

STATISTICAL CONVERGENCE FOR GENERAL BETA OPERATORS

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ABSTRACT. In this paper, we consider general Beta operators, which is a general sequence of integral type operators including Beta function. We study the King type Beta operators which preserves the third test function x^2 . We obtain some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators.

1. Introduction

Three classical operators L_n (Bernstein operators, Szász-Mirakjan operators and Baskakov operators) preserve $e_i(x) = x^i$ ($i = 0, 1$), i.e., $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x)$, $n \in \mathbb{N}$. For each of these operators, $L_n(e_2; x) \neq e_2(x) = x^2$. In the year 2003, J. P. King [10] presented a non-trivial sequence of positive linear operators $V_n : C[0, 1] \rightarrow C[0, 1]$, given as follows:

$$V_n(f; x) = \sum_{k=0}^n \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right), 0 \leq x \leq 1,$$

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where $r_n^*(x) : [0, 1] \rightarrow [0, 1]$, are defined by

$$r_n^*(x) = \begin{cases} x^2, & n=1, \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & n=2,3,\dots \end{cases}$$

This sequence preserves the test functions e_0, e_2 and $V_n(f, x) = r_n^*(x)$ holds. Replacing $r_n^*(x)$ by e_1 , then we obtain classical Bernstein operators.

Beta operators were introduced by Lupuş [11] and further modified and studied by Khan [9], Upreti [15], Divis [5] and others.

The Beta approximation $\beta_n(f)$ to a function $f : [0, 1] \rightarrow \mathbb{R}$ is the operator:

$$(1.1) \quad \beta_n(f; x) = \frac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t) dt$$

where $B(u, v)$ is the well-known beta probability density function

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt; \quad u, v > 0,$$

with the support $(0, 1)$ such that t denotes a value of the random variable T , where $n \in \mathbb{N}$, $x \in (0, 1)$ and f is any real measurable, Lebesgue integrable function defined on $[0, 1]$. When $x = 0$ or $x = 1$, then $\beta_n(f, x) = f(x)$ for all n .

Now the following Lemmas follow from [16], for the operators β_n mentioned by (1.1).

LEMMA 1.1 ([16]). *Let $e_i(x) = x^i$, $i = 0, 1, 2$. Then, for each $0 < x < 1$ and $n \in \mathbb{N}$, we have*

- (i) $\beta_n(e_0; x) = 1$,
- (ii) $\beta_n(e_1; x) = x$,
- (iii) $\beta_n(e_2; x) = \frac{x(1+nx)}{n+1}$.

LEMMA 1.2 ([5]). *For each $0 < x < 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) = t - x$, we have $\beta_n(\varphi_x^2; x) = \frac{x(1-x)}{n+1}$.*

The aim of this article is to construct a general Beta type operators including the King type Beta operators which preserves the third test function x^2 . We study some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators than the original Beta

operators $\beta_n(f, x)$. Note that rate of convergence and statistical convergence of many other approximation operators are available in literatures(See [1], [2], [4], [6], [7], [8], [12], [13], [14]).

2. King Type Beta operators

Let $\{\alpha_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, 1]$ with $0 < \alpha_n(x) < 1$. Now consider a sequence of positive linear operators:

$$(2.1) \quad \hat{\beta}_n(f, x) = \frac{1}{B(n\alpha_n(x), n(1 - \alpha_n(x)))} \int_0^1 t^{n\alpha_n(x)-1} (1 - t)^{n(1-\alpha_n(x))-1} f(t) dt,$$

where $x \in [0, 1]$, $f \in [0, 1]$ and $n \in \mathbb{N}$ (set of natural numbers). If $\alpha_n(x)$ is replaced by e_1 , then we obtain original beta operators (1.1). Note that

LEMMA 2.1. For each $0 \leq x \leq 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) := t - x$, we have

- (i) $\hat{\beta}_n(e_0; x) = 1,$
- (ii) $\hat{\beta}_n(e_1; x) = \alpha_n(x),$
- (iii) $\hat{\beta}_n(e_2; x) = \frac{\alpha_n(x)(1 + n\alpha_n(x))}{n + 1},$
- (iv) $\hat{\beta}_n(\varphi_x^2; x) = (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}.$

Now, if we replace $\alpha_n(x)$ by

$$\alpha_n^*(x) = \frac{-1 + \sqrt{1 + 4n(n + 1)x^2}}{2n}, \quad x \in [0, 1] \text{ and } n \in \mathbb{N},$$

then the operators $\hat{\beta}_n$ defined in (2.1) reduce to the operators

$$(2.2) \quad \beta_n^*(f; x) = \frac{1}{B(n\alpha_n^*(x), n(1 - \alpha_n^*(x)))} \int_0^1 t^{n\alpha_n^*(x)-1} (1 - t)^{n(1-\alpha_n^*(x))-1} f(t) dt.$$

These operators are the King type Beta operators. Furthermore, the following Lemma hold:

LEMMA 2.2. The operators defined by (2.2) verify the following identities

- (i) $\beta_n^*(e_0; x) = 1,$

$$(ii) \beta_n^*(e_1; x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2n},$$

$$(iii) \beta_n^*(e_2; x) = x^2.$$

LEMMA 2.3. For each $0 \leq x \leq 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) = t - x$, we have

$$(i) \beta_n^*(\varphi_x; x) = \frac{\sqrt{1 + 4n(n+1)x^2} - (1 + 2nx)}{2n},$$

$$(ii) \beta_n^*(\varphi_x^2; x) = \frac{(1 + 2nx)x - x\sqrt{1 + 4n(n+1)x^2}}{n}.$$

3. Rate of Convergence

In this section we study the rate of convergence of the operators $\hat{\beta}_n(f; x)$ to $f(x)$ by means of the modulus of continuity and Peetre's K -functional. For $f \in C[a, b]$, the modulus of continuity of f , denoted by $\omega(f; \delta)$, is defined to be

$$\omega(f; \delta) = \sup_{|y-x| < \delta, x, y \in [a, b]} |f(y) - f(x)|.$$

It is known that for any $\delta > 0$ and $x, y \in [a, b]$, we have

$$|f(y) - f(x)| \leq \omega(f; \delta) \left(\frac{|y-x|}{\delta} + 1 \right).$$

THEOREM 3.1. For every $f \in C[0, 1]$ and $0 \leq x \leq 1$, we have

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq 2\omega(f, \delta_{n,x})$$

where $\delta_{n,x} := \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}}$ and $\omega(f, \delta_{n,x})$ is the modulus of continuity of f .

Proof. Let $f \in C[0, 1]$ and $x \in [0, 1]$. Since $\hat{\beta}_n(e_0, x) = e_0(x)$, from Cauchy-Schwarz inequality for linear positive operators, we obtain for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq \left[\hat{\beta}_n(e_0; x) + \frac{1}{\delta_{n,x}} \left(\hat{\beta}_n((e_1 - x)^2; x) \right)^{1/2} \right] \omega(f, \delta_{n,x}).$$

Choosing $\delta_{n,x} = \sqrt{\hat{\beta}_n((e_1 - x)^2; x)} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}}$, we obtain

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq 2\omega(f, \delta_{n,x}).$$

□

For the King type Beta operators we have the following Corollary at once:

COROLLARY 3.2. *For every $f \in C[0, 1]$ and $0 \leq x \leq 1$, we have*

$$|\beta_n^*(f; x) - f(x)| \leq 2\omega(f, \delta_{n,x})$$

where $\delta_{n,x} = \sqrt{\frac{(1+2nx)x-x\sqrt{1+4n(n+1)x^2}}{n}}$.

Now we give the rate of convergence for the operators $\hat{\beta}_n(f; x)$ by using the Peetre’s K -functional in the space $C^2[0, 1]$. We recall some definitions and notations. The classical Peetre’s K -functional of a function $f \in C[0, 1]$ is defined by

$$K(f, \delta) = \inf \left\{ \|f - g\|_{C[0,1]} + \delta \|g''\|_{C[0,1]} : g \in C^2[0, 1] \right\}, \quad \delta > 0$$

where $C^2[0, 1] = \{g \in C[0, 1] : g', g'' \in C[0, 1]\}$.
and the norm

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$

THEOREM 3.3. *For each $f \in C[0, 1]$*

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq K \left(f; \left(|\alpha_n(x) - x| + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \right) \right).$$

Proof. Applying Taylor expansion to the function $g \in C^2[0, 1]$, we get

$$\hat{\beta}_n(g, x) - g(x) = g'(x)\hat{\beta}_n((e_1 - x), x) + \frac{1}{2}\hat{\beta}_n(g''(\xi)(e_1 - x)^2, x); \xi \in (t, x).$$

Hence

$$\begin{aligned} & \left| \hat{\beta}_n(g; x) - g(x) \right| \\ & \leq \|g'\|_{C[0,1]} \left| \hat{\beta}_n((e_1 - x), x) \right| + \|g''\|_{C[0,1]} \left| \hat{\beta}_n((e_1 - x)^2, x) \right| \\ & = \|g'\|_{C[0,1]} |\alpha_n(x) - x| + \|g''\|_{C[0,1]} \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right|. \end{aligned}$$

For each $f \in C[0, 1]$, we can write

$$\begin{aligned}
& \left| \hat{\beta}_n(f, x) - f(x) \right| \\
& \leq \left| \hat{\beta}_n(f, x) - \hat{\beta}_n(g, x) \right| + \left| \hat{\beta}_n(g, x) - g(x) \right| + |g - f| \\
& \leq 2 \|g - f\|_{C[0,1]} + \left| \hat{\beta}_n(g; x) - g(x) \right| \\
& \leq 2 \|g - f\|_{C[0,1]} \\
& \quad + \left(|\alpha_n(x) - x| + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \right) \|g''\|_{C[0,1]} \\
& \leq 2 \left(\|g - f\|_{C[0,1]} + |\alpha_n(x) - x| \right. \\
& \quad \left. + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \|g''\|_{C[0,1]} \right)
\end{aligned}$$

Taking infimum over $g \in C^2[0, 1]$, we get

$$\begin{aligned}
& \left| \hat{\beta}_n(f, x) - f(x) \right| \\
& \leq K \left(f; \left(|\alpha_n(x) - x| + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \right) \right).
\end{aligned}$$

□

For the King type Beta operators we immediately have the following Corollary:

COROLLARY 3.4. *For each $f \in C[0, 1]$*

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq K(f; \gamma_{n,x}),$$

where $\gamma_{n,x} = \frac{1}{2n} (2x - 1) (2nx - \sqrt{4n^2x^2 + 4nx^2 + 1} + 1)$.

4. Statistical convergence

In this part of the paper, we use concept of statistical convergence and study the Korovkin type approximation theorem for the operators $\hat{\beta}_n$. Before we present the main results, we shall recall some notation on the statistical convergence.

Let M be any subset of \mathbb{N} . The density of M is defined by

$$\delta(M) = \lim_n \frac{1}{n} \sum_{j=1}^n \chi_M(j)$$

provided the limit exists, where χ_M is the characteristic function of M . A sequence $x = (x_k)$ is said to be statistical convergence to the number l ,

$$\delta \{k \in \mathbb{N}: |x_k - l| \geq \varepsilon\} = 0$$

for every $\varepsilon > 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|x_k - l| < \varepsilon$. We write

$$st - \lim_n x_k = l$$

Assume that for each $x \in [0, 1], (\alpha_n(x))_{n \in \mathbb{N}}$ is a sequence in $(0, 1)$ satisfying

$$(4.1) \quad st - \lim_n \alpha_n(x) = x.$$

Then we have

$$(4.2) \quad st - \lim_n |x - \alpha_n(x)| = 0,$$

and

$$(4.3) \quad st - \lim_n \left| \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| = 0.$$

Such a sequence $(\alpha_n(x))_{n \in \mathbb{N}}$ can be constructed as follows. Choose

$$\alpha_n(x) = \begin{cases} 2 & , \text{if } n = m^2 \ (m \in \mathbb{N}) \\ \alpha_n^*(x) & , \text{otherwise} \end{cases}$$

where

$$\alpha_n^*(x) = \frac{-1 + \sqrt{1 + 4n(n + 1)x^2}}{2n}, \quad x \in [0, 1] \text{ and } n \in \mathbb{N}.$$

It is clear that (4.1) is satisfied.

THEOREM 4.1. For each $x \in [0, 1]$ and for every $f \in C[0, 1]$, we have

$$st - \lim_n \left| \hat{\beta}_n(f; x) - f(x) \right| = 0.$$

Proof. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Now define the sets:

$$\begin{aligned} U &:= \{n : \delta_{n,x}^2 \geq r\}, \\ U_1 &:= \left\{n : |x - \alpha_n(x)| \geq \sqrt{\frac{r - \varepsilon}{2}}\right\}, \\ U_2 &:= \left\{n : \left|\frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}\right| \geq \frac{r - \varepsilon}{2}\right\}, \end{aligned}$$

where $\delta_{n,x} := \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}}$. Then it follows that $U \subseteq U_1 \cup U_2$, which gives

$$(4.4) \quad \sum_{j=1}^n \chi_U(j) \leq \sum_{j=1}^n \chi_{U_1}(j) + \sum_{j=1}^n \chi_{U_2}(j)$$

Multiplying both sides of (4.4) by $\frac{1}{n}$ and letting $n \rightarrow \infty$, we get using (4.2) and (4.3) that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \chi_U(j) = 0.$$

This guarantees that $st\text{-}\lim_n \delta_{n,x}^2 = 0$ which implies $st\text{-}\lim_n \omega(f, \delta_{n,x}) = 0$. Using Theorem 3.1 completes the proof. \square

REMARK 4.2. If we choose the sequence $(\alpha_n(x))_{n \in \mathbb{N}}$ as in (4.1), then our statistical approximation result (Theorem 4.1) works; however its classical version does not work since

$$\alpha_n(x) \not\rightarrow x$$

in the usual sense.

5. Best Error Estimation

Let ψ_x be the first central moment function defined by $\psi_x(y) = y - x$. In order to get a better error estimation on a subinterval I of $[0, 1]$, in the approximation by means of the operators β_n , we are aimed to find a functional sequence (s_n) , $s_n : I \rightarrow A$, satisfying

$$(5.1) \quad \delta_{n,x}^* := \sqrt{\hat{\beta}_n(\psi_x^2; u_n(x))} \leq \sqrt{\beta_n(\psi_x^2; x)} =: \delta_{n,x} \quad \text{for } x \in I.$$

By Lemmas 1.2 and 2.1 (d), (5.1) takes the form

$$(5.2) \quad \frac{n}{n+1} s_n^2(x) + \left(\frac{1}{n+1} - 2x \right) s_n(x) - \left(\frac{n}{n+1} - 2 \right) x^2 - \frac{1}{n+1} x \leq 0.$$

Let

$$\Delta_n(x) := \left(\frac{1}{n+1} - 2x \right)^2 + 4 \frac{n}{n+1} \left\{ \left(\frac{n}{n+1} - 2 \right) x^2 + \frac{1}{n+1} x \right\}.$$

Then it is clear that

$$(5.3) \quad \Delta_n(x) \geq 0$$

and

$$(5.4) \quad x + \frac{x}{n} - \frac{1}{2n} \in [0, 1]$$

hold for every $x \in I = [\frac{1}{4}, \frac{3}{4}]$ and for every $n \geq 1$. Therefore, from (5.2), (5.3) and (5.4), we get

$$\frac{2x - \frac{1}{n+1} - \sqrt{\Delta_n(x)}}{2 \frac{n}{n+1}} \leq s_n(x) \leq \frac{2x - \frac{1}{n+1} + \sqrt{\Delta_n(x)}}{2 \frac{n}{n+1}}.$$

Then $s_n(x)$ takes its minimum when

$$s_n(x) := x + \frac{x}{n} - \frac{1}{2n}.$$

Therefore, for all $x \in [\frac{1}{4}, \frac{3}{4}]$, we define a new Beta type operator by

$$\begin{aligned} \beta_n^s(f; x) &= \beta_n(f; s_n(x)) \\ &= \frac{1}{B(ns_n(x), n(1-s_n(x)))} \int_0^1 t^{ns_n(x)-1} (1-t)^{n(1-s_n(x))-1} f(t) dt. \end{aligned}$$

Then, for all $x \in [\frac{1}{4}, \frac{3}{4}]$ and $n \geq 1$, we have

$$\beta_n^s(\psi_x^2; x) = \frac{x(1-x)}{n} - \frac{1}{4n(n+1)} \leq \frac{x(1-x)}{n+1} = \beta_n(\psi_x^2; x)$$

which shows that the operators $\beta_n^s(f; x)$ provides the better estimation than the operators $\beta_n(f; x)$.

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