Korean J. Math. **20** (2012), No. 4, pp. 395–402
http://dx.doi.org/10.11568/kjm.2012.20.4.395

DIAMETER OF THE DIRECT PRODUCT OF WIELANDT GRAPH

Sooyeon Kim and Byung Chul Song*

Abstract. A digraph D is primitive if there is a positive integer k such that there is a walk of length k between arbitrary two vertices of D. The exponent of a primitive digraph is the least such k. Wielandt graph W_n of order n is known as the digraph whose exponent is $n^2 - 2n + 2$, which is the maximum of all the exponents of the primitive digraphs of order n. It is known that the diameter of the multiple direct product of a digraph W_n strictly increases according to the multiplicity of the product. And it stops when it attains to the exponent of W_n . In this paper, we find the diameter of the direct product of Wielandt graphs.

1. Introduction

A digraph $D = (V, A)$ is primitive if there is a positive integer k such that for each pair u, v of vertices of D, there is a directed walk from u to v of length k in D. We say that the smallest such k to be the exponent of D and denote it by $\exp(D)$. For each pair of vertices $u,\ v$ of D if there is a directed walk from u to v of length k, then we use the notation $u \xrightarrow{k} v$. In [7], Wielandt stated that the maximum exponent of the primitive digraphs of order n is $n^2 - 2n + 2$. And he also provide the

Received August 1, 2012. Revised October 5, 2012. Accepted October 10, 2012.
2010 Mathematics Subject Classification: 05C12, 05C20, 05C50.
Key words and phrases: diameter, direct product, Wielandt graph.
This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.
*Corresponding author.

digraph, say the Wielandt graph W_n, which has the maximum exponent for all primitive digraph of order n. See also [5]. The Wielandt graph $W_n = (V, A)$ is a digraph with the vertex set $V = \{0, 1, 2, \ldots, n-1\}$ and the arc set $A = \{(i, i+1) | 0 \leq i \leq n-2\} \bigcup \{(n-1, 0), (n-1, 1)\}$. For example, Wielandt graph of order 5 is as in Figure 1.

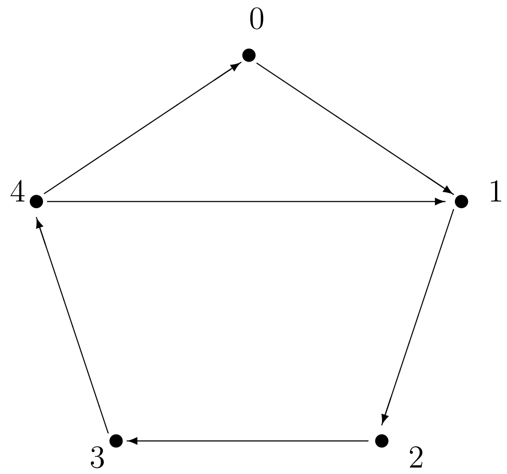

Figure 1. Wielandt graph W_5 of order 5

For a digraph $D = (V, A)$, the distance $\mathrm{dist}(u, v)$ from a vertex u to a vertex v is the smallest k such that $u \xrightarrow{k} v$.

The diameter of the digraph D is defined by

$$\mathrm{diam}(D) = \max_{u,v \in V} \{\mathrm{dist}(u, v)\}.$$

It is obvious that D is strongly connected if and only if $\mathrm{diam}(D)$ is finite. Moon [4] found a relation between the diameter, the minimum degree and the number of vertices of a graph. The relation implies that if a graph with n vertices has diameter $d \geq 3$ and has minimum degree $r \geq 2$, then $d \leq \frac{3n-2r-6}{r}$.

For two digraphs $D = (V_D, A_D)$, $E = (V_E, A_E)$, define the direct product $D \times E = (V, A)$ of D and E by a digraph where

$$V = V_D \times V_E$$

and

$$A = \{((u_1, u_2), (v_1, v_2)) | (u_1, v_1) \in A_D \text{ and } (u_2, v_2) \in A_E\}.$$

Weichsel [6] and MacAndrew [3] studied the connectivity of the direct product of graphs or digraphs. Lamprey and Barnes [2] showed that the exponent of the direct product of two digraphs D and E satisfies

$$\exp(D \times E) = \max\{\exp(D), \exp(E)\}.$$

As an example of the direct product of digraphs, we consider the direct product of Wielandt graph of order 5. $W_5 \times W_5$ is the digraph with vertex set $\{(i,j)|0 \le i,\ j \le 4\}$ and $((i,j),(i',j'))$ is an arc only when (1) $i' = i+1$ and $j' = j+1$ for $0 \le i, j \le 3$, (2) $i' = 0$ or 1 with $i = 4$ and $j' = j+1$ for $0 \le j \le 3$, (3) $j' = 0$ or 1 with $j = 4$, and $i' = i+1$ for $0 \le i \le 3$ or (4) $(i,j) = (4,4)$ and (i',j') is $(0,0)$, $(0,1)$, $(1,0)$ or $(1,1)$.

Kim, Song and Hwang [1] showed that the diameter of the multiple direct product of a primitive digraph D strictly increases and it stops when it attains to the exponent of D.

In this paper, we show that the diameter of the direct product of Wielandt graph W_n of order n is $\frac{n^2}{2}$ for even n, and $\frac{n^2+1}{2}$ for odd n.

2. Main theorem

Let W_n be the Wielandt graph of order n. Note that $\text{diam}(W_n) = \text{dist}(1,0) = n-1$.

For a Wielandt graph of order n, the following are strightforward.

1. For $i \neq 0$, $i \xrightarrow{k} i$ if and only if $k = pn + q(n-1)$ for nonnegative $p,\ q$;
2. $0 \xrightarrow{k} 0$ if and only if $k = pn + q(n-1)$ for positive p and nonnegative q.

From now on we use $\alpha_{(n)}$ if there is an integer p such that $\alpha - pn = \alpha_{(n)}$ with $0 \le \alpha_{(n)} \le n-1$.

Lemma 1. *Let W_n be the Wielandt graph of order n and i be vertex of W_n. If $0 \xrightarrow{k} i$ for some k, then for vertex d, $d \xrightarrow{k} (d+i)_{(n)}$.*

Proof. If $k < n$, then trivially $k = i$ and $d \xrightarrow{k} (d+i)_{(n)}$.

Let $k \ge n$. In this case since $d \overset{pn+q(n-1)}{\longrightarrow} d$ for some nonnegative p and q and $k \ge n > i$, $k = pn + q(n-1) + i$ where at least one of p and q is positive. If $p \ge 1$, then since

$$d \overset{(n-1)-d}{\longrightarrow} (n-1) \xrightarrow{2} 1 \overset{(p-1)n+q(n-1)}{\longrightarrow} 1 \overset{d+i-1}{\longrightarrow} (d+i)_{(n)},$$

we have $d \xrightarrow{k} (d+i)_{(n)}$. If $q \geq 1$, then since

$$d \overset{(n-1)-d}{\longrightarrow} (n-1) \xrightarrow{1} 1 \overset{pn+(q-1)(n-1)}{\longrightarrow} 1 \overset{d+i-1}{\longrightarrow} (d+i)_{(n)},$$

we have $d \xrightarrow{k} (d+i)_{(n)}$. □

Note that the converse of Lemma 1 doesn't hold. For example, $n - 1 \xrightarrow{2} 2 = ((n-1)+3)_{(n)}$, but $0 \xrightarrow{2} 3$ is impossible.

Lemma 2. *The diameter of the direct product $W_n \times W_n$ of Wielandt graphs satisfies*

$$\operatorname{diam}(W_n \times W_n) = \max_{0 \leq i,j \leq n-1} \operatorname{dist}((0,0),(i,j)).$$

Proof. By the definition

$$\operatorname{diam}(W_n \times W_n) \geq \max_{0 \leq i,j \leq n-1} \operatorname{dist}((0,0),(i,j)).$$

If $\operatorname{dist}((\alpha,\beta),((\alpha+i)_{(n)},(\beta+j)_{(n)})) = k$ and $l \leq k-1$, then at least one of $\alpha \xrightarrow{l} (\alpha+i)_{(n)}$ and $\beta \xrightarrow{l} (\beta+j)_{(n)}$ is impossible. By Lemma 1, at least one of $0 \xrightarrow{l} i$ and $0 \xrightarrow{l} j$ is impossible. Therefore $\operatorname{dist}((0,0),(i,j)) \geq k$ and we get

$$\begin{aligned}\operatorname{diam}(W_n \times W_n) &= \max_{0 \leq \alpha,\beta,i,j \leq n-1} \operatorname{dist}((\alpha,\beta),((\alpha+i)_{(n)},(\beta+j)_{(n)})) \\ &\leq \max_{0 \leq i,j \leq n-1} \operatorname{dist}((0,0),(i,j)).\end{aligned}$$

□

Proposition 1. *For n is even, the diameter of the direct product $W_n \times W_n$ of Wielandt graphs of order n is $\frac{n^2}{2}$.*

Proof. Since $(0,0) \xrightarrow{i} (i,i)$ and $i \leq n-1$, we have

$$\operatorname{dist}((0,0),(i,i)) = i \leq n-1 < \frac{n^2}{2}.$$

Since $\operatorname{dist}((0,0),(i,j)) = \operatorname{dist}((0,0),(j,i))$, there is no loss of generality we may assume that $0 \leq i < j \leq n-1$.

If $j - i < \frac{n}{2}$, then $j - i \leq \frac{n-2}{2}$. Since $0 \xrightarrow{i} i \overset{(j-i)n}{\longrightarrow} i$ and $0 \xrightarrow{j} j \overset{(j-i)(n-1)}{\longrightarrow} j$, there is a walk from $(0,0)$ to (i,j) of length $i + (j-i)n$. We have

$$i + (j-i)n \leq i + \frac{n-2}{2}n = (n+1) + \frac{n-2}{2}n < \frac{n^2}{2}.$$

If $j-i \geq \frac{n}{2}$, then $n-(j-i) \leq \frac{n}{2}$ and $i \leq j-\frac{n}{2} \leq \frac{n-2}{2}$. If we assume that $i \neq 0$, then $0 \xrightarrow{i} i \xrightarrow{(n-(j-i))(n-1)} i$ and $0 \xrightarrow{j} j \xrightarrow{n(n-(j-i)-1)} j$. Hence there is a walk from $(0,0)$ to (i,j) of length $i+(n-(j-i))(n-1)$. We have

$$i+(n-(j-i))(n-1) \leq i+\frac{n}{2}(n-1) = \frac{n-2}{2}+\frac{n}{2}(n-1) \leq \frac{n^2}{2}.$$

If $i=0$ and $j=\frac{n}{2}$, then $0 \xrightarrow{(\frac{n}{2})n} 0$ and $0 \xrightarrow{\frac{n}{2}} \frac{n}{2} \xrightarrow{\frac{n}{2}(n-1)} \frac{n}{2}$. There is a walk from $(0,0)$ to (i,j) of length $\frac{n^2}{2}$.

If $i=0$ and $j>\frac{n}{2}$, then $n-j<\frac{n}{2}$ and so $n-j \leq \frac{n-2}{2}$. Since $0 \xrightarrow{n+(n-j)(n-1)} 0$ and $0 \xrightarrow{j+(n-j)n} j$, we have $(0,0) \xrightarrow{j+(n-j)n} (0,j)$. In this case we have

$$j+n(n-j) \leq n-1+n(\frac{n-2}{2}) = \frac{n^2-2}{2} < \frac{n^2}{2}.$$

Therefore for $0 \leq i,j \leq n-1$, we have $\operatorname{dist}[(0,0),(i,j)] \leq \frac{n^2}{2}$. Moreover we have

$$\operatorname{diam}(W_n \times W_n) \geq \max_{0 \leq i,j \leq n-1} \operatorname{dist}((0,0),(i,j)) \leq \frac{n^2}{2}.$$

To prove the converse, assume that $(0,0) \xrightarrow{\alpha} (0,\frac{n}{2})$ for some $\alpha \leq \exp(W_n) = n^2-2n+2$. Then we have $\alpha = n+p_1n+p_2(n-1) = \frac{n}{2}+q_1n+q_2(n-1)$ for some nonnegative integers p_1,p_2,q_1,q_2 and with $0 \leq p_2, q-2 \leq n-1$. It is enough to show that $\alpha \geq \frac{n^2}{2}$. By (1), we have

$$n[1+2(p_1-q_1)-2(q_2-p_2)] = 2(q_2-p_2).$$

Since $2-2n \leq 2(p_2-q_2) \leq 2n-2$ and $1+2(p_1-q_1)-2(q_2-p_2) \neq 0$, we know that $2(p_2-q_2)$ is n or $-n$. If $p_2 = q_2+\frac{n}{2}$, then $p_2 \geq \frac{n}{2}$. We have

$$\alpha = n+p_1n+p_2(n-1) \geq n+\frac{n}{2}(n-1) = \frac{n^2+n}{2} > \frac{n^2}{2}.$$

If $q_2 = p_2+\frac{n}{2}$, then $q_2 \geq \frac{n}{2}$. As a consequence

$$\alpha = \frac{n}{2}+q_1n+q_2(n-1) \geq \frac{n}{2}+\frac{n}{2}(n-1) = \frac{n^2}{2}.$$

□

Proposition 2. *For n is odd, the diameter of the direct product $W_n \times W_n$ of Wielandt graphs of order n is $\frac{n^2+1}{2}$.*

Proof. If $j-i<\frac{n}{2}$, then $j-i\le\frac{n-1}{2}$. Since $0\xrightarrow{i}i\xrightarrow{(j-i)n}i$ and $0\xrightarrow{j}j\xrightarrow{(j-i)(n-1)}j$, there is a walk from $(0,0)$ to (i,j) of length $i+(j-i)n$. We have

$$\begin{aligned} i+(j-i)n=(j-i)(n-1)+j &\le \frac{(n-1)(n+1)}{2} \\ &\le \frac{n^2-1}{2}<\frac{n^2+1}{2}. \end{aligned}$$

If $j-i>\frac{n}{2}$, then $n-(j-i)<\frac{n}{2}$ and so $n-(j-i)\le\frac{n-1}{2}$. Since $j\le n-1$, $i\le(n-1)-(j-i)\le\frac{n-1}{2}-1\le\frac{n-3}{2}$. Let us assume that $i\ne 0$. Since $0\xrightarrow{i}i\xrightarrow{(n-(j-i))(n-1)}i$ and $0\xrightarrow{j}j\xrightarrow{n(n-(j-i)-1)}j$, there is a walk from $(0,0)$ to (i,j) of length $i+(n-(j-i))(n-1)$. We have

$$\begin{aligned} i+(n-(j-i))(n-1) &\le \frac{n-3}{2}+(\frac{n-1}{2})(n-1) \\ &= \frac{n^2-n-2}{2}\le\frac{n^2+1}{2}. \end{aligned}$$

If $i=0$ and $j=\frac{n+1}{2}$, then since

$$0\xrightarrow{1}1\xrightarrow{(\frac{n-1}{2})(n-1)}1\xrightarrow{n-1}0$$

and

$$0\xrightarrow{\frac{n+1}{2}}\frac{n+1}{2}\xrightarrow{(\frac{n-1}{2})n}\frac{n+1}{2},$$

there is a walk from $(0,0)$ to (i,j) of length $\frac{n^2+1}{2}$.

If $i=0$ and $j>\frac{n+1}{2}$, then $n-j<\frac{n-1}{2}$ and so $n-j\le\frac{n-3}{2}$. Since $0\xrightarrow{n+(n-j)(n-1)}0$ and $0\xrightarrow{j+(n-j)n}j$, we have $(0,0)\xrightarrow{j+(n-j)n}(0,j)$. In this case we have

$$\begin{aligned} j+n(n-j) &\le n-1+n(\frac{n-3}{2}) \\ &= \frac{n^2-n-2}{2}<\frac{n^2+1}{2}. \end{aligned}$$

Hence for $0\le i,j\le n-1$ with odd n, we conclude that $\mathrm{dist}((0,0),(i,j))\le\frac{n^2+1}{2}$.

If $(0,0)\xrightarrow{\alpha}(0,\frac{n+1}{2})$, then

$$\alpha=n+p_1n+p_2(n-1)=\frac{n+1}{2}+q_1n+q_2(n-1)$$

for some nonnegative integers p_1, p_2, q_1, q_2. It is enough to show that $\alpha \geq \frac{n^2+1}{2}$. We have

$$2n(p_1 - q_1) = (2q_2 - 2p_2 - 1)(n-1).$$

There are 2 possible cases.

1. $k(n-1) = 2(p_1 - q_1)$ and $2q_2 - 2p_2 - 1 = kn$, for $k \geq 1$.
2. $k(1-n) = 2(p_1 - q_1)$ and $-(2q_2 - 2p_2 - 1) = kn$, for $k \geq 1$.

If (1) holds, then since

$$p_1 = \frac{k(n-1)}{2} + q_1 \geq \frac{k(n-1)}{2},$$

we conclude that

$$\begin{aligned}\alpha &= n + p_1 n + p_2(n-1)\\ &\geq n + p_1 n \geq \frac{k(n-1)}{2}n\\ &\geq n + \frac{n-1}{2}n = \frac{n^2+n}{2} \geq \frac{n^2+1}{2}.\end{aligned}$$

If (2) holds, then since

$$p_2 = \frac{k(n-1)}{2} + q_2 \geq \frac{k(n-1)}{2},$$

we conclude that

$$\begin{aligned}\alpha &= n + p_1 n + p_2(n-1)\\ &\geq n + p_2(n-1) \geq n + \frac{k(n-1)}{2}(n-1)\\ &\geq n + \frac{n}{2}n - 1 = \frac{n^2+n}{2} \geq \frac{n^2+1}{2}.\end{aligned}$$

□

By combining Proposition 1 and 2, we have the following theorem.

THEOREM 1. *If W_n be the Wielandt graph of order n, then the diameter* $\mathrm{diam}(W_n \times W_n)$ *of the direct product of W_n satisfies*

$$\mathrm{diam}(W_n \times W_n) = \lfloor \frac{n^2+1}{2} \rfloor.$$

In [1], Kim, Song and Hwang showed that the diameter of $W_n \times \cdots \times W_n$ increases as the multiplicity of the product increases. And it finally stops when it reaches the value $\exp(W_n) = n^2 - 2n + 2$ at which the multiplicity of the product is $n-1$. It is worth computing the diameter of the multiple direct product of the Wielandt graph when the multiplicity varies from 3 to $n-2$.

References

[1] B. Kim, B. Song and W. Hwang, *The exponent of a digraph and the diameter of its multiple direct product*, Linear Algebra Appl. **437** (2012), 2601–2612.
[2] R. Lamprey, B. Barnes, *Primitivity of products of digraphs*, in: Proc. 10th Southeastern Conf. on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., Congress. Numer., XXIII–XXIV 637–644 (1979).
[3] M. McAndrew, *On the product of directed graphs*, Proc. Amer. Math. Soc. **14** (1963), 101–108.
[4] J. Moon, *On the diameter of a graph*, Michigan Math. J. **12** (1965), 349–351.
[5] H. Schneider, *Wielandt's proof of the exponent inequality for primitive nonnegative matrices*, Linear Algebra Appl. **353** (2002), 5–10.
[6] P. Weischel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. **12** (1962), 47–52.
[7] H. Wielandt, *Unzerleghare, nicht negative Matrizen*, Math. Z. **52** (1950), 642–645.

Graduate school of Education
Gangneung-wonju National University
Gangneung 210-702, Korea
E-mail: ksy316@freechal.com

Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail: bcsong@gwnu.ac.kr