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ON THE FIELD EQUATIONS IN $g - ESX_n$

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Abstract. This paper is a direct continuation of [1] and [2]. In this paper we investigate some properties of ES-curvature tensor and contracted ES-curvature tensor of $g - ESX_n$. Also, we study the field equations in the n-dimensional ES manifold $g - ESX_n$.

1. Preliminaries

This paper is a direct continuation of our previous paper [1] and [2], which will be denoted by I in the present paper. All considerations in this paper are based on our results and symbolism of I([1], [2], [3], [4], [5], [6], [7], [8], [9]). Whenever necessary, these results will be quoted in the text. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper.

(a) generalized n-dimensional Riemannian manifold X_n

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$\det\left(\frac{\partial x'}{\partial x}\right) \neq 0. \tag{1.1}$$

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In $n-g-UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}. \tag{1.2}$$

where

$$\mathfrak{g} = \det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \det(k_{\lambda\mu}). \tag{1.3}$$

In virtue of (1.3) we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{1.4}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner. There exists a unique tensor $*g^{\lambda\nu}$ satisfying

$$g_{\lambda\mu}{}^{*}g^{\lambda\nu} = g_{\mu\lambda}{}^{*}g^{\nu\lambda} = \delta^{\nu}_{\mu}. \tag{1.5}$$

It may be also decomposed into its symmetric part ${}^{*}h^{\lambda\nu}$ and skew-symmetric part ${}^{*}k^{\lambda\nu}$:

$${}^{*}g^{\lambda\nu} = {}^{*}h^{\lambda\nu} + {}^{*}k^{\lambda\nu}. \tag{1.6}$$

The manifold X_n is connected by a general real connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ with the following transformation rule:

$$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}}\left(\frac{\partial x^{\beta}}{\partial x^{\lambda'}}\frac{\partial x^{\gamma}}{\partial x^{\mu'}}\Gamma_{\beta}{}^{\alpha}{}_{\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x^{\lambda'}\partial x^{\mu'}}\right). \tag{1.7}$$

It may also be decomposed into its symmetric part $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ and its skew-symmetric part $S_{\lambda\nu}{}^{\nu}$, called the torsion tensor of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Lambda_{\lambda}{}^{\nu}{}_{\mu} + S_{\lambda\mu}{}^{\nu}; \quad \Lambda_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{(\lambda}{}^{\nu}{}_{\mu)}; \quad S_{\lambda\mu}{}^{\nu} = \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}. \tag{1.8}$$

A connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is said to be Einstein if it satisfies the following system of Einstein's equations:

$$\partial_{\omega}g_{\lambda\mu} - \Gamma_{\lambda}{}^{\alpha}{}_{\omega}g_{\alpha\mu} - \Gamma_{\omega}{}^{\alpha}{}_{\mu}g_{\lambda\alpha} = 0. \tag{1.9}$$

or equivalently

$$D_{\omega}g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}. \tag{1.10}$$

where D_{ω} is the symbolic vector of the covariant derivative with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ in (1.9), certain conditions are imposed. These conditions may be condensed to

$$S_{\lambda} = S_{\lambda\alpha}{}^{\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu}Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0. \tag{1.11}$$

where Y_λ is an arbitrary vector, and

$$R_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^{\nu}{}_{\omega]} + \Gamma_{\alpha}{}^{\nu}{}_{[\mu}\Gamma_{|\lambda|}{}^{\alpha}{}_{\omega]}). \tag{1.12}$$

If the system (1.10) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form (Hlavatý, 1957)

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}. \tag{1.13}$$

where $U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}$ and $\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$ are Christoffel symbols defined by $h_{\lambda\mu}$.

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}. \tag{1.14}$$

$$K_p = k_{[\alpha_1}{}^{\alpha_1}k_{\alpha_2}{}^{\alpha_2}\cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots). \tag{1.15}$$

$$ {}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \ {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^\alpha \ {}^{(p-1)}k_\alpha{}^\nu \quad (p = 1, 2, \cdots). \tag{1.16}$$

In X_n it was proved in [5] that

$$K_0 = 1, \ K_n = k \text{ if } n \text{ is even, and } \mathrm{K_p = 0 \text{ if } p \text{ is odd.}} \tag{1.17}$$

$$\begin{aligned} &\mathfrak{g} = \mathfrak{h}(1 + K_1 + K_2 + \cdots + K_n) \\ \text{or} \quad &g = 1 + K_1 + K_2 + \cdots + K_n. \end{aligned} \tag{1.18}$$

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s+p)}k_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \cdots). \tag{1.19}$$

We also use the following useful abbreviations for an arbitrary vector Y, for $p = 1, 2, 3, \cdots$:

$$ {}^{(p)}Y_\lambda = {}^{(p-1)} k_\lambda{}^\alpha Y_\alpha. \tag{1.20}$$

$$ {}^{(p)}Y^\nu = {}^{(p-1)} k^\nu{}_\alpha Y^\alpha. \tag{1.21}$$

(c) n-dimensional ES manifold ESX_n

In this subsection, we display an useful representation of the ES connection in n-g-UFT.

DEFINITION 1.1. A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} \tag{1.22}$$

for an arbitrary non-null vector X_μ.

A connection which is both semi-symmetric and Einstein is called an ES connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by $g_{\lambda\mu}$ by means of an ES connection, is called an n-dimensional ES manifold. We denote this manifold by $g - ESX_n$ in our further considerations.

THEOREM 1.2. *Under the condition* (1.22)*, the system of equations* (1.10) *is equivalent to*

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2k^\nu_{(\lambda} X_{\mu)} + 2\delta^\nu_{[\lambda} X_{\mu]}. \tag{1.23}$$

Proof. Substituting (1.22) for $S_{\lambda\mu}{}^\nu$ into (1.13), we have the representation (1.23). □

In $g - ESX_n$, the following theorem was proved in [1]:

THEOREM 1.3. *In $g - ESX_n$, the following relations hold for $p, q = 1, 2, 3, \cdots$:*

$$S_\lambda = (1-n) X_\lambda. \tag{1.24}$$

$$U_\lambda = \frac{1}{2}\partial_\lambda ln\mathfrak{g}. \tag{1.25}$$

$$^{(p+1)}S_\lambda = (1-n)^{(p)}U_\lambda. \tag{1.26}$$

$$^{(p)}U_\alpha{}^{(q)}X^\alpha = 0 \quad \textit{if} \quad p+q-1 \quad \textit{is} \quad \textit{odd}. \tag{1.27}$$

$$D_\lambda X_\mu = \nabla_\lambda X_\mu. \tag{1.28}$$

$$D_{[\lambda} X_{\mu]} = \nabla_{[\lambda} X_{\mu]} = \partial_{[\lambda} X_{\mu]}. \tag{1.29}$$

$$\nabla_{[\lambda} U_{\mu]} = 0, \qquad D_{[\lambda} U_{\mu]} = 2U_{[\lambda} X_{\mu]} = 2^{(2)}X_{[\lambda} X_{\mu]}. \tag{1.30}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols defined by $h_{\lambda\mu}$.

2. The ES curvature tensor and the contracted ES curvature tensor in $g - ESX_n$

This chapter is devoted to the study of the ES curvature tensor and the contracted ES curvature tensors in $g - ESX_n$ and of some useful identities involving them.

THEOREM 2.1. *In $g - ESX_n$, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = L_{\omega\mu\lambda}{}^{\nu} + M_{\omega\mu\lambda}{}^{\nu} + N_{\omega\mu\lambda}{}^{\nu}. \tag{2.1}$$

where

$$L_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\left\{ \begin{matrix} \nu \\ \omega]\lambda \end{matrix} \right\} + \left\{ \begin{matrix} \nu \\ \alpha[\mu \end{matrix} \right\}\left\{ \begin{matrix} \alpha \\ \omega]\lambda \end{matrix} \right\}\right). \tag{2.2}$$

$$M_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + \delta^{\nu}_{[\mu}\nabla_{\omega]}X_{\lambda} + \nabla_{[\mu}U^{\nu}{}_{\omega]\lambda}). \tag{2.3}$$

$$N_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{[\omega}X_{\mu]}X_{\lambda} +^{(2)} X_{\lambda}k_{[\mu}{}^{\nu}X_{\omega]}). \tag{2.4}$$

Proof. Substitute (1.13) into (1.12) and make use of (2.2) to obtain

$$\begin{aligned}
R_{\omega\mu\lambda}{}^{\nu} &= 2\partial_{[\mu}\left(\left\{ \begin{matrix} \nu \\ \omega]\lambda \end{matrix} \right\} + X_{\omega]}\delta^{\nu}_{\lambda} - \delta^{\nu}_{\omega]}X_{\lambda} + U^{\nu}{}_{\omega]\lambda}\right) \\
&+ 2\left(\left\{ \begin{matrix} \nu \\ \alpha[\mu \end{matrix} \right\} + \delta^{\nu}_{\alpha}X_{[\mu} - X_{\alpha}\delta^{\nu}_{[\mu} + U^{\nu}{}_{\alpha[\mu}\right) \\
&\times \left(\left\{ \begin{matrix} \alpha \\ \omega]\lambda \end{matrix} \right\} + X_{\omega]}\delta^{\alpha}_{\lambda} - \delta^{\alpha}_{\omega]}X_{\lambda} + U^{\alpha}{}_{\omega]\lambda}\right) \\
&= L_{\omega\mu\lambda}{}^{\nu} + 2\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + 2\left(\delta^{\nu}_{[\mu}\partial_{\omega]}X_{\lambda} - \delta^{\nu}_{[\mu}\left\{ \begin{matrix} \alpha \\ \omega]\lambda \end{matrix} \right\}X_{\alpha}\right) \\
&+ 2\left(\partial_{[\mu}U^{\nu}{}_{\omega]\lambda} + \left\{ \begin{matrix} \alpha \\ \lambda[\omega \end{matrix} \right\}U^{\nu}{}_{\mu]\alpha} + \left\{ \begin{matrix} \nu \\ \alpha[\mu \end{matrix} \right\}U^{\alpha}{}_{\omega]\lambda}\right) \\
&+ 2\left(\delta^{\nu}_{[\omega}X_{\mu]}X_{\lambda} - X_{\alpha}\delta^{\nu}_{[\mu}U^{\alpha}{}_{\omega]\lambda} + U^{\nu}{}_{\alpha[\mu}U^{\alpha}{}_{\omega]\lambda}\right)
\end{aligned} \tag{2.5}$$

In virtue of (1.22), the sum of the second, third and fourth terms on the right-hand side of (2.5) is $M_{\omega\mu\lambda}{}^{\nu}$. On the other hand, using (1.22), (1.25), and (1.27), we have

$$U^{\nu}{}_{\lambda\mu} = 2k_{(\lambda}{}^{\nu}X_{\mu)} \tag{2.6}$$

$$-X_{\alpha}\delta^{\nu}_{[\mu}U^{\alpha}{}_{\omega]\lambda} = 0 \tag{2.7}$$

$$U^{\nu}{}_{\alpha[\mu}U^{\alpha}{}_{\omega]\lambda} =^{(2)} X_{\lambda}k_{[\mu}{}^{\nu}X_{\omega]} \tag{2.8}$$

Substituting (2.7) and (2.8) into the fifth term of (2.5), we find that it is equal to $N_{\omega\mu\lambda}{}^{\nu}$. Consequently, our proof of the theorem is completed. □

The tensors

$$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}, \qquad V_{\omega\mu} = R_{\omega\mu\alpha}{}^{\alpha}. \tag{2.9}$$

are called *the first and second contracted ES curvature tensors* of the ES connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, respectively. We see in the following two theorems that they appear as functions of the vectors $X_{\lambda}, S_{\lambda}, U_{\lambda}$, and hence also as functions of $g_{\lambda\mu}$ and its first two derivatives in virtue of (1.24), (1.25) and (2.1).

THEOREM 2.2. *The first contracted ES curvature tensor $R_{\mu\lambda}$ in $g-ESX_n$ may be given by*

$$\begin{aligned} R_{\mu\lambda} = L_{\mu\lambda} + 2\partial_{[\mu}X_{\lambda]} + \nabla_{\mu}T_{\lambda} - \nabla_{\alpha}U^{\alpha}{}_{\mu\lambda} \\ +(n-1)X_{\mu}X_{\lambda} + U_{\mu}U_{\lambda}. \end{aligned} \tag{2.10}$$

where

$$L_{\mu\lambda} = L_{\alpha\mu\lambda}{}^{\alpha}. \tag{2.11}$$

$$T_{\lambda\mu}{}^{\nu} = S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}, \qquad T_{\lambda} = T_{\lambda\alpha}{}^{\alpha} = S_{\lambda} + U_{\lambda}. \tag{2.12}$$

Proof. Putting $\omega = \nu = \alpha$ in (2.1) and making use of (2.11), we have

$$R_{\mu\lambda} = L_{\mu\lambda} + M_{\alpha\mu\lambda}{}^{\alpha} + N_{\alpha\mu\lambda}{}^{\alpha}. \tag{2.13}$$

In virtue of (1.24) and (1.25), it follows from (2.3) that

$$\begin{aligned} M_{\alpha\mu\lambda}{}^{\alpha} &= 2\partial_{[\mu}X_{\lambda]} + (1-n)\nabla_{\mu}X_{\lambda} + \nabla_{\mu}U_{\lambda} - \nabla_{\alpha}U^{\alpha}{}_{\mu\lambda} \\ &= 2\partial_{[\mu}X_{\lambda]} + \nabla_{\mu}T_{\lambda} - \nabla_{\alpha}U^{\alpha}{}_{\mu\lambda}. \end{aligned} \tag{2.14}$$

On the other hand, in virtue of (1.25) the relation (2.4) gives

$$(2.15)\qquad \begin{aligned} N_{\alpha\mu\lambda}{}^{\alpha} &= (n-1)X_\mu X_\lambda +^{(2)} X_\mu^{(2)} X_\lambda -^{(2)} X_\lambda X_\mu k_\alpha{}^\alpha \\ &= (n-1)X_\mu X_\lambda + U_\mu U_\lambda. \end{aligned}$$

Our assertion follows immediately from (2.13), (2.14) and (2.15).

□

THEOREM 2.3. *The second contracted ES curvature tensor $V_{\omega\mu}$ in $g-ESX_n$ is a curl of the vector S_λ. That is,*

$$(2.16)\qquad V_{\omega\mu} = 2\partial_{[\omega}S_{\mu]}.$$

Proof. Putting $\lambda = \nu = \alpha$ in (2.1), we have

$$(2.17)\qquad V_{\omega\mu} = L_{\omega\mu\alpha}{}^{\alpha} + M_{\omega\mu\alpha}{}^{\alpha} + N_{\omega\mu\alpha}{}^{\alpha}.$$

In virtue of (1.11), (1.24), (1.25) and (1.30), the relations (2.2), (2.3) and (2.4) give

$$L_{\omega\mu\alpha}{}^{\alpha} = N_{\omega\mu\alpha}{}^{\alpha} = 0$$

$$M_{\omega\mu\alpha}{}^{\alpha} = 2(1-n)\partial_{[\omega}X_{\mu]} + 2\nabla_{[\mu}U_{\omega]} = 2(1-n)\partial_{[\omega}X_{\mu]} = 2\partial_{[\omega}S_{\mu]}$$

which together with (2.17) proves our assertion. □

THEOREM 2.4. *The tensor $R_{\mu\lambda}$ is symmetric when $n = 3$.*

Proof. The relation (2.10) may be written as

$$(2.18)\qquad \begin{aligned} R_{\mu\lambda} &= L_{\mu\lambda} + (3-n)\nabla_\mu X_\lambda - 2\nabla_{(\mu}X_{\lambda)} + \nabla_\mu U_\lambda \\ &\quad -\nabla_\alpha U^\alpha{}_{\mu\lambda} + (n-1)X_\mu X_\lambda + U_\mu U_\lambda. \end{aligned}$$

where use has been made of (1.24), (1.29) and (2.12). Hence, in virtue of (1.29) and (1.30) we have $R_{[\mu\lambda]} = 0$ if and only if $(3-n)\nabla_{[\mu}X_{\lambda]} = (3-n)\partial_{[\mu}X_{\lambda]} = 0$ □

REMARK 2.5. In the proof of the Theorem (2.4), we excluded the case that $\partial_{[\mu}X_{\lambda]} = 0$, because we assumed that X_λ is not a gradient vector in the definition of semi-symmetric connection in (1.22). In fact, the assumption that X_λ is not a gradient vector is essential in the discussions of the field equations in $g-ESX_n$.

THEOREM 2.6. *The contracted ES curvature tensors in $g-ESX_n$ are related by*

$$(2.19)\qquad 2R_{[\mu\lambda]} = 4\partial_{[\mu}X_{\lambda]} + V_{\mu\lambda}.$$

Proof. In virtue of (1.24), (1.29) and (1.30), the relation (2.19) may be proved from (2.18) as in the following way:

$$
\begin{aligned}
2R_{[\mu\lambda]} &= 2(3-n)\partial_{[\mu}X_{\lambda]} \\
&= 2(1-n)\partial_{[\mu}X_{\lambda]} + 4\partial_{[\mu}X_{\lambda]} \\
&= 2\partial_{[\mu}S_{\lambda]} + 4\partial_{[\mu}X_{\lambda]} \\
&= V_{\mu\lambda} + 4\partial_{[\mu}X_{\lambda]}.
\end{aligned} \tag{2.20}
$$

□

3. The field equations in $g - ESX_n$

By field equations we mean a set of partial equations for $g_{\lambda\mu}$. In the present section we are concerned with the geometry of field equations in $g - ESX_n$ and not with their physical applications. We saw in the previous section that ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ together with its contracted curvature tensor $R_{\mu\lambda}$ appear as a function of $g_{\lambda\mu}$. In order to obtain the tensor $g_{\lambda\mu}$ with which we started in dealing with (1.9), (1.10) and (1.11), we suggest the following conditions for it in terms of $R_{\mu\lambda}$

$$R_{[\mu\lambda]} = \partial_{[\mu}X_{\lambda]} \tag{3.1}$$

$$R_{(\mu\lambda)} = 0 \tag{3.2}$$

where X_λ is an arbitrary vector. The conditions (3.1) and (3.2) represent a system of n^2 differential equations of the second order for $g_{\lambda\mu}$.

The unified field theory in the n-dimensional ES manifold ESX_n is governed by the following set of equations: n^3 equations (1.10) under the conditions (1.22), which determine the unique ES connection $\Gamma_{\lambda\mu}{}^{\nu}$, and n^2 field equations (3.1) and (3.2) for n^2 unknowns $g_{\lambda\mu}$. In Theorem (3.3), it states that the unknowns Y_λ are uniquely determined in ESX_n. The conditions (3.1) and (3.2) are of a purely geometrical nature and physical interpretation is not involved in them a *priori*. Einstein suggested several different sets of field equations in his *four-dimensional unified field theory*. It would seem natural to follow the analogy of Einstein's field equations (1.11) in our manifold ESX_n, too. However, the restriction $S_\lambda = 0$ is too strong in our unified field theory in the ES manifold ESX_n, since this condition implies $X_\lambda = 0$ and hence $\Gamma_{\lambda\mu}{}^{\nu} = \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\}$ in virtue of (1.23) and (1.24). Therefore, we shall not adopt (1.11) as a starting point, exclude the condition $S_\lambda = 0$, and impose the field equations in ESX_n as given in (3.1) and (3.2).

REMARK 3.1. In our further considerations we restrict ourselves to the conditions

(3.3) $X_\lambda \neq 0$ *and* X_λ *not a gradient vector*

This restriction is quite natural in view of (3.1) and (3.2) and Remark (3.1). The first consequence of (3.3) is the following theorem.

THEOREM 3.2. *In* $g - ESX_n$ *we have*

(3.4) $$U^{\nu}{}_{\lambda\mu} \neq 0$$

Proof. Assume that $U^{\nu}{}_{\lambda\mu} \neq 0$. Then (1.22) implies that

(3.5) $$k_{\lambda\nu} X_\mu + k_{\mu\nu} X_\lambda = 0 \quad for\ every\ \lambda,\ \mu,\ \nu.$$

In virtue of the condition (3.3), there exists at least one fixed index δ such that $X_\delta \neq 0$. Hence

(3.6) $$k_{\lambda\nu} X_\delta + k_{\delta\nu} X_\lambda = 0 \quad for\ every\ \lambda,\ \nu.$$

Putting $\lambda = \delta$ in (3.6), we have $k_{\delta\nu} = 0$ for every ν. If $\lambda \neq \delta$, then $k_{\lambda\nu} = 0$ for every ν, since $k_{\delta\nu} = 0$. Hence we have

(3.7) $$k_{\lambda\nu} = 0 \quad for\ every\ \lambda,\ \nu$$

which is a contradiction to the non-symmetry of $g_{\lambda\mu}$. □

THEOREM 3.3. *In* $g - ESX_n$*, the field equation (3.1) is satisfied by a unique vector* Y_λ *given by*

(3.8) $$Y_\lambda = (3 - n) X_\lambda$$

when $n \neq 3$

Proof. In virtue of (2.18), we have

$$R_{[\mu\lambda]} = (3 - n)\partial_{[\mu} X_{\lambda]}$$

from which (3.8) follows.

□

THEOREM 3.4. *In $g - ESX_n$, the field equation (3.2) is equivalent to*

$$L_{\mu\lambda} + \nabla_{(\mu}T_{\lambda)} - \nabla_\alpha U^\alpha{}_{\mu\lambda} + (n-1)X_\mu X_\lambda + U_\mu U_\lambda = 0 \tag{3.9}$$

Proof. (3.9) is a immediate consequence of (2.10) and (3.2). □

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