|z_n\|^2$ and going to the limit, we get

$$\|P_+ P_{X_3} w_0\|^2 - \|P_- P_{X_1} w_0\|^2 = \langle (\lambda + \epsilon) w_0, P_{X_1 \oplus X_3} w_0 \rangle, \tag{3.12}$$

from which w_0 is the unique solution of the linear equation

$$A_\epsilon z = (\lambda + \epsilon)z.$$

Since $\lambda \notin \sigma(A)$, $w_0 = 0$, which is a contradiction to the fact $\|w_0\| = 1$. Thus $(z_n)_n$ is bounded. By the same arguments used for $(w_n)_n$, we get, up to a subsequence, $(z_n)_n$ converges strongly to a point $z \in X_1 \oplus X_3$ with $\inf_{z\in B_{23}(\rho)} I(z) \le I(z) \le \sup_{z\in\Sigma_{12}(R,R_1)} I(z) < 0$ and $DI|_{X_1\oplus X_3}(z) = 0$, which contradicts Lemma 3.3. Thus we prove the lemma. □

LEMMA 3.5. *The functional $-\tilde{I}$ satisfies the $(P.S.)^*_{-\tilde{c}}$ condition with respect to $(C_n)_n$ for any $-\tilde{c}$ such that*

$$0 < \inf_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} (-\tilde{I})(\tilde{z}) \le -\tilde{c} \le \sup_{\tilde{z}\in B_{23}(\rho)} (-\tilde{I})(\tilde{z}).$$

Proof. Let $(h_n)_n$ be a sequence in N with $h_n \to +\infty$ and $(\tilde{z_n})_n$ be a sequence in C with $\tilde{z_n} \in C_{h_n}$ for all n, $-\tilde{I}(\tilde{z_n}) \to -\tilde{c}$ and $\mathrm{grad}^-_{C_{h_n}}(-\tilde{I}|_{E_{h_n}})(\tilde{z_n}) \to 0$. Set $z_n = \Psi(\tilde{z_n})$. Then $I(z_n) \to c$. We first consider the case $\tilde{z_n} \notin \partial C_{h_n}$ for large n. Since for large n $P_{E_n} \circ P_{X_2} = P_{X_2} \circ P_{E_n} = P_{X_2}$, we have

$$\mathrm{grad}^-_{C_{h_n}}(-\tilde{I})(\tilde{z_n}) = P_{E_{h_n}}\Psi'(\tilde{z_n})D(-I)(z_n) = \Psi'(\tilde{z_n})(P_{E_{h_n}}D(-I)(z_n))$$
$$= P_{E_{h_n}}P_{X_1\oplus X_3}D(-I)(z_n) + P_{E_{h_n}}(1 - \frac{1}{\|P_{X_2}\tilde{z_n}\|_E})P_{X_2}D(-I)(z_n) \to 0.$$

Thus

$$P_{X_1\oplus X_3}P_{E_{h_n}}D(-I)(z_n) \to 0 \qquad \text{and} \tag{3.13}$$
$$(1 - \frac{1}{\|P_{X_2}\tilde{z_n}\|_E})P_{X_2}D(-I)(z_n) \to 0. \tag{3.14}$$

It is impossible that $\|P_{X_2}\tilde{z_n}\|_E \to 1$ because $\mathrm{dist}(z_n, X_2) \to 0$. Thus $P_{E_{h_n}}D(-I)(z_n) \to 0$. Using $(P.S.)^*_c$ for I of Lemma 3.1 it follows that $(z_n)_n$ has a subsequence $(z_{k_n})_n$ such that $z_{k_n} \to z$ for some z in X_2. Since Ψ is invertible in $\mathrm{int}(C)$, $\tilde{z_{k_n}} \to \Psi^{-1}(z)$. Next we consider the case $\tilde{z_n} \in \partial C_{h_n}$ for infinitely many n. We claim that this case cannot occur. If $\tilde{z_n} \in \partial C_{h_n}$, then $\|P_{X_2}\tilde{z_n}\|_E = 1$. Thus we have

$$\mathrm{grad}^-_{C_{h_n}}(-\tilde{I})(\tilde{z_n}) = P_{E_{h_n}}(P_{X_1\oplus X_3}D(-I)(z_n)$$
$$- \langle D(-I)(z_n), P_{X_2}\tilde{z_n}\rangle^+ P_{X_2}\tilde{z_n}) \to 0.$$

Using the properties of the projections, we get

$$P_{E_{h_n}}P_{X_1\oplus X_3}D(-I)(z_n) \to 0,$$

which contradicts to Lemma 3.3. In fact, let $\tilde{z}$ be the limit point of the subsequence $\tilde{z_{k_n}}$ of $\tilde{z_n}$, then $\tilde{z} \in \partial C$ and

$$\mathrm{grad}_C^-(-\tilde{I})(\tilde{z}) = P_{X_1 \oplus X_3}\mathrm{grad}(-I)(z) - \langle \mathrm{grad}(-I)(z), P_{X_2}\tilde{z}\rangle P_{X_2}\tilde{z}.$$

□

Proof of Theorem 1.1

Assume that G satisfies the conditions $(G1)-(G3)$, and that j_0, j_1 are negative integers with $j_1 < j_0 < 0$. Let $\lambda \notin Z$ and $\lambda < 0$. We note that the critical points of the functional $\tilde{I}$ coincide with the critical points of the functional $-\tilde{I}$. Thus it suffices to find the number of the critical points of $-\tilde{I}$, which is appropriate functional for applying Theorem 2.2, to find the number of the critical points of I. The common part of $(j_1 - \delta_1, j_1)$ and $(j_1 - 1 - \delta_2, j_1)$ is

$$(j_1 - \delta_1, j_1),$$

where δ_1 and δ_2 are small numbers introduced in Lemma 3.2 and Lemma 3.4 respectively. Let us set

$$\delta = \delta_1.$$

By Lemma 3.2, there exist $R > \rho > 0$, $R_1 > 0$, $R > 1$ and $\rho > 0$ with $R > \rho$ such that for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$,

$$\begin{aligned}\sup_{\tilde{z}\in S_{23}^{\tilde{}}(\rho)} (-\tilde{I})(\tilde{z}) = \sup_{z\in S_{23}(\rho)} (-I)(z) < 0 < \inf_{z\in\Sigma_{12}(R,R_1)} (-I)(z) \\ = \inf_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} (-\tilde{I})(z).\end{aligned}$$

and

$$\inf_{\tilde{z}\in\Delta_{12}(\tilde{R},R_1)} (-\tilde{I})(z) = -\sup_{\tilde{z}\in\Delta_{12}(\tilde{R},R_1)} \tilde{I}(z) > -\infty,$$

$$\sup_{\tilde{z}\in B_{23}^{\tilde{}}(\rho)} (-\tilde{I})(\tilde{z}) = -\inf_{\tilde{z}\in B_{23}^{\tilde{}}(\rho)} \tilde{I}(\tilde{z}) < \infty,$$

so the condition (a) of Theorem 2.2 for the functional $-\tilde{I}$ is satisfied. By Lemma 3.5, the functional $-\tilde{I}$ satisfies the $(P.S.)^*_{-\tilde{c}}$ condition with respect to $(C_n)_n$ for any $-\tilde{c} \in [\alpha, \beta]$, where $\alpha = \inf_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)}(-\tilde{I})(\tilde{z})$ and $\beta = \sup_{\tilde{z}\in B_{23}^{\tilde{}}(\rho)}(-\tilde{I})(\tilde{z})$, so the condition (b) of Theorem 2.2 is satisfied. By Lemma 3.4, for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$, $\tilde{I}$ has no critical point $\tilde{z}$ such that $\inf_{\tilde{z}\in B_{23}^{\tilde{}}(\rho)} \tilde{I}(\tilde{z}) \le \tilde{I}(\tilde{z}) \le \sup_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} \tilde{I}(\tilde{z}) < 0$ and $\tilde{z} \in \partial C$, so the condition (c) of Theorem 2.2 is satisfied. Thus by

Theorem 2.2, for any λ with $j_1 - \delta < \lambda < j_1 < j_0 < 0$ there exists two lower critical points $\tilde{z_1}$, $\tilde{z_2}$ for $-\tilde{I}$ on C such that

$$0 < \inf_{\tilde{z}\in\Sigma_{12}(\tilde{R},R_1)} (-\tilde{I})(\tilde{z}) \leq (-\tilde{I})(\tilde{z_i}) \leq \sup_{\tilde{z}\in B_{23}(\rho)} (-\tilde{I})(\tilde{z}), \qquad i = 1.2.$$

Thus the functional I has at least two lower critical points z_1, z_2 on X_2 with

$$\inf_{z\in B_{23}(\rho)} I(z) \leq I(z_i) \leq \sup_{z\in\Sigma_{12}(R,R_1)} I(z) < 0, \qquad i = 1.2.$$

Thus system (1.2) has at least two nontrivial solutions. Thus Theorem 1.1 is proved.

References

[1] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser, (1993).

[2] M. Degiovanni, *Homotopical properties of a class of nonsmooth functions,* Ann. Mat. Pura Appl. **156** (1990), 37–71.

[3] M. Degiovanni, A. Marino, and M. Tosques, *Evolution equation with lack of convexity,* Nonlinear Anal. **9** (1985), 1401–1433.

[4] G. Fournier, D. Lupo, M. Ramos, and M. Willem, *Limit relative category and critical point theory,* Dynam. Report, **3** (1993), 1–23.

[5] T. Jung and Q. H. Choi, *On the number of the periodic solutions of the nonlinear Hamiltonian system,* Nonlinear Analysis TMA, **Vol. 71**, No. 12 e1100–e1108 (2009).

[6] T. Jung and Q. H. Choi, *Periodic solutions for the nonlinear Hamiltonian systems,* Korean J. Math. **17** (3) (2009), 331–340.

[7] T. Jung and Q. H. Choi, *Existence of four solutions of the nonlinear Hamiltonian system with nonlinearity crossing two eigenvalues,* Boundary Value Problems, **Volume 2008**, 1–17.

[8] A. M. Micheletti and A. Pistoia, *On the number of solutions for a class of fourth order elliptic problems,* Communications on Applied Nonlinear Analysis **6** (2) (1999), 49–69.

[9] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS. Regional Conf. Ser. Math., **65**, Amer. Math. Soc., Providence, Rhode Island (1986).

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