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SET-VALUED CHOQUET-PETTIS INTEGRALS

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Abstract. In this paper, we introduce the Choquet-Pettis integral of set-valued mappings and investigate some properties and convergence theorems for the set-valued Choquet-Pettis integrals.

1. Introduction

Choquet [3] introduced the Choquet integral of real-valued functions with respect to a fuzzy measure which is a generalization of the Lebesgue integral. The notion of integral of set-valued mappings is very useful in many branches of mathematics like mathematical economics, control theory, convex analysis, etc. Several types of integrals of set-valued mappings were introduced and studied by Aumann [1], Di Piazza and Musial [6,7], El Amri and Hess [9], Jang, Kil, Kim and Kwon [10],Jang and Kwon [11], Zhang, Guo and Liu [17] and others. In [15] we introduced the Choquet-Pettis integral of Banach-valued functions in terms of the Choquet integral of real-valued functions.

In this paper, we introduce the Choquet-Pettis integral of set-valued mappings which is a generalization of the set-valued Pettis integral and investigate some properties and convergence theorems for the set-valued Choquet-Pettis integral.

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2. Preliminaries

Throughout this paper, Ω denotes an abstract nonempty set and Σ denotes a σ-algebra formed by subsets of Ω. X denotes a real Banach space with dual X^*. $C(X)$ denotes the family of all nonempty closed subsets of X, $CC(X)$ the family of all nonempty closed convex subsets of X, $CB(X)$ the family of all nonempty closed bounded convex subsets of X, $CWK(X)$ the family of all nonempty convex weakly compact subsets of X.

For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A.

For $A, B \in C(X)$, let $H(A, B)$ denote the Hausdorff metric of A and B defined by

$$H(A,B) = \max\left(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right),$$

where $d(a, B) = \inf_{b\in B} \|a - b\|$ and $d(b, A) = \inf_{a\in A} \|a - b\|$. Especially,

$$H(A,B) = \sup_{\|x^*\|\leq 1} |s(x^*,A) - s(x^*,B)|$$

whenever A, B are convex sets. The number $\|A\|$ is defined by

$$\|A\| = H(A, \{0\}) = \sup_{x\in A} \|x\|.$$

If $A \in CB(X)$ and $x_1^*, x_2^* \in X^*$, then

$$|s(x_1^*, A) - s(x_2^*, A)| \leq \|x_1^* - x_2^*\|\|A\|.$$

Note that $(CWK(X), H)$ is a complete metric space.

The mapping $F : [a, b] \to C(X)$ is called a *set-valued mapping.* F is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F)$ is measurable.

Definition 2.1. *([16]) A fuzzy measure* on a measurable space (Ω, Σ) is an extended real-valued set function $\mu : \Sigma \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \Sigma$.

When $\mu(\Omega) < \infty$, we say that μ is *finite.* When μ is finite, we define *the conjugate* μ^c of μ by

$$\mu^c(A) = \mu(\Omega) - \mu(A^C),$$

where A^C is the complement of $A \in \Sigma$.

A fuzzy measure μ is said to be *lower semi-continuous* if it satisfies

$$A_1 \subset A_2 \subset \cdots \text{ implies } \mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

A fuzzy measure μ is said to be *upper semi-continuous* if it satisfies

$$A_1 \supset A_2 \supset \cdots \text{ and } \mu(A_1) < \infty \text{ implies } \mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

A fuzzy measure μ is said to be *continuous* if it is both lower and upper semi-continuous.

The class of real-valued measurable functions is denoted by M and the class of nonnegative real-valued measurable functions is denoted by M^+.

Definition 2.2. ([3,12]) (i) *The Choquet integral* of $f \in M^+$ with respect to a fuzzy measure μ on $A \in \Sigma$ is defined by

$$(C)\int_A f d\mu = \int_0^{\infty} \mu((f \geq r) \cap A) dr,$$

where the right-hand side integral is the Lebesgue integral and $(f \geq r) = \{\omega \in \Omega \mid f(\omega) \geq r\}$ for all $r \geq 0$.

If $(C)\int_A f d\mu < \infty$, then we say that f is Choquet integrable on A with respect to μ. Instead of $(C)\int_\Omega f d\mu$, we will write $(C)\int f d\mu$.

(ii) Suppose $\mu(\Omega) < \infty$. The Choquet integral of $f \in M$ with respect to a fuzzy measure μ on $A \in \Sigma$ is defined by

$$(C)\int_A f d\mu = (C)\int_A f^+ d\mu - (C)\int_A f^- d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. When the right-hand side is $\infty - \infty$, the Choquet integral is not defined. If $(C)\int_A f d\mu$ is finite, then we say that f is Choquet integrable on A with respect to μ.

The Choquet integral is a generalization of the Lebesgue integral, since they coincide when μ is a classical $\sigma-$additive measure.

Definition 2.3. ([15]) *A function* $f : \Omega \to X$ *is called Choquet-Pettis integrable* if for each $x^* \in X^*$ the function $x^* f$ is Choquet integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. The vector x_A is called the *Choquet-Pettis integral* of f on A and is denoted by $(CP)\int_A f d\mu$.

Let $f, g \in M$. f and g are said to be *comonotonic* if $f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega')$ for $\omega, \omega' \in \Omega$. We denote $f \sim g$ when f and g are comonotonic [3]. A sequence $\{f_n\}$ of real-valued measurable functions is said to *converge* to f *in distribution,* in symbols $f_n \underrightarrow{D} f$, if

$$\lim_{n\to\infty} \mu((f_n \geq r)) = \mu(f \geq r)) \quad \text{e.c.},$$

where "e.c." stands "except at most countably many values of r" [5, 14].

THEOREM 2.4. *([18]) Let $A \in CC(X)$. Then the support function $s(\cdot, A) : X^* \to [-\infty, \infty]$ satisfies the followings:*

(1) *$s(\cdot, A)$ is positively homogeneous, i.e., $s(\lambda x^*, A) = \lambda s(x^*, A)$ for all $\lambda \geq 0$ and $x^* \in X^*$;*
(2) *$s(\cdot, A)$ is a convex function on X^*;*
(3) *$s(\cdot, A)$ is weak* lower semi-continuous on X^*.*

Conversely, if a function $\varphi : X^* \to [-\infty, \infty]$ satisfies the conditions (1)-(3), then there exists $A \in CC(X)$ such that $\varphi(x^*) = s(x^*, A)$ for each $x^* \in X^*$. The set A is unique and given by $A = \{x \in X : x^*(x) \leq \varphi(x^*)$ for all $x^* \in X^*\}$.

THEOREM 2.5. *([18]) If $A_n \in CWK(X)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s(x^*, A_n)$ exists for each $x^* \in X^*$, then there exists an $M > 0$ such that $\sup_{n\in\mathbb{N}} \|A_n\| \leq M$.*

3. Results

In this section, we introduce the Choquet-Pettis integral of set-valued mappings and obtain some properties and convergence theorems for the Choquet-Pettis integral. In the sequel, μ denotes a finite fuzzy measure on a measurable space (Ω, Σ).

DEFINITION 3.1. *A set-valued mapping $F : \Omega \to CWK(X)$ is said to be Choquet-Pettis integrable* on Ω if for each $x^* \in X^*$ $s(x^*, F)$ is Choquet integrable on Ω and for each $A \in \Sigma$ there exists $C_A \in CWK(X)$ such that $s(x^*, C_A) = (C) \int_A s(x^*, F) d\mu$ for all $x^* \in X^*$. In this case, we write $C_A = (CP) \int_A F d\mu$.

The set-valued Choquet-Pettis integral is a generalization of the set-valued Pettis integral. If μ is a classical complete $\sigma-$additive measure on (Ω, Σ), then the set-valued Choquet-Pettis integral coincides with the set-valued Pettis integral.

$F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are said to be *scalarly comonotonic* if for each $x^* \in X^*$ $s(x^*, F)$ and $s(x^*, G)$ are comonotonic. In this case, we write $F \sim_s G$.

THEOREM 3.2. (1) *If $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are scalarly comonotonic and Choquet-Pettis integrable on Ω, then $F+G$ is Choquet-Pettis integrable on Ω and for each $A \in \Sigma$*

$$(CP)\int_A (F+G)\,d\mu = (CP)\int_A F d\mu + (CP)\int_A G d\mu.$$

(2) *If $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω and $a \geq 0$, then aF is Choquet-Pettis integrable on Ω and for each $A \in \Sigma$*

$$(CP)\int_A aF d\mu = a(CP)\int_A F d\mu.$$

Proof. (1) Let $A \in \Sigma$. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Choquet-Pettis integrable on Ω, for each $x^* \in X^*$ $s(x^*, F)$ and $s(x^*, G)$ are Choquet integrable on Ω and there exists $C_A, D_A \in CWK(X)$ such that $s(x^*, C_A) = (C)\int_A s(x^*, F) d\mu$ and $s(x^*, D_A) = (C)\int_A s(x^*, G) d\mu$ for all $x^* \in X^*$. Hence for each $x^* \in X^*$ $s(x^*, F+G)$ is Choquet integrable on Ω. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are scalarly comonotonic,

$$(C)\int_A [s(x^*, F) + s(x^*, G)]\,d\mu = (C)\int_A s(x^*, F) d\mu + (C)\int_A s(x^*, G) d\mu$$

for all $x^* \in X^*$. Hence

$$\begin{aligned} s(x^*, C_A + D_A) &= s(x^*, C_A) + s(x^*, D_A) \\ &= (C)\int_A s(x^*, F) d\mu + (C)\int_A s(x^*, G) d\mu \\ &= (C)\int_A [s(x^*, F) + s(x^*, G)]\,d\mu \\ &= (C)\int_A s(x^*, F+G) d\mu \end{aligned}$$

for all $x^* \in X^*$. Hence $F+G$ is Choquet-Pettis integrable on Ω and

$$(CP)\int_A (F+G) d\mu = (CP)\int_A F d\mu + (CP)\int_A G d\mu.$$

(2) The proof is similar to (1).

□

A set $N \in \Sigma$ is called a *null set with respect to* μ if $\mu(A \cup N) = \mu(A)$ for all $A \in \Sigma$ [13]. The "almost everywhere" concept can be defined by using the "null set" in the same way as the classical measure theory.

THEOREM 3.3. *Let $f : \Omega \to X$ be Choquet-Pettis integrable on Ω and $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ be Choquet-Pettis integrable on Ω. Then*

(1) *if $f(\omega) \in F(\omega)$ on Ω, then $(CP)\int_\Omega f d\mu \in (CP)\int_\Omega F d\mu$;*
(2) *if $F(\omega) \subseteq G(\omega)$ on Ω, then $(CP)\int_\Omega F d\mu \subseteq (CP)\int_\Omega G d\mu$;*
(3) *if $F = G$ μ-a.e. and μ^c-a.e. on Ω, then $(CP)\int_\Omega F d\mu = (CP)\int_\Omega G d\mu$.*

Proof. (1) Since $f : \Omega \to X$ and $F : \Omega \to CWK(X)$ are Choquet-Pettis integrable on Ω, for each $x^* \in X^*$ x^*f and $s(x^*, F)$ are Choquet integrable on Ω and $(C)\int x^* f d\mu = x^*\big((CP)\int f d\mu\big)$ and $(C)\int s(x^*, F)d\mu = s\big(x^*, (CP)\int F d\mu\big)$. Since $f(\omega) \in F(\omega)$ on Ω, $x^* f \le s(x^*, F)$ on Ω for all $x^* \in X^*$ and so $(C)\int x^* f d\mu \le (C)\int s(x^*, F)d\mu$ for all $x^* \in X^*$. Hence $x^*\big((CP)\int f d\mu\big) \le s\big(x^*, (CP)\int F d\mu\big)$ for all $x^* \in X^*$. Since $(CP)\int F d\mu \in CWK(X)$, by the separation theorem $(CP)\int_\Omega f d\mu \in (CP)\int_\Omega F d\mu$.

(2) The proof is similar to (1).

(3) Since $F = G$ μ-a.e. and μ^c-a.e. on Ω, $s(x^*, F)^+ = s(x^*, G)^+$ μ-a.e. on Ω and $s(x^*, F)^- = s(x^*, G)^-$ μ^c-a.e. on Ω for all $x^* \in X^*$. Hence

$$\begin{aligned}(C)\int s(x^*, F)d\mu &= (C)\int s(x^*, F)^+ d\mu - (C)\int s(x^*, F)^- d\mu^c \\ &= (C)\int s(x^*, G)^+ d\mu - (C)\int s(x^*, G)^- d\mu^c \\ &= (C)\int s(x^*, G)d\mu\end{aligned}$$

for all $x^* \in X^*$. Thus $s\big(x^*, (CP)\int F d\mu\big) = s\big(x^*, (CP)\int G d\mu\big)$ for all $x^* \in X^*$. Since $(CP)\int F d\mu, (CP)\int G d\mu \in CWK(X)$, by the separation theorem $(CP)\int_\Omega F d\mu = (CP)\int_\Omega G d\mu$.

□

A set-valued mapping $F : \Omega \to C(X)$ is said to be *Choquet integrably bounded* on Ω if there exists a Choquet integrable function $g : \Omega \to \mathbb{R}^+$ such that $\|F(\omega)\| = \sup_{x \in F(\omega)} \|x\| \le g(\omega)$ for all $\omega \in \Omega$.

THEOREM 3.4. *Let μ be a continuous fuzzy measure and let X be a reflexive Banach space. If $F : \Omega \to CWK(X)$ is a scalarly measurable* *and Choquet integrably bounded set-valued mapping on Ω such that $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$, then $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω.*

Proof. Since $F : \Omega \to CWK(X)$ is scalarly measurable, $s(x^*, F)$ is measurable for all $x^* \in X^*$. Since $F : \Omega \to CWK(X)$ is Choquet integrably bounded on Ω, there exists a Choquet integrable function $g : \Omega \to \mathbb{R}^+$ such that $\|F(\omega)\| \leq g(\omega)$ for all $\omega \in \Omega$. Since g is Choquet integrable on Ω, $\|x^*\|g$ is also Choquet integrable on Ω for all $x^* \in X^*$. Since $s(x^*, F)^+ \leq \|x^*\|g$ on Ω for all $x^* \in X^*$, by [15, Remark 3.9] $s(x^*, F)$ is Choquet integrable on Ω for all $x^* \in X^*$. For each $A \in \Sigma$ we define a function $\varphi_A : X^* \to \mathbb{R}$ by $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$. Then φ_A is positively homogeneous and convex since $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$. $\{x^* \in X^* : \|x^*\| < 1\}$ is an open subset of X^* and for each $x^* \in X^*$ with $\|x^*\| < 1$, $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu \leq (C)\int_A \|x^*\|gd\mu = \|x^*\|(C)\int_A fs\mu < (C)\int_A gd\mu$. Thus φ_A is bounded on $\{x^* \in X^* : \|x^*\| < 1\}$. By [2, Proposition 19.9] φ_A is continuous on X^*. By Theorem 2.4 there exists $C_A \in CC(X)$ such that $\varphi_A(x^*) = s(x^*, C_A)$ for each $x^* \in X^*$. Since $|\varphi_A(x^*)| = \left|(C)\int_A s(x^*, F)d\mu\right| < \infty$ for each $x^* \in X^*$, $C_A \in CB(X)$ by the Resonance Theorem. Since X is reflexive, $C_A \in CWK(X)$ and $s(x^*, C_A) = \varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$ for each $x^* \in X^*$. Hence $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω. □

THEOREM 3.5. *Let $F : \Omega \to CWK(X)$ be a set-valued mapping on Ω such that $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$. Then the followings are equivalent:*

(1) *$F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω.*
(2) *$s(x^*, F)$ is Choquet integrable on Ω for all $x^* \in X^*$ and for each $A \in \Sigma$ the mapping $\varphi_A : X^* \to \mathbb{R}$, $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$, is $\tau(X^*, X)$-continuous, where $\tau(X^*, X)$ stands for the Mackey topology on X^*.*

Proof. (1) ⇒ (2). If $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω, then $s(x^*, F)$ is Choquet integrable on Ω for all $x^* \in X^*$ and for each $A \in \Sigma$ there exists $C_A \in CWK(X)$ such that $s(x^*, C_A) = (C)\int_A s(x^*, F)d\mu$ for all $x^* \in X^*$. Thus $\varphi_A(x^*) = s(x^*, C_A)$ for all $x^* \in X^*$. Since $C_A \in CWK(X)$, the mapping $x^* \mapsto s(x^*, C_A)$ is $\tau(X^*, X)$-continuous. Hence

$$\varphi_A : X^* \to \mathbb{R}, \varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu,$$

is $\tau(X^*, X)$-continuous.

(2) $\Rightarrow$ (1). Assume that (2) holds. For each $A \in \Sigma$ $\varphi_A : X^* \to \mathbb{R}$, $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$, is positively homogeneous. Since $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$, $\varphi_A : X^* \to \mathbb{R}$, $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$, is convex. Since φ_A is $\tau(X^*, X)$-continuous, for each $t \in \mathbb{R}$ the set $\{x^* \in X^* : \varphi_A(x^*) \le t\}$ is convex and $\tau(X^*, X)$-closed. Hence $\{x^* \in X^* : \varphi_A(x^*) \le t\}$ is weak* closed. Thus φ_A is weak* lower semi-continuous. By Theorem 2.4 there exists $C_A \in CC(X)$ such that $\varphi_A(x^*) = s(x^*, C_A)$ for all $x^* \in X^*$. Since $|\varphi_A(x^*)| = \left|(C)\int s(x^*, F)d\mu\right| < \infty$ for all $x^* \in X^*$, $C_A \in CB(X)$ by the Resonance Theorem. Since φ_A is $\tau(X^*, X)$-continuous, C_A is weakly compact, i.e., $C_A \in CWK(X)$. Thus there exists $C_A \in CWK(X)$ such that $s(x^*, C_A) = (C)\int_A s(x^*, F)d\mu$ for all $x^* \in X^*$. Therefore $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω.

□

Note that if $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω then $F : \Omega \to CWK(X)$ is scalarly measurable on Ω.

A sequence $\{F_n\}$ of scalarly measurable set-valued mappings is said to *converge scalarly* to F *in distribution,* in symbols $F_n \underrightarrow{sD} F$, if $s(x^*, F_n)$ converges to s^*F in distribution for all $x^* \in X^*$.

A sequence $\{A_n\}$ in $C(X)$ is said to *converge scalarly* to $A \in C(X)$, denoted by $\lim_{n\to\infty} A_n = A$ scalarly or $A_n \to A$ scalarly, if $\lim_{n\to\infty} s(x^*, A_n) = s(x^*, A)$ for all $x^* \in X^*$.

THEOREM 3.6. *Let X be a reflexive Banach space and let $\{F_n\}$ be a sequence of Choquet-Pettis integrable set-valued mappings on Ω and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$. If $\{F_n\}$ converges scalarly to F in distribution on Ω and $G : \Omega \to CWK(X)$ and $H : \Omega \to CWK(X)$ are Choquet-Pettis integrable set-valued mappings on Ω such that $\mu((s(x^*, H) \ge r)) \le \mu((s(x^*, F_n) \ge r)) \le \mu((s(x^*, G) \ge r))$ e.c. for $n = 1, 2, \cdots$ and $x^* \in X^*$, then F is Choquet-Pettis integrable on Ω and $(CP)\int F_n d\mu \to (CP)\int F d\mu$ scalarly.*

Proof. Since $G : \Omega \to CWK(X)$ and $H : \Omega \to CWK(X)$ are Choquet-Pettis integrable set-valued mappings on Ω, for each $x^* \in X^*$

$s(x^*, G)$ and $s(x^*, H)$ are Choquet integrable on Ω. Since $\{F_n\}$ converges scalarly to F in distribution on Ω, for each $x^* \in X^*$ $\{s(x^*, F_n)\}$ converges to $s(x^*, F)$ in distribution on Ω. Since $\mu((s(x^*, H) \geq r)) \leq \mu((s(x^*, F_n) \geq r)) \leq \mu((s(x^*, G) \geq r))$ e.c. for $n = 1, 2, \cdots$ and $x^* \in X^*$, by [5, Theorem 8.9] $s(x^*, F)$ is Choquet ihtegrable on Ω and $\lim_{n\to\infty}(C)\int_A s(x^*, F_n)d\mu = (C)\int_A s(x^*, F)d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$. Since F_n is Choquet-Pettis integrable on Ω for $n = 1, 2, \cdots$, for each $A \in \Sigma$ there exists $C_{n,A} \in CWK(X)$ such that $s(x^*, C_{n,A}) = (C)\int_A s(x^*, F_n)d\mu$ for $n = 1, 2, \cdots$ and $x^* \in X^*$.

For each $A \in \Sigma$ we define a function $\varphi_A : X^* \to \mathbb{R}$ by $\varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$. Then φ_A is positively homogeneous and convex since $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$.
Since $C_{n,A} = (CP)\int_A F_n d\mu \in CWK(X)$ for $n = 1, 2, \cdots$ and $\lim_{n\to\infty} s(x^*, (CP)\int_A F_n d\mu) = \lim_{n\to\infty}(C)\int_A s(x^*, F_n)d\mu = (C)\int_A s(x^*, F)d\mu$ exists for each $x^* \in X^*$, by Theorem 2.5 there exists $M > 0$ such that $\sup_{n\in\mathbb{N}}\|(CP)\int_A F_n d\mu\| \leq M$. For given $\epsilon > 0$ let $\delta = \epsilon/M$. If $x^*, y^* \in X^*$ and $\|x^* - y^*\| < \delta$, then

$$
\begin{aligned}
&|\varphi_A(x^*) - \varphi_A(y^*)| \\
&= \left|(C)\int_A s(x^*, F)d\mu - (C)\int_A s(y^*, F)d\mu\right| \\
&= \lim_{n\to\infty}\left|(C)\int_A s(x^*, F_n)d\mu - (C)\int_A s(y^*, F_n)d\mu\right| \\
&= \lim_{n\to\infty}\left|s(x^*, (CP)\int_A F_n d\mu) - s(y^*, (CP)\int_A F_n d\mu)\right| \\
&\leq \lim_{n\to\infty}\|x^* - y^*\|\left\|(CP)\int_A F_n d\mu\right\| \\
&\leq M\|x^* - y^*\| \\
&< M\delta = \epsilon.
\end{aligned}
$$

Thus φ_A is continuous on X^*. By Theorem 2.4 there exists $C_A \in CC(X)$ such that $\varphi_A(x^*) = s(x^*, C_A)$ for each $x^* \in X^*$.
Since $|\varphi_A(x^*)| = \left|(C)\int_A s(x^*, F)d\mu\right| < \infty$ for each $x^* \in X^*$, $C_A \in CB(X)$ by the Resonance Theorem. Since X is reflexive, $C_A \in CWK(X)$ and $s(x^*, C_A) = \varphi_A(x^*) = (C)\int_A s(x^*, F)d\mu$ for each $x^* \in X^*$.
Hence $F : \Omega \to CWK(X)$ is Choquet-Pettis integrable on Ω and

$$\lim_{n\to\infty} s(x^*, (CP)\int_A F_n d\mu) = \lim_{n\to\infty}(C)\int_A s(x^*, F_n)d\mu$$
$$= (C)\int_A s(x^*, F)d\mu = s(x^*, (CP)\int_A F d\mu).$$

Thus $(CP)\int_A F_n\, d\mu \to (CP)\int_A F d\mu$ scalarly.
In particular, $(CP)\int F_n d\mu \to (CP)\int F d\mu$ scalarly.

□

THEOREM 3.7. *Let μ be a continuous fuzzy measure and let X be a reflexive Banach space and let $\{F_n\}$ be a sequence of Choquet-Pettis integrable set-valued mappings on Ω and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$.*

(1) *If $F_n \uparrow F$ scalarly on Ω and there exsits a Choquet integrable function g such that $(s(x^*, F_1))^- \leq g$ on Ω for all $x^* \in X^*$, then F is Choquet-Pettis integrable on Ω and $(CP)\int F_n d\mu \uparrow (CP)\int F d\mu$ scalarly.*

(2) *If $F_n \downarrow F$ scalarly on Ω and there exsits a Choquet integrable function g such that $(s(x^*, F_1))^+ \leq g$ on Ω for all $x^* \in X^*$, then F is Choquet-Pettis integrable on Ω and $(CP)\int F_n d\mu \downarrow (CP)\int F d\mu$ scalarly.*

Proof. Since $F_n \uparrow F$ scalarly on Ω and there exsits a Choquet integrable function g such that $(s(x^*, F_1))^- \leq g$ on Ω for all $x^* \in X^*$, by [15, Remark 3.9] $s(x^*, F)$ is Choquet integrable on Ω and $(C)\int_A s(x^*, F_n)d\mu \uparrow (C)\int_A s(x^*, F)d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$. Since F_n is Choquet-Pettis integrable on Ω for $n = 1, 2, \cdots$, for each $A \in \Sigma$ there exists $C_{n,A} \in CWK(X)$ such that

$$s(x^*, C_{n,A}) = (C)\int_A s(x^*, F_n)d\mu, \text{ for } n = 1, 2, \cdots \text{ and } x^* \in X^*.$$

Using the same method as in the proof of Theorem 3.6, we can obtain that F is Choquet-Pettis integrable on Ω.
Since $(C)\int_A s(x^*, F_n)d\mu \uparrow (C)\int_A s(x^*, F)d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$, $(CP)\int_A F_n d\mu \uparrow (CP)\int_A F d\mu$ scalarly for all $A \in \Sigma$. In particular, $(CP)\int F_n d\mu \uparrow (CP)\int F d\mu$ scalarly.

(2) The proof is similar to (1).

□

THEOREM 3.8. *Let μ be a continuous fuzzy measure and let X be a reflexive Banach space and let $\{F_n\}$ be a sequence of Choquet-Pettis integrable set-valued mappings on Ω and let $F : \Omega \to CWK(X)$ be a set-valued mapping such that $s(x^*, F) \sim s(y^*, F)$ for each $x^*, y^* \in X^*$. If $\{F_n\}$ converges scalarly to F μ-a.e. and μ^c-a.e. on Ω and there exsit Choquet integrable functions g and h such that $h \leq s(x^*, F_n) \leq g$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, then F is Choquet-Pettis integrable on Ω and $(CP)\int F_n d\mu \to (CP)\int F d\mu$ scalarly.*

Proof. Since $\{F_n\}$ converges scalarly to F μ-a.e. on Ω, $(s(x^*, F_n))^+ \to (s(x^*, F))^+$ μ-a.e. on Ω for all $x^* \in X^*$. Since $s(x^*, F_n) \leq g$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, $(s(x^*, F_n))^+ \leq g^+$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$. By [17, Theorem 2.7] $(s(x^*, F))^+$ is Choquet integrable on Ω with respect to μ and $\lim_{n\to\infty}(C)\int_A (s(x^*, F_n))^+ \, d\mu = (C)\int_A (s(x^*, F))^+ d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$. Since $\{F_n\}$ converges scalarly to F μ^c-a.e. on Ω and $h \leq s(x^*, F_n)$ on Ω for $n = 1, 2, \cdots$ and $x^* \in X^*$, $(s(x^*, F))^-$ is also Choquet integrable on Ω with respect to μ^c and $\lim_{n\to\infty}(C)\int_A (s(x^*, F_n))^- \, d\mu^c = (C)\int_A (s(x^*, F))^- d\mu^c$ for all $A \in \Sigma$ and $x^* \in X^*$. Hence $s(x^*, F)$ is Choquet integrable on Ω with respect to μ and

$$\begin{aligned}
&\lim_{n\to\infty}(C)\int_A s(x^*, F_n) d\mu \\
&= \lim_{n\to\infty}\left[(C)\int_A (s(x^*, F_n))^+ d\mu - (C)\int_A (s(x^*, F_n))^- d\mu^c\right] \\
&= (C)\int_A (s(x^*, F))^+ d\mu - (C)\int_A (s(x^*, F))^- d\mu^c \\
&= (C)\int_A (s(x^*, F)) d\mu
\end{aligned}$$

for all $A \in \Sigma$ and $x^* \in X^*$. Since F_n is Choquet-Pettis integrable on Ω for $n = 1, 2, \cdots$, for each $A \in \Sigma$ there exists $C_{n,A} \in CWK(X)$ such that $s(x^*, C_{n,A}) = (C)\int_A s(x^*, F_n) d\mu$ for all $x^* \in X^*$, i.e., $C_{n,A} = (CP)\int_A F_n d\mu$.

Using the same method as in the proof of Theorem 3.6, we can obtain that F is Choquet-Pettis integrable on Ω and for each $A \in \Sigma$ $\lim_{n\to\infty}(C)\int_A s(x^*, F_n) d\mu = (C)\int_A s(x^*, F) d\mu$ for all $x^* \in X^*$. Thus $(CP)\int_A F_n d\mu \to (CP)\int_A F d\mu$ scalarly. In particular, $(CP)\int F_n d\mu \to (CP)\int F d\mu$ scalarly.

□

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