

## AT LEAST TWO SOLUTIONS FOR THE SEMILINEAR BIHARMONIC BOUNDARY VALUE PROBLEM

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ABSTRACT. We get one theorem that there exists a unique solution for the fourth order semilinear elliptic Dirichlet boundary value problem when the number 0 and the coefficient of the semilinear part belong to the same open interval made by two successive eigenvalues of the fourth order elliptic eigenvalue problem. We prove this result by the contraction mapping principle. We also get another theorem that there exist at least two solutions when there exist  $n$  eigenvalues of the fourth order elliptic eigenvalue problem between the coefficient of the semilinear part and the number 0. We prove this result by the critical point theory and the variation of linking method.

### 1. Introduction

Let  $\Omega$  be a bounded domain in  $R^n$  with smooth boundary  $\partial\Omega$  and let  $b \in R$  be a constant. Let  $\lambda_k (k = 1, 2, \dots)$  denote the eigenvalues and  $\phi_k (k = 1, 2, \dots)$  the corresponding eigenfunctions, suitably normalized with respect to  $L^2(\Omega)$  inner product, of the eigenvalue problem  $\Delta u + \lambda u =$

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0 in  $\Omega$  with  $u = 0$  on  $\partial\Omega$ , where each eigenvalue  $\lambda_k$  is repeated as often as its multiplicity. We recall that  $\lambda_1 < \lambda_2 \leq \lambda_3 \dots \rightarrow +\infty$ , and that  $\phi_1(x) > 0$  for  $x \in \Omega$ . In this paper we investigate the existence and the multiplicity of the solutions for the following fourth order semilinear elliptic equation with Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u &= g(u), & \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

where  $c \in R$ . Let us set

$$g(u) = b((u + 1)^+ - 1),$$

where  $u^+ = \max\{u, 0\}$  and  $b \in R$ . Tarantello [10] studied problem (1.1) when  $c < \lambda_1$  and  $b \geq \lambda_1(\lambda_1 - c)$ . She showed that (1.1) has at least two solutions, one of which is a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [8] also proved that if  $c < \lambda_1$  and  $b \geq \lambda_2(\lambda_2 - c)$ , then (1.1) has at least three solutions by the Leray-Schauder degree theory. Choi and Jung [3] showed that the problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= bu^+ + s & \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 & \text{on } \partial\Omega \end{aligned} \quad (1.2)$$

has at least two nontrivial solutions when  $c < \lambda_1$ ,  $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$  and,  $s < 0$  or when  $\lambda_1 < c < \lambda_2$ ,  $b < \lambda_1(\lambda_1 - c)$  and  $s > 0$ . The authors obtained these results by using the variational reduction method. The authors [5] also proved that when  $c < \lambda_1$ ,  $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$  and  $s < 0$ , (1.2) has at least three nontrivial solutions by using degree theory.

The eigenvalue problem  $\Delta^2 u + c\Delta u = \mu u$  in  $\Omega$  with  $u = 0$ ,  $\Delta u = 0$  on  $\partial\Omega$  has also infinitely many eigenvalues  $\mu_k = \lambda_k(\lambda_k - c)$ ,  $k \geq 1$  and corresponding eigenfunctions  $\phi_k$ ,  $k \geq 1$ . We note that  $\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) \leq \lambda_3(\lambda_3 - c) < \dots$ .

Our main results are as follows:

**THEOREM 1.1.** *Let  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_k(\lambda_k - c) < 0$ ,  $b < \lambda_{k+1}(\lambda_{k+1} - c)$ . Then (1.1) has a unique solution.*

**THEOREM 1.2.** *Let  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c) < \dots < \lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$ ,  $k \geq 1$ ,  $n \geq 1$ . Then (1.1) has at least two nontrivial solutions.*

Under the assumptions of Theorem 1.1 and Theorem 1.2, we cannot use the Leray-Schauder degree theory to prove main results because we cannot show the existence of a positive solution or a negative solution and because we can not find the unsolvability condition of the problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= b((u + 1)^+ - 1) + s\phi_1(x), & \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

By these reasons we use the critical point theory and variation of linking method for the proof of Theorem 1.2. In section 2, we introduce the Hilbert space and prove Theorem 1.1. In section 3, we prove Theorem 1.2.

### 2. Proof of Theorem 1.1

Let  $H$  be a subspace of  $L^2(\Omega)$  defined by

$$H = \{u \in L^2(\Omega) \mid \sum |\lambda_k(\lambda_k - c)|h_k^2 < \infty\},$$

where  $u = \sum h_k\phi_k \in L^2(\Omega)$  with  $\sum h_k^2 < \infty$ . Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)|h_k^2]^{\frac{1}{2}}.$$

Since  $\lambda_k(\lambda_k - c) \rightarrow +\infty$  and  $c$  is fixed, we have

- (i)  $\Delta^2 u + c\Delta u \in H$  implies  $u \in H$ .
- (ii)  $\|u\| \geq C\|u\|_{L^2(\Omega)}$ , for some  $C > 0$ .
- (iii)  $\|u\|_{L^2(\Omega)} = 0$  if and only if  $\|u\| = 0$ .

For the proof of the above results we refer [2].

LEMMA 2.1. *Assume that  $c$  is not an eigenvalue of  $-\Delta$ ,  $b \neq \lambda_k(\lambda_k - c)$  and bounded. Then all solutions in  $L^2(\Omega)$  of*

$$\Delta^2 u + c\Delta u = b((u + 1)^+ - 1) \quad \text{in } L^2(\Omega)$$

*belong to  $H$ .*

*Proof.* Let us write  $b((u + 1)^+ - 1) = \sum h_k\phi_k \in L^2(\Omega)$ .

$$(\Delta^2 + c\Delta)^{-1}b((u + 1)^+ - 1) = \sum \frac{1}{\lambda_k(\lambda_k - c)}h_k\phi_k \in L^2(\Omega).$$

$$\begin{aligned} \|(\Delta^2 + c\Delta)^{-1}b((u + 1)^+ - 1)\| &= \sum |\lambda_k(\lambda_k - c)|\frac{1}{(\lambda_k(\lambda_k - c))^2}h_k^2 \\ &\leq C\sum h_k^2 = C\|u\|_{L^2(\omega)}^2 < \infty \end{aligned}$$

for some  $C > 0$ . Thus  $(\Delta^2 + c\Delta)^{-1}(b((u+1)^+ - 1)) \in H$ .  $\square$

With the aid of Lemma 2.1 it is enough that we investigate the existence of the solutions of (1.1) in the subspace  $H$  of  $L^2(\Omega)$ .

**PROOF OF THEOREM 1.1.**

Assume that  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_k(\lambda_k - c) < 0$ ,  $b < \lambda_{k+1}(\lambda_{k+1} - c)$ . Let  $r = \frac{1}{2}\{\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)\}$ . We can rewrite (1.1) as

$$(\Delta^2 + c\Delta - r)u = b(u+1)^+ - r(u+1)^+ + r(u+1)^+ - ru - b, \quad \text{in } L^2(\Omega), \quad (2.1)$$

$$u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega.$$

or

$$u = (\Delta^2 + c\Delta - r)^{-1}[b(u+1)^+ - r(u+1)^+ + r(u+1)^+ - ru - b], \quad \text{in } L^2(\Omega), \quad (2.2)$$

$$u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega.$$

We note that the operator  $(\Delta^2 + c\Delta - r)^{-1}$  is a compact, self-adjoint and linear map from  $L^2(\Omega)$  into  $L^2(\Omega)$  with norm  $\frac{2}{\lambda_{k+1}(\lambda_{k+1}-c) - \lambda_k(\lambda_k-c)}$ , and

$$\begin{aligned} \|(b-r)\{(u_2+1)^+ - (u_1+1)^+\} + r\{(u_2+1)^+ - (u_1+1)^+\} - r(u_2 - u_1)\| \\ \leq \max\{|b-r|, |r|\}\|u_2 - u_1\| \\ < \frac{1}{2}\{\lambda_{k+1}(\lambda_{k+1}-c) - \lambda_k(\lambda_k-c)\}\|u_2 - u_1\|. \end{aligned}$$

Thus the right hand side of (2.2) defines a Lipschitz mapping from  $L^2(\Omega)$  into  $L^2(\Omega)$  with Lipschitz constant  $< 1$ . By the contraction mapping principle, there exists a unique solution  $u \in L^2(\Omega)$  of (2.2). By Lemma 2.1, the solution  $u \in H$ . We complete the proof.  $\square$

### 3. Proof of Theorem 1.2

Throughout this section we assume that  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$ . We shall prove Theorem 1.2 by applying the variational linking method (cf. Theorem 4.2 of [8]). Now, we recall a variation of linking theorem of the existence of critical levels for a functional.

Let  $X$  be a subspace of  $H$ ,  $\rho > 0$  and  $e \in H \setminus X$ ,  $e \neq 0$ . Let us set

$$B_\rho(X) = \{x \in X \mid \|x\| \leq \rho\},$$

$$S_\rho(X) = \{x \in X \mid \|x\| = \rho\},$$

$$\Delta_\rho(e, X) = \{\sigma e + v \mid \sigma \geq 0, v \in X, \|\sigma e + v\| \leq \rho\},$$

$$\Sigma_\rho(e, X) = \{\sigma e + v \mid \sigma \geq 0, v \in X, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in X, \|v\| \leq \rho\}.$$

**THEOREM 3.1.** (“A Variation of Linking”) *Let  $H$  be an Hilbert space, which is topological direct sum of the subspaces  $H_1$  and  $H_2$ . Let  $F \in C^1(H, R)$ . Moreover assume:*

- (a)  $\dim H_1 < +\infty$ ;
- (b) *there exist  $\rho > 0, R > 0$  and  $e \in H_1, e \neq 0$  such that  $\rho < R$  and*

$$\sup_{S_\rho(H_1)} F < \inf_{\Sigma_R(e, H_2)} F;$$

- (c)  $-\infty < a = \inf_{\Delta_R(e, H_2)} F$ ;
- (d) *(P.S.)<sub>c</sub> holds for any  $c \in [a, b]$ , where  $b = \sup_{B_\rho(H_1)} F$ .*

*Then there exist at least two critical levels  $c_1$  and  $c_2$  for the functional  $F$  such that :*

$$\inf_{\Delta_R(e, H_2)} F \leq c_1 \leq \sup_{S_\rho(H_1)} F < \inf_{\Sigma_R(e, H_2)} F \leq c_2 \leq \sup_{B_\rho(H_1)} F.$$

With the aid of Lemma 2.1 it is enough that we investigate the existence of the solutions of (1.1) in the subspace  $H$  of  $L^2(\Omega)$

Let us define the functional

$$F(u) = \int_\Omega \left[ \frac{1}{2} |\Delta u|^2 - \frac{c}{2} |\nabla u|^2 - \frac{b}{2} |u + 1|^+ - bu \right] dx. \tag{3.1}$$

By the assumption of Theorem 1.2,  $F(u)$  is well defined. By the following lemma,  $F(u) \in C^1$ . Thus the critical points of the functional  $F(u)$  coincide with the weak solutions of (1.1).

**LEMMA 3.1.** *Assume that  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$ . Then the functional  $F(u)$  is continuous and Frechét differentiable in  $H$  and*

$$DF(u)(h) = \int_\Omega [\Delta u \cdot \Delta h - c \nabla u \cdot \nabla h - b(u + 1)^+ h - bh] dx \tag{3.2}$$

for  $h \in H$ .

*Proof.* First we shall prove that  $F(u)$  is continuous at  $u$ . Let  $u \in H$ .

$$F(u+v) - F(u) = \int_\Omega \left[ \frac{1}{2} |\Delta(u+v)|^2 - \frac{c}{2} |\nabla(u+v)|^2 - \frac{b}{2} |(u+v+1)^+|^2 - b(u+v) \right] dx$$

$$- \int_\Omega \left[ \frac{1}{2} |\Delta u|^2 - \frac{c}{2} |\nabla u|^2 - \frac{b}{2} |(u+1)^+|^2 - bu \right] dx$$

$$= \int_{\Omega} [u \cdot (\Delta^2 v + c\Delta v) + \frac{1}{2}v \cdot (\Delta^2 v + c\Delta v) - (\frac{b}{2}|(u+v+1)^+|^2 - \frac{b}{2}|(u+1)^+|^2 - bv)] dx.$$

Let  $u = \sum h_k \phi_k$ ,  $v = \sum \tilde{h}_k \phi_k$ . Then we have

$$| \int_{\Omega} u \cdot (\Delta^2 v + c\Delta v) dx | = | \sum \int_{\Omega} \lambda_k (\lambda_k - c) h_k \tilde{h}_k | \leq \|u\| \|v\|$$

$$| \int_{\Omega} v \cdot (\Delta^2 v + c\Delta v) dx | = | \sum \lambda_k (\lambda_k - c) \tilde{h}_k^2 | \leq \|v\|^2.$$

On the other hand, by Mean Value Theorem, we have

$$\| \frac{b}{2}|(u+v+1)^+|^2 - \frac{b}{2}|(u+1)^+|^2 - bv \| \leq b\|v\|.$$

Thus we have

$$\| \frac{b}{2}|(u+v+1)^+|^2 - \frac{b}{2}|(u+1)^+|^2 - bv \| \leq 2b\|v\| = O(\|v\|).$$

Thus  $F(u)$  is continuous at  $u$ . Next we shall prove that  $F(u)$  is *Fréchet* differentiable at  $u \in H$ . We consider

$$\begin{aligned} |F(u+v) - F(u) - DF(u)v| &= | \int_{\Omega} \frac{1}{2}v(\Delta^2 v + c\Delta v) \\ &\quad - (\frac{b}{2}|(u+v+1)^+|^2 - \frac{b}{2}|(u+1)^+|^2 + b(u+1)^+v) | \\ &\leq \frac{1}{2}\|v\|^2 + b\|v\| + b(\|u\| + 1)\|v\| \\ &= \|v\|(\frac{1}{2}\|v\| + b + b(\|u\| + 1)) = O(\|v\|). \end{aligned}$$

Thus  $F(u)$  is *Fréchet* differentiable at  $u \in H$ . □

Let  $H^+$  be the subspace of  $H$  spanned by the eigenfunctions corresponding to the eigenvalues  $\lambda(\lambda - c) > 0$  and  $H^-$  the subspace of  $H$  spanned by the eigenfunctions corresponding to the eigenvalues  $\lambda(\lambda - c) < 0$ . Then  $H = H^+ \oplus H^-$ . Let  $H_{k+n}$  be the subspace of  $H$  spanned by  $\phi_1, \dots, \phi_{k+n}$  whose eigenvalues are  $\lambda_1(\lambda_1 - c), \dots, \lambda_{k+n}(\lambda_{k+n} - c)$ . Let  $H_{k+n}^\perp$  be the orthogonal complement of  $H_{k+n}$  in  $H$ . Then

$$H = H_{k+n} \oplus H_{k+n}^\perp.$$

Let  $L : H \rightarrow H$  be the linear continuous operator such that

$$(Lu)v = \int_{\Omega} (\Delta^2 u + c\Delta u) \cdot v dx - r \int_{\Omega} uv dx. \tag{3.3}$$

Then  $L$  is an isomorphism and  $H_{k+n}, H_{k+n}^\perp$  are the negative space and the positive space of  $L$ . Thus we have

$$\begin{aligned} \forall u \in H_{k+n} & : & (Lu)u & \leq (\lambda_{k+n}(\lambda_{k+n} - c) - r)\|u\|^2, \\ \forall u \in H_{k+n}^\perp & : & (Lu)u & \geq (\lambda_{k+n+1}(\lambda_{k+n+1} - c) - r)\|u\|^2. \end{aligned}$$

Thus there exists  $\nu > 0$  such that

$$(Lu)u \leq -\nu\|u\|^2, \tag{3.4}$$

$$(Lu)u \geq \nu\|u\|^2. \tag{3.5}$$

We can write

$$F(u) = \frac{1}{2}(Lu)u - \psi(u),$$

where

$$\psi(u) = \int_{\Omega} \left[ \frac{b}{2}|(u+1)^+|^2 - bu - \frac{1}{2}ru^2 \right] dx.$$

Since  $H$  is compactly embedded in  $L^2$ , the map  $D\psi : H \rightarrow H$  is compact.

**LEMMA 3.2.** *Let  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$ . Then  $F(u)$  satisfies the  $(P.S.)_\gamma$  condition for any  $\gamma \in \mathbb{R}$ .*

*Proof.* Let  $(u_n)$  be a sequence in  $H$  with  $DF(u_n) \rightarrow 0$  and  $F(u_n) \rightarrow \gamma$ . Since  $L$  is an isomorphism and  $D\psi$  is compact, it is sufficient to show that  $(u_n)$  is bounded in  $H$ . We argue by contradiction. we suppose that  $\|u_n\| \rightarrow +\infty$ . Let  $v_n = \frac{u_n}{\|u_n\|}$ . Up to a subsequence, we have  $v_n \rightarrow v$  in  $H$ . Since  $DF(u_n) \rightarrow 0$ , we get

$$\frac{DF(u_n)u_n}{\|u_n\|^2} = (Lv_n)v_n - \int_{\Omega} \left[ b\left(v_n + \frac{1}{\|u_n\|}\right)^+ v_n - b\frac{v_n}{\|u_n\|} - rv_n^2 \right] dx \rightarrow 0. \tag{3.6}$$

Let  $P^+ : H \rightarrow H_{k+n}^\perp$  and  $P^- : H \rightarrow H_{k+n}$  denote the orthogonal projections. Since  $P^+v_n - P^-v_n$  is bounded in  $H$ , we have

$$\begin{aligned} (LP^+v_n)P^+v_n - (LP^-v_n)P^-v_n - \int_{\Omega} \left[ b\left(v_n + \frac{1}{\|u_n\|}\right)^+ (P^+v_n - P^-v_n) \right. \\ \left. - b\frac{P^+v_n - P^-v_n}{\|u_n\|} - rv_n(P^+v_n - P^-v_n) \right] dx \rightarrow 0. \end{aligned} \tag{3.7}$$

Since  $P^+v_n - P^-v_n \rightarrow P^+v - P^-v$  in  $H$ , from (3.4) and (3.5) we get

$$0 < \nu\|v\|^2 \leq \int_{\Omega} [(bv^+ - rv)(P^+v - P^-v)] dx.$$

Hence  $v \neq 0$ . On the other hand, from (3.7), we get

$$\begin{aligned} & (LP^+v)P^+v - (LP^-v)P^-v - \int_{\Omega} [bv^+(P^+v - P^-v) \\ & \quad - r(P^+v + P^-v)(P^+v - P^-v)]dx = 0. \end{aligned} \quad (3.8)$$

Let us choose  $v \in H^+$ . Then from (3.8), we have

$$\begin{aligned} 0 &= (LP^+v)P^+v - (LP^-v)P^-v - \int_{\Omega} [bv^+(P^+v - P^-v) \\ & \quad - r(P^+v + P^-v)(P^+v - P^-v)]dx \\ &\geq (LP^+v)P^+v - (LP^-v)P^-v - \int_{\Omega} [bv(P^+v - P^-v) \\ & \quad - r(P^+v + P^-v)(P^+v - P^-v)]dx \\ &= (LP^+v)P^+v - (LP^-v)P^-v - \int_{\Omega} [b(P^+v + P^-v)(P^+v - P^-v) \\ & \quad - r(P^+v + P^-v)(P^+v - P^-v)]dx \\ &\geq \{(\lambda_{k+n+1}(\lambda_{k+n+1} - c) - r) - (b - r)\} \\ & \quad \int_{\Omega} (P^+v)^2 - \{(\lambda_{k+n}(\lambda_{k+n} - c)) - r) - (b - r)\} \int_{\Omega} (P^-v)^2 \\ &\geq 0 \end{aligned} \quad (3.9)$$

Since  $(\lambda_{k+n+1}(\lambda_{k+n+1} - c) - r) - (b - r) > 0$  and  $-(\lambda_{k+n}(\lambda_{k+n} - c)) - r) - (b - r) > 0$ , the left hand side of (3.9) is positive or equal to 0, so the only possibility to hold (3.9) is that  $P^+v = 0$  and  $P^-v = 0$ . Thus  $v = 0$ . This is a contradiction. We complete the proof.  $\square$

**LEMMA 3.3.** *Let  $\lambda_k < c < \lambda_{k+1}$  and  $\lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$ . Then*

(i) *there exists  $R_{k+n} > 0$  such that the functional  $F(u)$  is bounded from below on  $H_{k+n}^{\perp}$ ;*

$$\inf_{\substack{u \in H_{k+n}^{\perp} \\ \|u\| = R_{k+n}}} F(u) > 0 \quad \text{and} \quad \inf_{\substack{u \in H_{k+n}^{\perp} \\ \|u\| < R_{k+n}}} F(u) > -\infty. \quad (3.10)$$

(ii) *there exists  $\rho_{k+n} > 0$  such that*

$$\sup_{\substack{u \in H_{k+n} \\ \|u\| = \rho_{k+n}}} F(u) < 0 \quad \text{and} \quad \sup_{\substack{u \in H_{k+n} \\ \|u\| \leq \rho_{k+n}}} F(u) < \infty. \quad (3.11)$$



*Proof.* (i) Let  $u \in H_{k+n}^\perp$ . Then we have

$$\begin{aligned} \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow +\infty}} F(u) &\geq \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow \infty}} \frac{1}{2} \left(1 - \frac{r}{\lambda_{k+n+1}(\lambda_{k+n+1} - c)}\right) \|u\|^2 \\ &\quad - \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow +\infty}} \int_{\Omega} \left[\frac{b}{2} |(u+1)^+|^2 - bu - \frac{r}{2} u^2\right] dx \\ &\geq \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow \infty}} \frac{1}{2} \left(1 - \frac{r}{\lambda_{k+n+1}(\lambda_{k+n+1} - c)}\right) \|u\|^2 - \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow +\infty}} \int_{\Omega} \left[\frac{b}{2} (u^2+1) - \frac{r}{2} u^2\right] dx \\ &\geq \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow +\infty}} \frac{1}{2} \left(1 - \frac{r}{\lambda_{k+n+1}(\lambda_{k+n+1} - c)}\right) \|u\|^2 - \lim_{\substack{u \in H_{k+n}^\perp \\ \|u\| \rightarrow +\infty}} \frac{1}{2} (b-r) \int_{\Omega} u^2 - \frac{b}{2} |\Omega| \\ &\rightarrow +\infty, \end{aligned}$$

since  $b - r < \lambda_{k+n+1}(\lambda_{k+n+1} - c) - r = \frac{\lambda_{k+n+1}(\lambda_{k+n+1}-c) - \lambda_{k+n}(\lambda_{k+n}-c)}{2}$ . Thus there exists  $R_{k+n} > 0$  such that  $\inf_{\substack{u \in H_{k+n}^\perp \\ \|u\|=R_{k+n}}} F(u) > 0$ . Moreover if

$u \in H_{k+n}^\perp$  with  $\|u\| < R_{k+n}$ , then we have

$$\begin{aligned} F(u) &\geq \frac{1}{2} (\lambda_{k+n+1}(\lambda_{k+n+1} - c)) \|u\|_{L^2(\Omega)}^2 - \int_{\Omega} \left[\frac{b}{2} (u+1)^2 - bu\right] dx \\ &> \frac{1}{2} \{(\lambda_{k+n+1}(\lambda_{k+n+1} - c)) - b\} \|u\|_{L^2(\Omega)}^2 - \int_{\Omega} \left[\frac{b}{2}\right] dx > -\infty. \end{aligned}$$

Thus we have  $\inf_{\substack{u \in H_{k+n}^\perp \\ \|u\| < R_{k+n}}} F(u) > -\infty$ .

(ii) Let  $u \in H_{k+n}$ , Then

$$\begin{aligned} (Lu)u &\leq (\lambda_{k+n}(\lambda_{k+n} - c) - r) \int_{\Omega} u^2 dx \\ &\leq \frac{\lambda_{k+n}(\lambda_{k+n} - c) - \lambda_{k+n+1}(\lambda_{k+n+1} - c)}{2} \int_{\Omega} u^{+2}, \\ \int_{\Omega} \left[\frac{1}{2} b |(u+1)^+|^2 - bu - ru^2\right] dx &\geq \int_{\Omega} \left[\frac{1}{2} b |u^+|^2 - bu - ru^{+2}\right] dx, \end{aligned}$$

so that

$$\begin{aligned} F(u) &\leq \frac{1}{2} \frac{\lambda_{k+n}(\lambda_{k+n} - c) - \lambda_{k+n+1}(\lambda_{k+n+1} - c)}{2} \int_{\Omega} u^{+2} - \frac{b-r}{2} \int_{\Omega} u^{+2} + \int_{\Omega} budx \end{aligned}$$

$$\leq \frac{1}{2} \left\{ \frac{\lambda_{k+n}(\lambda_{k+n} - c) - \lambda_{k+n+1}(\lambda_{k+n+1} - c)}{2} - (b-r) \right\} \|u^+\|_{L^2(\Omega)}^2 + b \|u\|_{L^2(\Omega)}.$$

Since  $\frac{\lambda_{k+n}(\lambda_{k+n} - c) - \lambda_{k+n+1}(\lambda_{k+n+1} - c)}{2} - (b-r) < 0$ , there exists  $\rho_{k+n} > 0$  such that if  $u \in H_{k+n}$  and  $\|u\| = \rho_{k+n}$ , then  $\sup F(u) < 0$ . Moreover, if  $u \in H_{k+n}$  and  $\|u\| \leq \rho_{k+n}$ , then we have  $F(u) \leq \frac{1}{2} \left\{ \frac{\lambda_{k+n}(\lambda_{k+n} - c) - \lambda_{k+n+1}(\lambda_{k+n+1} - c)}{2} - (b-r) \right\} \|u^+\|_{L^2(\Omega)}^2 + b \|u\|_{L^2(\Omega)} \leq b \|u\|_{L^2(\Omega)} < \infty$ .  $\square$

LEMMA 3.4. *Let  $\lambda_k < c < \lambda_{k+1}$ ,  $\lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$  and let  $e_1 \in \text{span}\{\phi_{k+1}, \dots, \phi_{k+n}\}$  with  $\|e_1\| = 1$ . Then there exists  $R_{k+n}^-$  such that  $R_{k+n}^- > \rho_{k+n}$ ,*

$$\inf_{u \in \Sigma_{R_{k+n}^-}(e_1, H_{k+n}^\perp)} F(u) \geq 0 \quad \text{and} \quad \inf_{u \in \Delta_{R_{k+n}^-}(e_1, H_{k+n}^\perp)} F(u) \geq -\infty.$$

*Proof.* Let us chose  $u \in H_{k+n}^\perp$  and  $\sigma \geq 0$  and  $e_1 \in \text{span}\{\phi_{k+1}, \dots, \phi_{k+n}\}$  with  $\|e_1\| = 1$ . Then we get

$$\begin{aligned} F(u + \sigma e_1) &\geq \frac{1}{2} \lambda_{k+n+1}(\lambda_{k+n+1} - c) \|u\|_{L^2(\Omega)}^2 + \frac{\sigma^2}{2} \|e_1\|^2 \\ &\quad - \int_{\Omega} \left[ \frac{b}{2} (u + \sigma e_1 + 1)^2 - b(u + \sigma e_1) \right] dx \\ &= \frac{1}{2} \{ \lambda_{k+n+1}(\lambda_{k+n+1} - c) - b \} \|u\|_{L^2(\Omega)}^2 + \frac{\sigma^2}{2} (\Lambda - b) \|e_1\|_{L^2(\Omega)}^2 \\ &\quad - b\sigma^2 \|u\|_{L^2(\Omega)} \|e_1\|_{L^2(\Omega)} - \frac{b}{2} |\Omega|, \end{aligned}$$

where  $\lambda_{k+1}(\lambda_{k+1} - c) \leq \Lambda \leq \lambda_{k+n}(\lambda_{k+n} - c)$ . Choose  $\sigma > 0$  so mall that  $\frac{\sigma}{2} \|e_1\|^2$  is small. We can choose a number  $R_{k+n}^- > 0$  such that  $R_{k+n}^- > \sigma$ ,  $R_{k+n}^- > \rho_{k+n}$ , and  $\inf_{\substack{u \in H_{k+n}^\perp, \sigma \geq 0 \\ \|u + \sigma e_1\| = R_{k+n}^-}} F(u + \sigma e_1) \geq 0$ . Moreover if

$u \in H_{k+n}^\perp$ ,  $\sigma \geq 0$   $\|u + \sigma e_1\| \leq R_{k+n}^-$ , then  $F(u) \geq \frac{\sigma^2}{2} (\Lambda - b) \|e_1\|_{L^2(\Omega)}^2 - b\sigma \|u\|_{L^2(\Omega)} \|e_1\|_{L^2(\Omega)} - \frac{b}{2} |\Omega| \geq -\infty$ . Thus we prove the lemma.  $\square$

PROOF OF THEOREM 1.2.

By Lemma 3.1,  $F(u)$  is continuous and *Frechet* differentiable in  $H$ . By Lemma 3.2.  $F(u)$  satisfies the  $(P.S.)_\gamma$  condition for any  $\gamma \in R$ . We note that the subspace  $S_{\rho_{k+n}} \cap H_{k+n}$  and the subspace  $\Sigma_{R_{k+n}^-}(e_1, H_{k+n}^\perp)$  link at the subspace  $\{e_1\}$ . By Lemma 3.3 and Lemma 3.4, we have

$$\sup_{u \in S_{\rho_{k+n}} \cap H_{k+n}} F(u) < \inf_{u \in \Sigma_{R_{k+n}^-}(e_1, H_{k+n}^\perp)} F(u).$$

By Lemma 3.4, we also have  $\inf_{u \in \Delta_{R_{k+n}^-}(e_1, H_{k+n}^1)} F(u) > -\infty$ . Thus by the variation of linking Theorem 3.1, there exists at least two nontrivial solutions of (1.1). Thus we complete the Theorem 1.2.  $\square$

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