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BOUNDED WEAK SOLUTION FOR THE HAMILTONIAN SYSTEM

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Abstract. We investigate the bounded weak solutions for the Hamiltonian system with bounded nonlinearity decaying at the origin and periodic condition. We get a theorem which shows the existence of the bounded weak periodic solution for this system. We obtain this result by using variational method, critical point theory for indefinite functional.

1. introduction

Let $G(t, z(t))$ be a C^2 function defined on $R^1 \times R^{2n}$ which is $2\pi-$periodic with respect to the first variable t. In this paper we investigate the number of 2π-periodic solutions of the following Hamiltonian system

$$
\begin{aligned}
\dot{p(t)} &= -G_q(t, p(t), q(t)), \\
\dot{q(t)} &= G_p(t, p(t), q(t)),
\end{aligned} \tag{1.1}
$$

where $p,\ q \in R^n,\ z = (p, q)$. Let J be the standard symplectic structure on R^{2n}, i. e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

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where I_n is the $n \times n$ identity matrix. Then (1.1) can be rewritten as

$$-J\dot{z} = G_z(t, z(t)), \tag{1.2}$$

where $\dot{z} = \frac{dz}{dt}$ and G_z is the gradient of G. We assume that $G \in C^2(R^1 \times R^{2n}, R^1)$ satisfies the following conditions:
$(G1)$ $G \in C^2(R^1 \times R^{2n}, R^1)$,
$(G2)$ $G(t, z(t)) = O(|z|^2)$ as $|z| \to 0$, $G(t, \theta) = 0$, $G_z(t, \theta) = \theta$, where $\theta = (0, \cdots, 0)$,
$(G3)$ there exists $C > 0$ such that $|G(t, \xi)| < C$ $\forall t \in R$, $\xi \in R^3$.
Several authors ([1], [3], [4], [5] etc.) studied the nonlinear hamiltonian system. Jung and Choi ([3], [4]) considered (1.1) with nonsingular potential nonlinearity or jumping nonlinearity crossing one eigenvalue, or two eigenvalues, or several eigenvalues. Chang ([1]) proved that (1.1) has at least two nontrivial $2\pi-$periodic weak solutions under some asymptotic nonlinearity. Jung and Choi ([3]) proved that (1.1) has at least m weak solutions, which are geometrically distinct and nonconstant under some jumping nonlinearity.

We are looking for the weak solutions of (1.1) with the conditions $(G1)$-$(G3)$. The 2π-periodic weak solution $z = (p, q) \in E$ of (1.1) satisfies

$$\int_0^{2\pi} (\dot{z} - JG_z(t, z(t))) \cdot Jw dt = 0 \qquad \text{for all } w \in E,$$

i. e.,

$$\int_0^{2\pi} [(\dot{p} + G_q(t, z(t))) \cdot \psi - (\dot{q} - G_p(t, z(t))) \cdot \phi] dt = 0$$
$$\text{for all } \zeta = (\phi, \psi) \in E,$$

where E is introduced in section 2.

Our main result is as follows:

THEOREM 1.1. *Assume that G satisfies the conditions $(G1) - (G3)$. Then system (1.1) has at least one bounded 2π-periodic solution.*

For the proof of our main result we approach the variational method and apply the critical point theory to indefinite functional. The outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce the perturbed operator A_ϵ such that A_ϵ^{-1} is a compact operator, and the associated functional $I(z)$ corresponding to the operator A_ϵ, prove that

$I(z)$ satisfies *Fréchet* differentiability, and state the critical point theorem for indefinite functional. In section 3, we show that the associated functional $I(z)$ satisfies the geometrical assumptions of the critical point theorem for indefinite functional, and prove Theorem 1.1.

2. Compact operator and variational approach

Let $L^2([0,2\pi],R^{2n})$ denote the set of $2n$-tuples of the square integrable $2\pi-$periodic functions and choose $z \in L^2([0,2\pi],R^{2n})$. Then it has a Fourier expansion $z(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{ikt}$, with $a_k = \frac{1}{2\pi}\int_0^{2\pi} z(t)e^{-ikt}dt \in C^{2n}$, $a_{-k} = \bar{a}_k$ and $\sum_{k\in Z}|a_k|^2 < \infty$. Let

$$A : z(t) \mapsto -J\dot{z}(t)$$

with domain

$$\begin{aligned} D(A) &= \{z(t) \in H^1([0,2\pi],R^{2n}) |\ z(0) = z(2\pi)\} \\ &= \{z(t) \in L^2([0,2\pi],R^{2n}) |\ \sum_{k\in Z}(\epsilon + |k|)^2|a_k|^2 < +\infty\}, \end{aligned}$$

where ϵ is a positive small number. Then A is a self-adjoint operator. Let $\{M_\lambda\}$ be the spectral resolution of A, and let α be a positive number such that $\alpha \notin \sigma(A)$ and $[-\alpha,\alpha]$ contains only one element 0 of $\sigma(A)$. Let

$$P_0 = \int_{-\alpha}^{\alpha} dM_\lambda, \qquad P_+ = \int_{\alpha}^{+\infty} dM_\lambda, \qquad P_- = \int_{-\infty}^{-\alpha} dM_\lambda.$$

Let

$$\begin{aligned} L_0 &= P_0 L^2([0,2\pi],R^{2n}), \\ L_+ &= P_+ L^2([0,2\pi],R^{2n}), \\ L_- &= P_- L^2([0,2\pi],R^{2n}). \end{aligned}$$

For each $u \in L^2([0,2\pi],R^{2n})$, we have the decomposition

$$u = u_0 + u_+ + u_-,$$

where $u_0 \in L_0$, $u_+ \in L_+$, $u_- \in L_-$. According to A, there exists a small number $\epsilon > 0$ such that $-\epsilon \notin \sigma(A)$. Let us define the space E as follows:

$$E = D(|A|^{\frac{1}{2}}) = \{z \in L^2([0,2\pi],R^{2n}) |\ \sum_{k\in Z}(\epsilon + |k|)|a_k|^2 < \infty\}$$

with the scalar product

$$(z,w)_E = \epsilon(z,w)_{L^2} + (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_{L^2}$$

and the norm

$$\|z\| = (z,z)_E^{\frac{1}{2}} = (\sum_{k\in Z}(\epsilon+|k|)|a_k|^2)^{\frac{1}{2}}.$$

The space E endowed with this norm is a real Hilbert space continuously embedded in $L^2([0,2\pi],R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-\frac{1}{2},2}([0,2\pi],R^{2n})$. We note that the operator $(\epsilon+|A|)^{-1}$ is a compact linear operator from $L^2([0,2\pi],R^{2n})$ to E such that

$$((\epsilon+|A|)^{-1}w, z)_E = \int_0^{2\pi}(w(t),z(t))dt.$$

Let

$$A_\epsilon = \epsilon I + A.$$

Let

$$E_0 = |A_\epsilon|^{-\frac{1}{2}}L_0, \quad E_+ = |A_\epsilon|^{-\frac{1}{2}}L_+, \quad E_- = |A_\epsilon|^{-\frac{1}{2}}L_-.$$

Then $E = E_0 \oplus E_+ \oplus E_-$ and for $z \in E$, z has the decomposition $z = z_0 + z_+ + z_- \in E$, where

$$z_0 = |A_\epsilon|^{-\frac{1}{2}}u_0, \qquad z_+ = |A_\epsilon|^{-\frac{1}{2}}u_+, \qquad z_- = |A_\epsilon|^{-\frac{1}{2}}u_-.$$

Thus we have

$$\|z_0\|_{E_0} = \|u_0\|_{L_0}, \qquad \|z_+\|_{E_+} = \|u_+\|_{L_+}, \qquad \|z_-\|_{E_-} = \|u_-\|_{L_-}$$

and that E_0, E_+, E_- are isomorphic to L_0, L_+, L_-, respectively. The associated functional of (1.2) on E is as follows:

$$\begin{aligned}(2.1)\ I(z) &= \frac{1}{2}(\|A_\epsilon^{\frac{1}{2}}z_+\|_{L^2}^2 + \|A_\epsilon^{\frac{1}{2}}M_+z_0\|^2 - \|(-A_\epsilon)^{\frac{1}{2}}z_-\|_{L^2}^2 \\ &\qquad -\|(-A_\epsilon)^{\frac{1}{2}}M_-z_0\|^2) - \psi_\epsilon(z), \\ &= \frac{1}{2}(\|u_+\|^2 + \|M_+u_0\|^2 - \|M_-u_0\|^2 - \|u_-\|^2) - \psi_\epsilon(z),\end{aligned}$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2}\|z\|_{L^2}^2$, $\psi(z) = \int_0^{2\pi}G(t,z(t))dt$. Let

$$F(z) = G_z(t,z(t)).$$

By $G \in C^2$, $\psi(z) = \int_0^{2\pi} G(t, z(t))dt \in C^2(S^1 \times D, R^1)$. Let

$$F_\epsilon(z) = \epsilon I + F(z) = \epsilon I + G_z(t, z(t)).$$

Then (1.2) can be rewritten as

$$A_\epsilon(z) = F_\epsilon(z). \tag{2.2}$$

The Euler equation of the functional $I(z)$ is the system

$$u_+ = |A_\epsilon|^{-\frac{1}{2}} P_+ F_\epsilon(z), \tag{2.3}$$

$$u_- = -|A_\epsilon|^{-\frac{1}{2}} P_- F_\epsilon(z), \tag{2.4}$$

$$\begin{aligned} M_+ u_0 &= |A_\epsilon|^{-\frac{1}{2}} M_+ P_0 F_\epsilon(z) \\ M_- u_0 &= -|A_\epsilon|^{-\frac{1}{2}} M_- P_0 F_\epsilon(z). \end{aligned} \tag{2.5}$$

Thus $z = z_0 + z_+ + z_-$ is a solution of (2.2) if and only if $u = u_0 + u_+ + u_-$ is a critical point of I. The system (2.3)-(2.5) is reduced to

$$A_\epsilon z_+ = P_+ F_\epsilon(z_0 + z_+ + z_-) \qquad \text{or} \quad z_+ = (A_\epsilon)^{-1} P_+ F_\epsilon(z_0 + z_+ + z_-), \tag{2.6}$$

$$A_\epsilon z_- = P_- F_\epsilon(z_0 + z_+ + z_-) \qquad \text{or} \quad z_- = (A_\epsilon)^{-1} P_- F_\epsilon(z_0 + z_+ + z_-), \tag{2.7}$$

$$\begin{aligned} A_\epsilon M_+ z_0 &= M_+ P_0 F_\epsilon(z_0 + z_+ + z_-), \\ A_\epsilon M_- z_0 &= M_- P_0 F_\epsilon(z_0 + z_+ + z_-). \end{aligned} \tag{2.8}$$

By the following Lemma 2.1, the weak solutions of (1.2) coincide with the critical points of the functional $I(z)$.

Lemma 2.1. *Assume that G satisfies the conditions $(G1) - (G3)$. Then $I(z)$ is continuous and Fréchet differentiable in E with Fréchet derivative*

$$DI(z)w = \int_0^{2\pi} (A_\epsilon z - F_\epsilon(z)) \cdot w dt \qquad \text{for all } w \in E, \tag{2.9}$$

Moreover $DI \in C$. That is, $I \in C^1$.

Proof. First we prove that $I(z) = \int_0^{2\pi}[\frac{1}{2}A_\epsilon z - G(t, z(t)) - \frac{\epsilon}{2}z^2]dt$ is continuous and *Fréchet* differentiable in E. For $z, w \in E$,

$$
\begin{aligned}
&|I(z+w) - I(z)| \\
&= |\int_0^{2\pi} \frac{1}{2}A_\epsilon(z+w)\cdot(z+w)dt - \int_0^{2\pi}[G(t, z+w) + \frac{\epsilon}{2}(z+w)^2]dt \\
&\qquad - \int_0^{2\pi}\frac{1}{2}A_\epsilon(z)\cdot z dt + \int_0^{2\pi}[G(t,z) + \frac{\epsilon}{2}z^2]dt \\
&= |\int_0^{2\pi}\frac{1}{2}[A_\epsilon(z)\cdot w + A_\epsilon(w)\cdot z + A_\epsilon(w)\cdot w]dt \\
&\qquad - \int_0^{2\pi}[G(t, z+w) - G(t,z) + \frac{\epsilon}{2}(2z\cdot w + w^2))]dt|.
\end{aligned}
$$

We have

$$
\begin{aligned}
(2.10) \qquad &|\int_0^{2\pi}[G(t, z+w) - G(t,z)]dt| \\
&\leq |\int_0^{2\pi}[G_z(t, z(t))\cdot w + O(\|w\|_{R^{2n}})]dt| = O(\|w\|_{R^{2n}}).
\end{aligned}
$$

Thus we have

$$|I(z+w) - I(z)| = O(\|w\|_{R^{2n}}).$$

Next we shall prove that $I(z)$ is *Fréchet* differentiable in E. For $z, w \in E$,

$$
\begin{aligned}
&|I(z+w) - I(z) - DI(z)w| \\
&= |\int_0^{2\pi}\frac{1}{2}A_\epsilon(z+w)\cdot(z+w)dt - \int_0^{2\pi}[G(t, z+w) + \frac{\epsilon}{2}(z+w)^2]dt \\
&\qquad - \int_0^{2\pi}\frac{1}{2}A_\epsilon(z)\cdot z dt + \int_0^{2\pi}[G(t,z) + \frac{\epsilon}{2}z^2]dt \\
&\qquad - \int_0^{2\pi}A_\epsilon(z)\cdot w dt + \int_0^{2\pi}[G_z(t,z) + \epsilon z\cdot w]dt \\
&= |\int_0^{2\pi}\frac{1}{2}[A_\epsilon(w)\cdot z dt + A_\epsilon(w)\cdot w]dt \\
&\qquad - \int_0^{2\pi}[G(t, z+w) - G(t,z) - G_z(t,z) + \frac{\epsilon}{2}w^2)]dt|.
\end{aligned}
$$

By (2.10), we have

$$\int_0^{2\pi} [G(t, z+w) - G(t, z) - G_z(t, z)]dt = O(\|w\|_{R^{2n}}).$$

Thus

$$|I(z+w) - I(z) - DI(z)w| = O(\|w\|_{R^{2n}}).$$

□

Now, we recall the critical point theorem for the indefinite functional (cf. [2]).

Let

$$B_r = \{u \in E | \ \|u\| \le r\},$$

$$\partial B_r = \{u \in E | \ \|u\| = r\}.$$

Theorem 2.1. *Critical point theorem for the indefinite functional) Let E be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^{\perp}$. Suppose that $I \in C^1(E, R)$, satisfies (PS), and*
(I1) $I(u) = \frac{1}{2}(Lu, u) + bu$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : E_i \to E_i$ is bounded and self adjoint, $i = 1, 2$,
(I2) b' is compact, and
(I3) there exists a subspace $\tilde{E} \subset E$ and sets $S \subset E$, $Q \subset \tilde{E}$ and constants $\alpha > \omega$ such that
(i) $S \subset E_1$ and $I|_S \ge \alpha$,
(ii) Q is bounded and $I|_{\partial Q} \le \omega$,
(iii) S and ∂Q link.
Then I possesses a critical value $c \ge \alpha$.

3. Proof of Theorem 1.1

We shall show that the functional $I(z)$ satisfies the geometric assumptions of the critical point theorem for indefinite functional.

Lemma 3.1. *Palais-Smale condition) Assume that G satisfies $(G1)-(G3)$. Then $I(z)$ satisfies the Palais-Smale condition: If for a sequence (z_k), $I(z_k)$ is bounded from above and $DI(z_k) \to 0$ as $k \to \infty$, then (z_k) has a convergent subsequence.*

Proof. Let (z_k) be a sequence with

$$I(z_k) \leq M \tag{3.1}$$

and

$$DI(z_k) = z_k - A_\epsilon^{-1}(G_z(t, z_k) + \epsilon z_k) \to \theta \qquad \text{as } m \to \infty, \tag{3.2}$$

where A_ϵ^{-1} is compact operator and $\theta = (0, \cdots, 0)$. We claim that $\{z_k\}$ has a convergent subsequence. Since $G_z(t, z(t)) + \epsilon z_k$ is bounded for a small constant ϵ and A_ϵ^{-1} is compact operator, by (3.2), $\{z_k\}$ has a convergent subsequence. Thus we prove the lemma. □

Let

$$B_\rho = \{z \in E | \ \|z\| \leq \rho\},$$

$$\partial B_\rho = \{z \in E | \ \|z\| = \rho\},$$

$$Q = (\bar{B}_R \cap E_-) \oplus \{re| \ e \in \partial B_1 \cap E_+ \ \ 0 < r < R\}.$$

LEMMA 3.2. *Assume that G satisfies the conditions $(G1) - (G3)$. there exist a constant $\rho > 0$ and sets $S \subset E$, $Q \subset E$ such that*
(i) $\partial B_\rho \subset E_+$ and $I|_{\partial B_\rho} > 0$,
(ii) Q is bounded and $I|_{\partial Q} < 0$,
(iii) ∂B_ρ and ∂Q link.

Proof. (i) Let $z \in E_+ \subset E$. Since $G(x, t, z)$ is bounded, there exists a constant $C > 0$ such that

$$\begin{aligned} I(z) &= \frac{1}{2}(\|A_\epsilon^{\frac{1}{2}} z_+\|_{L^2}^2 + \|A_\epsilon^{\frac{1}{2}} M_+ z_0\| - \|(-A_\epsilon)^{\frac{1}{2}} z_-\|_{L^2}^2 - \|(-A_\epsilon)^{\frac{1}{2}} M_- z_0\|^2) \\ &\quad - \int_0^{2\pi} G(t, z) dt - \frac{\epsilon}{2}\|z\|_{L^2}^2 \\ &\geq \frac{1}{2}(\|A_\epsilon^{\frac{1}{2}} z_+\|_{L^2}^2 - C - \frac{\epsilon}{2}\|z\|_{L^2}^2 \end{aligned}$$

for $C > 0$. Then there exist a constant $\rho > 0$ such that if $z \in \partial B_\rho \cap E_+$, then $I(z) > 0$.
(ii) Let us choose $e \in B_1 \cap E_+$. Let $z \in \bar{B}_r \cap E_- \oplus \{re| \ 0 < r\}$. Then $z = w + y$, $w \in \bar{B}_r \cap E_-$, $y = re$. We note that

$$\text{If } w \in \bar{B}_r \cap E_-, \text{then } \int_0^{2\pi} A_\epsilon(z) \cdot z dt = -\|(-A_\epsilon)^{\frac{1}{2}} z_-\|_{L^2}^2 \leq 0.$$

By (G3), $G(t, w + re)$ is bounded from below. Thus, there exists a constant $C_1 > 0$ such that if $z = w + re$, then we have

$$\begin{aligned} I(z) &= \frac{1}{2}r^2 - \|(-A_\epsilon)^{\frac{1}{2}} z_-\|^2_{L^2} - \int_0^{2\pi} G(t, w + re)dt - \frac{\epsilon}{2}\|z\|^2_{L^2} \\ &\leq \frac{1}{2}r^2 - \|(-A_\epsilon)^{\frac{1}{2}} z_-\|^2_{L^2} + C_1 - \frac{\epsilon}{2}\|z\|^2_{L^2}. \end{aligned}$$

We can choose a constant $R > r$ such that if $z = w + re \in Q = (\bar{B}_r \cap E_-) \oplus \{re|\ e \in B_1 \cap E_+,\ 0 < r < R\}$, then $I(z) < 0$. Thus we prove the lemma. □

Proof of Theorem 1.1

By Lemma 2.1, $I(z)$ is continuous and *Fréchet* differentiable in E and moreover $DI \in C$. By Lemma 3.1, $I(z)$ satisfies the $(P.S.)$ condition. By Lemma 3.2, there exist a constant $\rho > 0$ and sets $\partial B_\rho \subset E_+$ with radius $\rho > 0$, $Q \subset E$ such that $I|_{\partial B_\rho} > 0$, Q is bounded and $I|_{\partial Q} < 0$, and ∂B_ρ and ∂Q link. By Theorem 2.1, $I(z)$ possesses a critical value $c > 0$. Thus (1.1) has at least one nontrivial periodic weak solution. Thus we prove Theorem 1.1

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