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DYNAMICAL BIFURCATION OF THE ONE-DIMENSIONAL CONVECTIVE CAHN-HILLIARD EQUATION

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Abstract. In this paper, we study the dynamical behavior of the one-dimensional convective Cahn-Hilliard equation(CCHE) on a periodic cell $[-\pi, \pi]$. We prove that as the control parameter passes through the critical number, the CCHE bifurcates from the trivial solution to an attractor. We describe the bifurcated attractor in detail which gives the final patterns of solutions near the trivial solution.

1. Introduction

In this paper, we consider the one-dimensional convective Cahn-Hilliard equation(CCHE):

$$
\begin{aligned}
u_t &= -(\lambda u - u^3 + \alpha u_{xx})_{xx} + uu_x \\
&= -\alpha u_{xxxx} - \lambda u_{xx} + uu_x + 6uu_x^2 + 3u^2 u_{xx}.
\end{aligned}
\tag{1.1}
$$

Here, $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, $\lambda \in \mathbb{R}$ is a control parameter related to the driving force of the system, and α is a positive real number. The

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CCHE has been suggested recently for the description of several physical phenomena, including spinodal decomposition of (driven) phase separating systems in an external field, instability of steps moving on a crystal surface, and thermodynamically unstable surfaces [11, 13]. The CCHE contains nonlinearities typical of both the Kuramoto-Sivashinsky equation(KSE) and the Cahn-Hilliard equation(CHE): uu_x for KSE and $(u^3)_{xx}$ for CHE. Thus, one may expect the dynamical aspects of both equations in CCHE [3].

In this paper, we are interested in the dynamical behavior of CCHE which provides us the final patterns of the evolutionary solutions. The final patterns near the trivial solution $u \equiv 0$ is closely related to the value of the control parameter λ. As soon as λ crosses the critical value, the trivial solution becomes unstable and bifurcates to an attractor. This attractor is responsible for the long-time dynamics of solutions near the trivial solution. So, the key ingredient in this study is to verify the structure of the attractor. Recently, there have been lots of studies in this direction for various phase transition equations. For example, see [2, 4, 5, 6, 7, 8, 9, 10, 12]. To set up our problem, we consider the CCHE (1.1) under the periodic boundary condition on $\Omega = [-\pi, \pi]$. For the functional setting of the periodic CCHE, let

$$\begin{aligned} H &= \Big\{ u \in L^2(\Omega;\mathbb{R}) : u(-\pi) = u(\pi) \ \text{ and } \int_{-\pi}^{\pi} u(x)dx = 0 \Big\}, \\ H^4_{per}(\Omega;\mathbb{R}) &= \Big\{ u \in H^4(\Omega;\mathbb{R}) : \frac{\partial^j u}{\partial x^j}(-\pi) = \frac{\partial^j u}{\partial x^j}(\pi) \text{ for } j = 0,1,2,3 \Big\}, \\ H_1 &= H^4_{per}(\Omega;\mathbb{R}) \cap H. \end{aligned}$$

On the other hand, it is easy to see that the CCHE (1.1) is invariant under the odd periodic condition. So, we also study the CCHE on Ω under the odd periodic condition. For this, we define $\tilde{H} = H \cap \{u \in L^2(\Omega;\mathbb{R}) : u(-x) = -u(x),\ x \in [0,\pi]\}$ and $\tilde{H}_1 = \tilde{H} \cap H_1$.

We formulate (1.1) in an abstract equation

$$\begin{cases} \dfrac{du}{dt} &= \mathcal{L}u + G(u), \\ u(0) &= u_0, \end{cases} \tag{1.2}$$

by setting

$$\mathcal{L}u = -\Big(\alpha \frac{\partial^4}{\partial x^4} + \lambda \frac{\partial^2}{\partial x^2} \Big) u,$$

and the nonlinear operator $G(u) = G_2(u, u) + G_3(u, u, u)$, where

$$G_2(u, v) = uv_x \quad \text{and} \quad G_3(u, v, w) = 3uvw_{xx} + 6uv_x w_x.$$

Then, $\mathcal{L} : H_1 \to H$ is well-defined. Similarly, $\mathcal{L} : \tilde{H}_1 \to \tilde{H}$ is also well-defined.

The signs of eigenvalues of the linear operator $\mathcal{L}$ plays an important role in the dynamical bifurcation. If all the eigenvalues are negative, then the trivial solution is asymptotically stable. As λ varies and passes through a certain number, some of the eigenvalues are positive and the trivial solution becomes unstable. In the sequel, the CCHE bifurcates to an attractor which determines the final patterns of solutions near the trivial solution. A direct calculation show that $\mathcal{L}$ allows an eigenvalue sequence

$$\beta_n(\lambda) = n^2(\lambda - \alpha n^2), \quad n = 1, 2, \cdots$$

with the corresponding eigenvectors

$$\phi_n(x) = \sin nx, \quad \psi_n(x) = \cos nx$$

for $n \geq 1$. In $\tilde{H}$, ϕ_n are only eigenvectors for $n \geq 1$. We note that the eigenvectors are orthogonal to each other and

$$\|\phi_n\|_H = \|\psi_n\|_H = \sqrt{\pi} \quad (\|\phi_n\|_{\tilde{H}} = \sqrt{\pi}, \text{ resp.})$$

for all $n \geq 1$. We are interested in the first instant that some eigenvalues are positive. For the periodic case, this happens when $n = 1$. Indeed, if λ is slightly bigger than α, then

$$\beta_1 > 0 \ \text{ and } \ \beta_n < 0 \text{ for all } n \geq 2.$$

The final pattern of solutions near the trivial solution are determined by the center manifold of the trivial solution. Thus, it is important to reduce the CCHE on the center manifold of the trivial solution. Generally, it is not easy to find a center manifold function in exact form. Recently, Ma and Wang derive a formula of a center manifold function (see Theorem 3.8 in [7]). We will analyze the behavior of solutions on the center manifold by use of this formula. The main results of this paper are the following.

Theorem 1.1. *As λ passes through α, CCHE* (1.1) *defined in $\tilde{H}$ bifurcates to two steady points*

$$u^{\pm} = \pm\rho_\alpha \phi_1 + o(\lambda - \alpha), \tag{1.3}$$

where $\rho_\alpha > 0$ *and*

$$\rho_\alpha^2 = \frac{48\alpha\beta_1}{36\alpha+1} \ + \ o(\lambda-\alpha). \tag{1.4}$$

So, we have a pitchfork bifurcation.

THEOREM 1.2. *As* λ *passes through* α*, CCHE* (1.1) *defined in* H *bifurcates to an attractor* $\mathcal{A}_1(\lambda,\alpha)$ *which is homeomorphic to* S^1 *and consists of steady solutions given by*

$$\big\{u = w_1\phi_1 + w_2\psi_1 + o(\lambda-\alpha) \ : \ w_1^2 + w_2^2 = \rho_\alpha^2\big\}.$$

We prove Theorem 1.1 and Theorem 1.2 in subsequent sections. We follow the method in [1] where the center manifold reduction was made by using of Theorem 3.8 in [7].

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We assume that λ is slightly bigger than α. We note that

$$\beta_1(\alpha) = \lambda - \alpha. \tag{2.1}$$

Let $\tilde{E}_1 = \operatorname{span}\{\phi_1\}$ and $\tilde{E}_2 = \tilde{E}_1^\perp$ in $\tilde{H}$. Let $\tilde{P}_j : \tilde{H} \to \tilde{E}_j$ be the canonical projections and $\tilde{\mathcal{L}}_j = \mathcal{L}|_{\tilde{E}_j}$, for $j = 1, 2$. For $u \in \tilde{H}$, we write

$$u = \sum_{n=1}^{\infty} y_n\phi_n = y_1\phi_1 + \sum_{n=2}^{\infty} y_n\phi_n \equiv v + \tilde{\Phi}(v),$$

where $\tilde{\Phi} : \tilde{E}_1 \to \tilde{E}_2$ is a center manifold function and $v = \tilde{P}_1 u = y_1\phi_1$. The reduced equation of (1.2) on the center manifold is

$$\frac{dv}{dt} = \tilde{\mathcal{L}}_1 v + \tilde{P}_1 G\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big). \tag{2.2}$$

By taking the inner product of (2.2) with ϕ_1, we have the following:

$$\frac{dy_1}{dt} = \beta_1 y_1 + g(y_1), \tag{2.3}$$

where

$$g(y_1) = \frac{1}{\pi}\left\langle G_2(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)), \phi_1\right\rangle + \frac{1}{\pi}\left\langle G_3(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)), \phi_1\right\rangle.$$

By means of Theorem 3.8 in [7], the center manifold function $\tilde{\Phi}$ can be expressed as

$$\begin{aligned}\tilde{\Phi}(y_1\phi_1) &= (-\tilde{\mathcal{L}}_2)^{-1}\tilde{P}_2 G_2(y_1\phi_1) + O(|\beta_1|\cdot\pi|y_1|^2) + o(\pi|y_1|^2)\\ &= (-\tilde{\mathcal{L}}_2)^{-1}\tilde{P}_2 G_2(y_1\phi_1) + o(|y_1|^2),\end{aligned}\tag{2.4}$$

where the last equality comes from (2.1).

By direct computation, we have

$$\begin{aligned}G_2(y_1\phi_1) &= (y_1\phi_1)(y_1\phi_1)_x = y_1^2\sin x\cos x\\ &= \frac{y_1^2}{2}\sin 2x = \frac{y_1^2}{2}\phi_2.\end{aligned}$$

From (2.4), since $\mathcal{L}\phi_2 = \beta_2\phi_2$, the center manifold function becomes

$$\tilde{\Phi}(y_1\phi_1) = -\frac{y_1^2}{2}\frac{\phi_2}{\beta_2} + o(|y_1|^2).$$

Then

$$\begin{aligned}&G_2(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1))\\ &= \big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)_x\\ &= \Big(y_1\phi_1 - \frac{y_1^2}{2}\frac{\phi_2}{\beta_2} + o(|y_1|^2)\Big)\Big(y_1\psi_1 - \frac{y_1^2}{\beta_2}\psi_2 + o(|y_1|^2)\Big)\\ &= y_1^2\phi_1\psi_1 - \frac{y_1^3}{\beta_2}\phi_1\psi_2 - \frac{y_1^3}{2\beta_2}\phi_2\psi_1 + o(|y_1|^3)\\ &= \frac{y_1^2}{2}\phi_2 - \frac{y_1^3}{2\beta_2}(-\phi_1+\phi_3) - \frac{y_1^3}{4\beta_2}(\phi_1+\phi_3) + o(|y_1|^3)\\ &= \frac{y_1^3}{4\beta_2}\phi_1 + \frac{y_1^2}{2}\phi_2 - \frac{3y_1^3}{4\beta_2}\phi_3 + o(|y_1|^3),\end{aligned}$$

and

$$\begin{aligned}&G_3(y_1\phi_1 + \tilde{\Phi}(y_1))\\ &= 3\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)^2\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)_{xx}\\ &\qquad + 6\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)\big(y_1\phi_1 + \tilde{\Phi}(y_1\phi_1)\big)_x^2\end{aligned}$$

$$
\begin{aligned}
&= 3\Big(y_1\phi_1 - \frac{y_1^2}{2}\frac{\phi_2}{\beta_2} + o(|y_1|^2)\Big)^2\Big(-y_1\phi_1 + \frac{2y_1^2}{\beta_2}\phi_2 + o(|y_1|^2)\Big) \\
&\quad + 6\Big(y_1\phi_1 - \frac{y_1^2}{2}\frac{\phi_2}{\beta_2} + o(|y_1|^2)\Big)\Big(y_1\psi_1 - \frac{y_1^2}{\beta_2}\psi_2 + o(|y_1|^2)\Big)^2 \\
&= -3y_1^3\phi_1^3 + 6y_1^3\phi_1\psi_1^2 + o(|y_1|^3) = -3y_1^3\frac{3\phi_1 - \phi_3}{4} + 6y_1^3\frac{\phi_1 + \phi_3}{4} + o(|y_1|^3) \\
&= -\frac{3y_1^3}{4}\phi_1 + \frac{9y_1^3}{4}\phi_3 + o(|y_1|^3).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\frac{1}{\pi}\Big\langle G_2(y_1\phi_1 + \tilde{\Phi}(y_1)), \phi_1\Big\rangle &= \frac{1}{4\beta_2}y_1^3 + o(|y_1|^3), \\
\frac{1}{\pi}\Big\langle G_3(y_1\phi_1 + \tilde{\Phi}(y_1)), \phi_1\Big\rangle &= -\frac{3}{4}y_1^3 + o(|y_1|^3).
\end{aligned}
$$

Hence, (2.3) becomes

$$
\frac{dy_1}{dt} = \beta_1 y_1 - d_1 y_1^3 + o(|y_1|^3). \tag{2.5}
$$

where

$$
d_1 = d_1(\alpha, \lambda) = \frac{3}{4} - \frac{1}{4\beta_2} = \frac{3\beta_2 - 1}{4\beta_2}. \tag{2.6}
$$

We note that (2.5) has two steady points $y_1 = \pm\rho_\alpha$ with $\rho_\alpha > 0$, where

$$
\rho_\alpha^2 = \frac{\beta_1}{d_1} = \frac{4\beta_1\beta_2}{3\beta_2 - 1}. \tag{2.7}
$$

Since λ is slightly bigger than α, we have

$$
\beta_2 = 4(\lambda - 4\alpha) < 0 \tag{2.8}
$$

and hence ρ_α is well-defined. The formulas (2.8) also provides an exact form of (2.7) as

$$
\rho_\alpha^2 = \frac{48\alpha\beta_1}{36\alpha + 1} - \frac{16\beta_1(\lambda - \alpha)}{(36\alpha + 1)^2} \ + \ o(\lambda - \alpha) = \frac{48\alpha\beta_1}{36\alpha + 1} \ + \ o(\lambda - \alpha),
$$

which yields (1.4). Now we have two solutions given by (1.3). It is easy to check that the solutions $u^\pm$ are stable. This completes the proof. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We proceed as in the proof of Theorem 1.1. Let $E_1 = \text{span}\{\phi_1, \psi_1\}$ and $E_2 = E_1^{\perp}$ in H. Let $P_j : H \to E_j$ be the canonical projection and $\mathcal{L}_j = \mathcal{L}|_{E_j}$, for $j = 1, 2$. For $u \in H$, we write

$$u = \sum_{n=1}^{\infty}(y_n\phi_n + z_n\psi_n).$$

If $\Phi : E_1 \to E_2$ is a center manifold function and $v = P_1u = y_1\phi_1 + z_1\psi_1$, then the reduced equation of (1.1) on the center manifold is

$$\frac{dv}{dt} = \mathcal{L}_1^{\alpha}v + P_1G\big(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)\big). \tag{3.1}$$

By taking the inner product of (3.1) with ϕ_1 and ψ_1, we have the following:

$$\begin{cases} \dfrac{dy_1}{dt} = \beta_1 y_1 + F_1(y_1, z_1), \\ \dfrac{dz_1}{dt} = \beta_1 z_1 + F_2(y_1, z_1). \end{cases} \tag{3.2}$$

Here,

$$\begin{aligned} F_1(y_1, z_1) = {}& \frac{1}{\pi}\left\langle G_2(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \phi_1\right\rangle \\ & + \frac{1}{\pi}\left\langle G_3(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \phi_1\right\rangle \end{aligned}$$

and

$$\begin{aligned} F_2(y_1, z_1) = {}& \frac{1}{\pi}\left\langle G_2(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \psi_1\right\rangle \\ & + \frac{1}{\pi}\left\langle G_3(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \psi_1\right\rangle . \end{aligned}$$

For the computation of F_1 and F_2, we need to derive a formula for Φ. As in the proof of Theorem 1.1 we utilize Theorem 3.8 in [7]. The center manifold function Φ can be expressed as

$$\begin{aligned} \Phi(y_1\phi_1 + z_1\psi_1) &= (-\mathcal{L}_2)^{-1}P_2G_2(y_1\phi_1 + z_1\psi_1) \\ &\qquad + O\big(|\beta_1| \cdot \pi(y_1^2 + z_1^2)\big) + o(\pi(y_1^2 + z_1^2)) \\ &= (-\mathcal{L}_2)^{-1}P_2G_2(y_1\phi_1 + z_1\psi_1) + o(y_1^2 + z_1^2) \end{aligned} \tag{3.3}$$

where the last equality comes from (2.1). By direct computation, we have

$$\begin{aligned}G_2(y_1\phi_1+z_1\psi_1) &= (y_1\phi_1+z_1\psi_1)(y_1\phi_1+z_1\psi_1)_x\\ &= \Big(y_1\phi_1+z_1\psi_1\Big)\Big(y_1\psi_1-z_1\phi_1\Big)\\ &= (y_1^2-z_1^2)\phi_1\psi_1+y_1z_1(\psi_1^2-\phi_1^2)\\ &= \frac{y_1^2-z_1^2}{2}\phi_2+y_1z_1\psi_2.\end{aligned}$$

Hence, from (3.3), we obtain

$$\Phi(y_1\phi_1+z_1\psi_1) = -\frac{y_1^2-z_1^2}{2}\frac{\phi_2}{\beta_2}-y_1z_1\frac{\psi_2}{\beta_2}+o(y_1^2+z_1^2).$$

As a consequence,

$$\begin{aligned}&G_2(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1))\\ &= \big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)\big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)_x\\ &= \Big(y_1\phi_1+z_1\psi_1-\frac{y_1^2-z_1^2}{2}\frac{\phi_2}{\beta_2}-y_1z_1\frac{\psi_2}{\beta_2}+o(y_1^2+z_1^2)\Big)\\ &\qquad\times\Big(y_1\psi_1-z_1\phi_1-\frac{y_1^2-z_1^2}{\beta_2}\psi_2+\frac{2y_1z_1}{\beta_2}\phi_2+o(y_1^2+z_1^2)\Big)\\ &= (y_1^2-z_1^2)\phi_1\psi_1+y_1z_1(\psi_1^2-\phi_1^2)-\frac{y_1^3-2y_1z_1^2}{\beta_2}\phi_1\psi_2+\frac{5y_1^2z_1-z_1^3}{2\beta_2}\phi_1\phi_2\\ &\qquad+\frac{5y_1z_1^2-y_1^3}{2\beta_2}\psi_1\phi_2+\frac{z_1^3-2y_1^2z_1}{\beta_2}\psi_1\psi_2+o(y_1^3+z_1^3).\end{aligned}$$

Using elementary properties of the trigonometric functions, we obtain

$$\begin{aligned}&G_2(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1))\\ &= (y_1^2-z_1^2)\frac{\phi_2}{2}+y_1z_1\psi_2-\frac{y_1^3-2y_1z_1^2}{\beta_2}\frac{-\phi_1+\phi_3}{2}+\frac{5y_1^2z_1-z_1^3}{2\beta_2}\frac{\psi_1-\psi_3}{2}\\ &\qquad+\frac{5y_1z_1^2-y_1^3}{2\beta_2}\frac{\phi_1+\phi_3}{2}+\frac{z_1^3-2y_1^2z_1}{\beta_2}\frac{\psi_1+\psi_3}{2}+o(|y_1|^3+|z_1|^3)\\ &= \frac{y_1^3+y_1z_1^2}{4\beta_2}\phi_1+\frac{y_1^2-z_1^2}{2}\phi_2+\frac{y_1^2z_1+z_1^3}{4\beta_2}\psi_1+y_1z_1\psi_2\\ &\qquad+\frac{-3y_1^3+9y_1z_1^2}{4\beta_2}\phi_3+\frac{-9y_1^2z_1+3z_1^3}{4\beta_2}\psi_3+o(|y_1|^3+|z_1|^3).\end{aligned}$$

As a consequence, we are led to

$$\begin{aligned}
&\frac{1}{\pi}\left\langle G_2(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)),\phi_1\right\rangle\\
&\qquad=\ \frac{y_1^3+y_1z_1^2}{4\beta_2}+o(|y_1|^3+|z_1|^3),\\
&\frac{1}{\pi}\left\langle G_2(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)),\psi_1\right\rangle\\
&\qquad=\ \frac{y_1^2z_1+z_1^3}{4\beta_2}+o(|y_1|^3+|z_1|^3).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&G_3(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1))\\
=&3\big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)^2\big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)_{xx}\\
&+6\big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)\big(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1)\big)_x^2\\
=&3\Big(y_1\phi_1+z_1\psi_1-\frac{\mu(y_1^2-z_1^2)}{2}\frac{\phi_2}{\beta_2}+\mu y_1z_1\frac{\psi_2}{\beta_2}+o(y_1^2+z_1^2)\Big)^2\\
&\qquad\times\Big(-y_1\phi_1-z_1\psi_1+\frac{2\mu(y_1^2-z_1^2)}{\beta_2}\phi_2-\frac{4\mu y_1z_1}{\beta_2}\psi_2+o(y_1^2+z_1^2)\Big)\\
&+6\Big(y_1\phi_1+z_1\psi_1-\frac{\mu(y_1^2-z_1^2)}{2}\frac{\phi_2}{\beta_2}+\mu y_1z_1\frac{\psi_2}{\beta_2}+o(y_1^2+z_1^2)\Big)\\
&\qquad\times\Big(y_1\psi_1-z_1\phi_1-\frac{\mu(y_1^2-z_1^2)}{\beta_2}\psi_2-\frac{2\mu y_1z_1}{\beta_2}\phi_2+o(y_1^2+z_1^2)\Big)^2.
\end{aligned}$$

Using elementary properties of trigonometric functions, we deduce that

$$\begin{aligned}
&G_3(y_1\phi_1+z_1\psi_1+\Phi(y_1\phi_1+z_1\psi_1))\\
&=3\Big(y_1^2\phi_1^2+2y_1z_1\phi_1\psi_1+z_1^2\psi_1^2\Big)\Big(-y_1\phi_1-z_1\psi_1\Big)+o(y_1^2+z_1^2)\\
&\qquad+6\Big(y_1\phi_1+z_1\psi_1\Big)\Big(y_1^2\psi_1^2-2y_1z_1\phi_1\psi_1+z_1^2\phi_1^2\Big)+o(y_1^2+z_1^2)\\
&=-3\Big(y_1^3\phi_1^3+3y_1^2z_1\phi_1^2\psi_1+3y_1z_1^2\phi_1\psi_1^2+z_1^3\psi_1^3\Big)\\
&\qquad+6\Big(y_1z_1^2\phi_1^3+(-2y_1^2z_1+z_1^3)\phi_1^2\psi_1+(y_1^3-2y_1z_1^2)\phi_1\psi_1^2+y_1^2z_1\psi_1^3\Big)\\
&\qquad+o(y_1^2+z_1^2)
\end{aligned}$$

$$\begin{aligned}
&= -3\Big((y_1^3 - 2y_1 z_1^2)\phi_1^3 + (7y_1^2 z_1 - 2z_1^3)\phi_1^2\psi_1 \\
&\quad + (z_1^3 - 2y_1^2 z_1)\psi_1^3\Big) + o(y_1^2 + z_1^2) \\
&= -3\Big((y_1^3 - 2y_1 z_1^2)\frac{3\phi_1 - \phi_3}{4} + (7y_1^2 z_1 - 2z_1^3)\frac{\psi_1 - \psi_3}{4} \\
&\quad + (7y_1 z_1^2 - 2y_1^3)\frac{\phi_1 + \phi_3}{4} + (z_1^3 - 2y_1^2 z_1)\frac{3\psi_1 + \psi_3}{4}\Big) + o(y_1^2 + z_1^2) \\
&= -3\Big(\frac{y_1^3 + y_1 z_1^2}{4}\phi_1 + \frac{9y_1 z_1^2 - 3y_1^3}{4}\phi_3 + \frac{y_1^2 z_1 + z_1^3}{4}\psi_1 + \frac{3z_1^3 - 9y_1^2 z_1}{4}\psi_3\Big) \\
&\quad + o(y_1^2 + z_1^2)
\end{aligned}$$

which yields that

$$\begin{aligned}
&\frac{1}{\pi}\langle G_3(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \phi_1\rangle \\
&\qquad = -\frac{3}{4}(y_1^3 + y_1 z_1^2) + o(|y_1|^3 + |z_1|^3), \\
&\frac{1}{\pi}\langle G_3(y_1\phi_1 + z_1\psi_1 + \Phi(y_1\phi_1 + z_1\psi_1)), \psi_1\rangle \\
&\qquad = -\frac{3}{4}(y_1^2 z_1 + z_1^3) + o(|y_1|^3 + |z_1|^3).
\end{aligned}$$

In the sequel, (3.2) becomes

$$\frac{d\mathbf{y}}{dt} = \beta_1 \mathbf{y} - \mathbf{F}(\mathbf{y}) + o(|\mathbf{y}|^3), \tag{3.4}$$

where $\mathbf{y} = (y_1, z_1)$ and

$$\mathbf{F}(\mathbf{y}) = d_1(y_1^3 + y_1 z_1^2,\ y_1^2 z_1 + z_1^3).$$

Here, d_1 is the number defined by (2.6). The equation (3.4) also appears as a bifurcation equation of the one-dimensional modified Swift-Hohenberg equation [1] and produces a similar patterns of solutions. Here, we provide the analysis of (3.4) for the sake of completeness.

Since $\beta_2 < 0$, we obtain that $d_1 > 0$. Furthermore, since

$$\langle \mathbf{F}(\mathbf{y}), \mathbf{y}\rangle = d_1(y_1^2 + z_1^2)^2 = d_1|\mathbf{y}|^4,$$

we have the following:

$$d_1|\mathbf{y}|^4 \le \langle \mathbf{F}(\mathbf{y}), \mathbf{y}\rangle \le 2d_1|\mathbf{y}|^4.$$

This implies by Theorem 5.10 of [7] that (3.4) bifurcates from the trivial solution to an attractor $\mathcal{A}_1(\lambda, \alpha)$ as λ passes through α which is homeomorphic to S^1.

We recall that the CCHE (1.1) is invariant under the odd periodic condition. We have seen that the CCHE bifurcates an attractor in $\widetilde{H}$ consisting of two steady solutions $\pm\rho_\alpha\phi_1 + o(\lambda-\alpha)$. Since the CCHE is invariant in H under the spatial translation, the static solution $u = \rho_\alpha\phi_1 + o(\lambda-\alpha)$ generates one parameter family of static solutions as follows: for $\theta \in \mathbb{R}$,

$$\begin{aligned} & \rho_\alpha \cos(x+\theta) + o(\lambda-\alpha) \\ = \ & \rho_\alpha \cos\theta \cdot \sin x \ + \ \rho_\alpha \sin\theta \cdot \cos x \ + o(\lambda-\alpha) \\ = \ & w_1\phi_1 + w_2\psi_1 + o(\lambda-\alpha). \end{aligned}$$

Since $w_1^2 + w_2^2 = \rho_\alpha^2$, this set of static solutions form an invariant circle. It is obvious that this circle is contained in the attractor $\mathcal{A}_1(\lambda, \alpha)$. Since $\mathcal{A}_1(\lambda, \alpha)$ is already homeomorphic to S^1, we conclude that $\mathcal{A}_1(\lambda, \alpha)$ consists of static solutions. This finishes the proof. □

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