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TRAVELLING WAVE SOLUTIONS FOR SOME NONLINEAR EVOLUTION EQUATIONS

Hyunsoo Kim and Jin Hyuk Choi*

Abstract. Nonlinear partial differential equations are more suitable to model many physical phenomena in science and engineering. In this paper, we consider three nonlinear partial differential equations such as Novikov equation, an equation for surface water waves and the Geng-Xue coupled equation which serves as a model for the unidirectional propagation of the shallow water waves over a flat bottom. The main objective in this paper is to apply the generalized Riccati equation mapping method for obtaining more exact traveling wave solutions of Novikov equation, an equation for surface water waves and the Geng-Xue coupled equation. More precisely, the obtained solutions are expressed in terms of the hyperbolic, the trigonometric and the rational functional form. Solutions obtained are potentially significant for the explanation of better insight of physical aspects of the considered nonlinear physical models.

1. Introduction

Nonlinear partial differential equations (PDEs) are widely used as model to describe complex physical phenomena in several branches of

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*Corresponding author.

science and engineering such as solid state physics, plasma wave, thermodynamics, soil mechanics, civil engineering, population ecology, infectious disease epidemiology, neural networks and so on. More precisely, it is complicated to solve partial differential equations and even if an exact solution is obtainable, the required calculations may be too complicated in derivation or it may be difficult to interpret the outcome. Therefore, for the past few decades, much attention has been paid for finding exact solutions of nonlinear partial differential equations [2, 3, 6, 8]. With the use of these solutions, one may give better insight into the physical aspects of the particular nonlinear models. In this connection, a considerable number of analytic methods have been successfully developed and applied for finding exact traveling wave solutions to nonlinear evolution equations such as extended Jacobi elliptic function expansion method [1, 5], (G'/G)-expansion method [9, 10, 15], Adomian decomposition method [4], modified F-expansion method [11], Kudryashov Method [16] and so on. However, there is no unified method in the literature that can be used to find exact solution to all kinds of nonlinear evolution equations.

Consider the Novikov equation in the following form [19];

$$u_{xxt} - u_t = 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} \tag{1}$$

which is an integrable and can be regarded as a generalization for the Camassa-Holm type equation with cubic nonlinearity. It is a model for the unidirectional propagation of the shallow water waves over a flat bottom and is attracted much attention due to its interesting properties such as complete integrability, existence of peaked solitons and multipeakons. Next, we consider the Geng-Xue system [20];

$$\begin{cases} u_{xxt} - u_t = (u_x - u_{xxx})\, uv + 3\,(u - u_{xx})\, vu_x, \\ v_{xxt} - v_t = (v_x - v_{xxx})\, uv + 3\,(v - v_{xx})\, uv_x. \end{cases} \tag{2}$$

It should be noted that u and v play the same role and can be interchanged without changing the system. Further, we focus on an equation for surface water waves of moderate amplitude in the shallow water regime [13];

$$\begin{aligned} u_t + u_x + 6uu_x - 6u^2 u_x + 12u^3 u_x + u_{xxx} - u_{xxt} \\ +14uu_{xxx} + 28u_x u_{xx} = 0 \end{aligned} \tag{3}$$

which arises as an approximation to the Euler equations and modeling the unidirectional propagation of surface water waves.

One of most effectively straightforward method to construct exact solution of nonlinear partial differential equations is the generalized Riccati equation mapping method [12, 14]. This method is a very powerful one for finding exact solutions of nonlinear partial differential equations. The key idea of this method is to take full advantages of a generalized Riccati equation involving a parameter and use its solutions to replace the tanh function which is similar to the improved Riccati equation mapping method [17, 18]. Recently, Naher and Abdullah [7] obtained traveling wave solutions for the (2+1)-dimensional modified Zakharov-Kuznetsov equation by applying the generalized Riccati equation mapping method. In this paper, we implement the generalized Riccati equation mapping method [17,18] to find the exact traveling wave solutions such as hyperbolic and trigonometric function solutions to the models Eq.(1), Eq.(2) and Eq.(3). In particular, hyperbolic and trigonometric function solutions of the considered models are obtained.

2. Summary of the Riccati equation mapping method

The generalized Riccati equation mapping method is a direct technique to find more and new traveling wave solutions for nonlinear partial differential equations. Now, let us present the algorithm of the generalized Riccati equation mapping method for finding exact solutions of nonlinear partial differential equation (PDE) [17,18]. The main idea behind this method is to use the solution of the generalized Riccati equation mapping method to replace the tanh function in the tanh method. Consider a nonlinear PDE in the following form:

$$P(u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, ...) = 0. \tag{4}$$

Now, the first step is to unite the independent variables x and t into one particular variable through the wave transformation

$$\xi = l(x - \omega t), u(x, t) = u(\xi). \tag{5}$$

By using the wave transformation (5), nonlinear PDE (4) can be reduced to the following ordinary differential equation (ODE)

$$Q(u, u', u'', u''', ...) = 0. \tag{6}$$

Suppose that the equation (6) has the following solution

$$u(x,t) = u(\xi) = \sum_{i=-m}^{m} a_i \psi^i, \tag{7}$$

where a_{-m} and a_m are constants to be determined later such that $a_{-m} \neq 0$ or $a_m \neq 0$. Now, we introduce a new variable $\psi = \psi(\xi)$ which is a solution of the generalized Riccati equation

$$\psi' = r + p\psi + q\psi^2, \tag{8}$$

where r, p and q are constants and $q \neq 0$. The parameter m is determined by balancing the linear terms of highest order with the nonlinear ones of highest order. Normally m is a positive integer, so that an analytical solution in closed form may be obtained. Substituting Eq.(8) into Eq.(7) and comparing the coefficients of each power of ψ in both sides, we get an over-determined system of nonlinear algebraic equations with respect to $r, a_{-m}, a_{-m+1}, \cdots, a_m$. Finally, we obtain over-determined systems of equations. Solving this resulting system, we get values for the unknown parameters. Further, it is well known that Eq.(8) has many families of solutions which are provided in [17, 18]. Finally, we substitute these values together with the well known solutions of Eq.(8) into Eq.(7), we can construct explicit traveling wave solutions of Eq.(4).

3. Exact solutions of models

In this section, we obtain the exact solutions for the Novikov equation, Geng-Xue coupled equations and an equation for surface water waves by using the generalized Riccati equation mapping method. In order to seek traveling wave solutions to the Novikov equation [19], by applying the wave transformation defined as in Eq.(5) into the Eq.(1), we can obtain an ordinary differential equation in the form

$$l^2\omega u''' - \omega u' + 4u^2u' - 3l^2uu'u'' - l^2u^2u''' = 0. \tag{9}$$

Now, we employ the improved Riccati equation mapping method, to solve the ODE (9) and as a result we obtain the exact solutions of Novikov equation Eq.(1). To determine the parameter m, we balance the linear terms of highest order in OED (9) with the highest order non-linear terms. The balancing procedure yields $m = 1$, so the solution of the OED (9) is of the form

$$u(\eta) = \frac{a_{-1}}{\psi(\xi)} + a_0 + a_1\psi(\xi). \tag{10}$$

By substituting Eq.(10) into (9) and making use of Eq.(8), we obtain the system of algebraic equations for $a_{-1}, a_0, a_1, \omega, r, p, q$ and l by equating all coefficients of the functions $\psi(\xi)$ to zero. Solving the system of algebraic equations with the aid of MAPLE, we can obtain the following set of solutions;

$$\left\{ l = \pm\tfrac{1}{p}, r = 0, p = p, q = q, \omega = \omega, a_{-1} = \tfrac{pa_0}{q}, a_0 = a_0, a_1 = 0. \right. \tag{11}$$

Substituting Eq.(11) into Eq.(10), the traveling wave solution of Eq.(1) can be obtained. For the coefficient set Eq.(11), the value $\Delta = p^2 - 4qr$ in the generalized Riccati equation mapping method is positive because $p^2 - 4qr = p^2 > 0$ for all real number q and $r = 0$. Therefore, we can get solutions based on Type.1 which is provided in [18] and also when $r = 0$ and $qp \neq 0$, we can obtain the solutions from Type.3 as in [18]. According to the solutions of Type.1 as in [18], we obtain the following hyperbolic wave solutions for Eq.(1):

$$u_1(x,t) = -2pa_0 \left[p + |p| \tanh\left(\pm\frac{|p|}{2p}(x - \omega t) \right) \right]^{-1} + a_0,$$

$$u_2(x,t) = -2pa_0 \left[p + |p| \coth\left(\pm\frac{|p|}{2p}(x - \omega t) \right) \right]^{-1} + a_0,$$

$$u_3(x,t) = -2pa_0 \left[p + |p| \left(\tanh\left(\pm\frac{|p|}{p}(x - \omega t) \right) \pm i\,\mathrm{sech}\left(\pm\frac{|p|}{p}(x - \omega t) \right) \right) \right]^{-1} + a_0,$$

$$u_4(x,t) = -2pa_0 \left[p + |p| \left(\coth\left(\pm\frac{|p|}{p}(x - \omega t) \right) \pm \mathrm{csch}\left(\pm\frac{|p|}{p}(x - \omega t) \right) \right) \right]^{-1} + a_0,$$

$$u_5(x,t) = -4pa_0 \left[2p + |p| \left(\tanh\left(\pm\frac{|p|}{4p}(x - \omega t) \right) \pm \coth\left(\pm\frac{|p|}{4p}(x - \omega t) \right) \right) \right]^{-1} + a_0,$$

$$u_6(x,t) = 2pa_0 \left[-p + \frac{\sqrt{p^2(A^2+B^2)} - A|p|\cosh\left(\pm\frac{|p|}{p}(x-\omega t)\right)}{A\sinh\left(\pm\frac{|p|}{p}(x-\omega t)\right)+B} \right]^{-1} + a_0,$$

$$u_7(x,t) = 2pa_0 \left[-p - \frac{\sqrt{p^2(B^2-A^2)} + A|p|\cosh\left(\pm\frac{|p|}{p}(x-\omega t)\right)}{A\sinh\left(\pm\frac{|p|}{p}(x-\omega t)\right)+B} \right]^{-1} + a_0,$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$. Further, we obtain the following set of exact solutions for Eq.(1);

$$\begin{aligned} u_{25}(x,t) &= -\frac{a_0}{d}\left(d + \cosh\left(\pm(x-\omega t)\right) - \sinh\left(\pm(x-\omega t)\right)\right), \\ u_{26}(x,t) &= -\frac{a_0\left(d + \cosh\left(\pm(x-\omega t)\right) + \sinh\left(\pm(x-\omega t)\right)\right)}{\cosh\left(\pm(x-\omega t)\right) + \sinh\left(\pm(x-\omega t)\right)}, \end{aligned} \tag{12}$$

where d is an arbitrary constant.

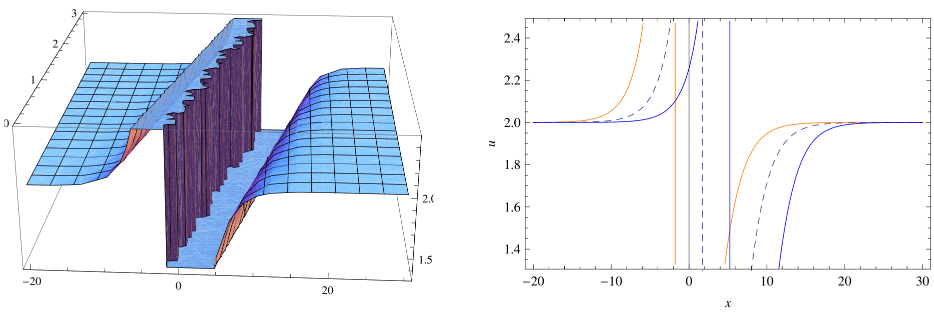

FIGURE 1. The first figure represents three dimensional plot of the solution u_5 and the second figure represents two dimensional plot for the Novikov equation (1) under the given parameters $\omega = 3.5, a_0 = -2, p = -10$ when $t = 0, 1, 2$.

Next, we obtain wave solutions for the Geng-Xue coupled equations Eq.(2). By applying the wave transformation defined as in Eq.(5), the Eq.(2) becomes a system of ordinary differential equation which can be written as

$$\begin{cases} l^2\omega u''' - \omega u' + (u' - l^2u''')uv + 3(u - l^2u'')vu' = 0, \\ l^2\omega v''' - \omega v' + (v' - l^2v''')uv + 3(v - l^2v'')uv' = 0. \end{cases} \tag{13}$$

Now, we employ the improved Riccati equation mapping method, to solve the ODE Eq.(13) and as a result we can obtain exact traveling wave solutions of Geng-Xue system Eq.(2). To determine parameters m and n of u and v, we balance the linear terms of highest order in Eq.(13) with the highest order nonlinear terms. The balancing procedure yields $m+n=2 \Rightarrow m=1, n=1$, so the solution of the ordinary differential equation Eq.(13) can be written in form

$$\begin{cases} u(\eta) = \frac{a_{-1}}{\psi(\xi)} + a_0 + a_1\psi(\xi), \\ v(\eta) = \frac{b_{-1}}{\psi(\xi)} + b_0 + b_1\psi(\xi). \end{cases} \tag{14}$$

By substituting Eq.(14) into (13) and making use of Eq.(8), we obtain the system of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, \omega, r, p, q$ and l by equating all coefficients of the functions $\psi(\xi)$ to zero. Solving the system of algebraic equations with the aid of MAPLE, two possible sets of solutions obtained;

Case 1.

$$\begin{cases} l = l, r = \frac{1}{2l^2q}, p = 0, q = q, \omega = \omega, a_{-1} = \frac{\omega}{2b_1}, \\ a_0 = 0, a_1 = 0, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1. \end{cases} \tag{15}$$

Case 2.

$$\begin{cases} l = l, r = \frac{1}{2l^2q}, p = 0, q = q, \omega = \omega, a_{-1} = a_{-1}, \\ a_0 = 0, a_1 = a_1, b_{-1} = \frac{\omega}{2a_1}, b_0 = 0, b_1 = 0. \end{cases} \tag{16}$$

For Case 1 and Case 2, the value $\Delta = p^2 - 4qr$ in the generalized Riccati equation mapping method is negative because $p^2 - 4qr = -\frac{2}{l^2} < 0$ for all real number l. Therefore, we have to apply solutions of each case in Type 2 of [18] and write the family of solutions for the two cases of coefficient sets.

According to the solution Type.2 as in [18], we obtain the following trigonometric solutions for Eq.(2) with $\xi = l(x-\omega)t$:

$$u_{13}(x,t) = \frac{\omega q}{b_1}\sqrt{\frac{l^2}{2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right),$$
$$v_{13}(x,t) = b_{-1}q\sqrt{2l^2}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right) + \frac{b_1}{q\sqrt{2l^2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right);$$
$$u_{14}(x,t) = -\frac{\omega q}{b_1}\sqrt{\frac{l^2}{2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right),$$
$$v_{14}(x,t) = -b_{-1}q\sqrt{2l^2}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right) + \frac{b_1}{q\sqrt{2l^2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right);$$

$$
\begin{aligned}
u_{15}(x,t) &= \tfrac{\omega q}{b_1}\sqrt{\tfrac{l^2}{2}}\left[\tan\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\sec\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right]^{-1},\\
v_{15}(x,t) &= b_{-1}q\sqrt{2l^2}\left[\tan\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\sec\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right]^{-1}\\
&\qquad +\tfrac{b_1}{q\sqrt{2l^2}}\left[\tan\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\sec\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right];\\
u_{16}(x,t) &= -\tfrac{\omega q}{b_1}\sqrt{\tfrac{l^2}{2}}\left[\cot\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\csc\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right]^{-1},\\
v_{16}(x,t) &= -b_{-1}q\sqrt{2l^2}\left[\cot\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\csc\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right]^{-1}\\
&\qquad +\tfrac{b_1}{q\sqrt{2l^2}}\left[\cot\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\pm\csc\left(\sqrt{\tfrac{2}{l^2}}\xi\right)\right];\\
u_{17}(x,t) &= \tfrac{2\omega q}{b_1}\sqrt{\tfrac{l^2}{2}}\left[\tan\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)-\cot\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)\right]^{-1},\\
v_{17}(x,t) &= 4b_{-1}q\sqrt{\tfrac{l^2}{2}}\left[\tan\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)-\cot\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)\right]^{-1}\\
&\qquad +\tfrac{b_1}{4q}\sqrt{\tfrac{l^2}{2}}\left[\tan\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)-\cot\left(\tfrac{1}{\sqrt{8l^2}}\xi\right)\right];\\
u_{18}(x,t) &= \tfrac{\omega q}{b_1}\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)},\\
v_{18}(x,t) &= 2b_{-1}q\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}\\
&\qquad +\tfrac{b_1}{2q}\frac{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}{A\sin\left(\sqrt{2/l^2}\xi\right)+B};\\
u_{19}(x,t) &= -\tfrac{\omega q}{b_1}\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)},\\
v_{19}(x,t) &= -2b_{-1}q\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}\\
&\qquad +\tfrac{b_1}{2q}\frac{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}{A\sin\left(\sqrt{2/l^2}\xi\right)+B},
\end{aligned}
$$

where A and B are two non-zero real constants satisfying $A^2 - B^2 > 0$.

$$
\begin{aligned}
u_{20}(x,t) &= \tfrac{\omega}{4b_1 r}\sqrt{\tfrac{2}{l^2}}\tan\left(\tfrac{1}{\sqrt{2l^2}}\xi\right),\\
v_{20}(x,t) &= \tfrac{b_{-1}}{2r}\sqrt{\tfrac{2}{l^2}}\tan\left(\tfrac{1}{\sqrt{2l^2}}\xi\right)+b_1 r\sqrt{2l^2}\cot\left(\tfrac{1}{\sqrt{2l^2}}\xi\right);\\
u_{21}(x,t) &= \tfrac{\omega}{4b_1 r}\sqrt{\tfrac{2}{l^2}}\cot\left(\tfrac{1}{\sqrt{2l^2}}\xi\right),\\
v_{21}(x,t) &= \tfrac{b_{-1}}{2r}\sqrt{\tfrac{2}{l^2}}\cot\left(\tfrac{1}{\sqrt{2l^2}}\xi\right)+b_1 r\sqrt{2l^2}\tan\left(\tfrac{1}{\sqrt{2l^2}}\xi\right);
\end{aligned}
$$

$$u_{22}(x,t) = \frac{\omega}{2b_1 r\sqrt{2l^2}} \frac{\sin\left(\sqrt{2/l^2}\xi\right) \pm 1}{\cos\left(\sqrt{2/l^2}\xi\right)},$$
$$v_{22}(x,t) = \frac{b_{-1}}{r\sqrt{2l^2}} \frac{\sin\left(\sqrt{2/l^2}\xi\right) \pm 1}{\cos\left(\sqrt{2/l^2}\xi\right)} + b_1 r\sqrt{2l^2} \frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{2/l^2}\xi\right) \pm 1};$$
$$u_{23}(x,t) = \frac{\omega}{2b_1 r\sqrt{2l^2}} \frac{\cos\left(\sqrt{2/l^2}\xi\right) \pm 1}{\sin\left(\sqrt{2/l^2}\xi\right)},$$
$$v_{23}(x,t) = \frac{b_{-1}}{r\sqrt{2l^2}} \frac{\cos\left(\sqrt{2/l^2}\xi\right) \pm 1}{\sin\left(\sqrt{2/l^2}\xi\right)} + b_1 r\sqrt{2l^2} \frac{\sin\left(\sqrt{2/l^2}\xi\right)}{\cos\left(\sqrt{2/l^2}\xi\right) \pm 1};$$
$$u_{24}(x,t) = \frac{\omega}{2b_1 r\sqrt{2l^2}} \frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{1/2l^2}\xi\right)},$$
$$v_{24}(x,t) = \frac{b_{-1}}{r\sqrt{2l^2}} \frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{1/2l^2}\xi\right)} + b_1 r\sqrt{2l^2} \frac{\sin\left(\sqrt{1/2l^2}\xi\right)}{\cos\left(\sqrt{2/l^2}\xi\right)}.$$

Further, we obtain the following exact traveling wave solutions for Eq.(2) with $\xi = l(x-\omega)t$:

$$u_{13}(x,t) = a_{-1}q\sqrt{2l^2}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right) + \frac{a_1}{q\sqrt{2l^2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right),$$
$$v_{13}(x,t) = \frac{\omega q}{a_1}\sqrt{\frac{l^2}{2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right);$$
$$u_{14}(x,t) = -a_{-1}q\sqrt{2l^2}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right) + \frac{a_1}{q\sqrt{2l^2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right),$$
$$v_{14}(x,t) = -\frac{\omega q}{a_1}\sqrt{\frac{l^2}{2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right);$$
$$u_{15}(x,t) = a_{-1}q\sqrt{2l^2}\left[\tan\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \sec\left(\sqrt{\frac{2}{l^2}}\xi\right)\right]^{-1} + \frac{a_1}{q\sqrt{2l^2}}\left[\tan\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \sec\left(\sqrt{\frac{2}{l^2}}\xi\right)\right],$$
$$v_{15}(x,t) = \frac{\omega q}{a_1}\sqrt{\frac{l^2}{2}}\left[\tan\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \sec\left(\sqrt{\frac{2}{l^2}}\xi\right)\right]^{-1};$$
$$u_{16}(x,t) = -a_{-1}q\sqrt{2l^2}\left[\cot\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \csc\left(\sqrt{\frac{2}{l^2}}\xi\right)\right]^{-1} + \frac{a_1}{q\sqrt{2l^2}}\left[\cot\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \csc\left(\sqrt{\frac{2}{l^2}}\xi\right)\right],$$
$$v_{16}(x,t) = -\frac{\omega q}{a_1}\sqrt{\frac{l^2}{2}}\left[\cot\left(\sqrt{\frac{2}{l^2}}\xi\right) \pm \csc\left(\sqrt{\frac{2}{l^2}}\xi\right)\right]^{-1};$$
$$u_{17}(x,t) = 4a_{-1}q\sqrt{\frac{l^2}{2}}\left[\tan\left(\frac{1}{\sqrt{8l^2}}\xi\right) - \cot\left(\frac{1}{\sqrt{8l^2}}\xi\right)\right]^{-1} + \frac{a_1}{4q}\sqrt{\frac{l^2}{2}}\left[\tan\left(\frac{1}{\sqrt{8l^2}}\xi\right) - \cot\left(\frac{1}{\sqrt{8l^2}}\xi\right)\right],$$
$$v_{17}(x,t) = \frac{2\omega q}{a_1}\sqrt{\frac{l^2}{2}}\left[\tan\left(\frac{1}{\sqrt{8l^2}}\xi\right) - \cot\left(\frac{1}{\sqrt{8l^2}}\xi\right)\right]^{-1};$$

$$
\begin{aligned}
u_{18}(x,t) &= 2a_{-1}q\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)} \\
&\qquad +\frac{a_1}{2q}\frac{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}{A\sin\left(\sqrt{2/l^2}\xi\right)+B}, \\
v_{18}(x,t) &= \frac{\omega q}{a_1}\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}-A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}; \\
u_{19}(x,t) &= -2a_{-1}q\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)} \\
&\qquad +\frac{a_1}{2q}\frac{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)}{A\sin\left(\sqrt{2/l^2}\xi\right)+B}, \\
v_{19}(x,t) &= -\frac{\omega q}{a_1}\frac{A\sin\left(\sqrt{2/l^2}\xi\right)+B}{\pm\sqrt{2(A^2-B^2)/l^2}+A\sqrt{2/l^2}\cos\left(\sqrt{2/l^2}\xi\right)},
\end{aligned}
$$

where A and B are two non-zero real constants satisfying $A^2-B^2>0$.

$$
\begin{aligned}
u_{20}(x,t) &= \frac{a_{-1}}{2r}\sqrt{\frac{2}{l^2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right)+a_1r\sqrt{2l^2}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right), \\
v_{20}(x,t) &= \frac{\omega}{4a_1r}\sqrt{\frac{2}{l^2}}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right); \\
u_{21}(x,t) &= \frac{a_{-1}}{2r}\sqrt{\frac{2}{l^2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right)+a_1r\sqrt{2l^2}\tan\left(\frac{1}{\sqrt{2l^2}}\xi\right), \\
v_{21}(x,t) &= \frac{\omega}{4a_1r}\sqrt{\frac{2}{l^2}}\cot\left(\frac{1}{\sqrt{2l^2}}\xi\right); \\
u_{22}(x,t) &= \frac{a_{-1}}{r\sqrt{2l^2}}\frac{\sin\left(\sqrt{2/l^2}\xi\right)\pm1}{\cos\left(\sqrt{2/l^2}\xi\right)}+a_1r\sqrt{2l^2}\frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{2/l^2}\xi\right)\pm1}, \\
v_{22}(x,t) &= \frac{\omega}{2a_1r\sqrt{2l^2}}\frac{\sin\left(\sqrt{2/l^2}\xi\right)\pm1}{\cos\left(\sqrt{2/l^2}\xi\right)}; \\
u_{23}(x,t) &= \frac{a_{-1}}{r\sqrt{2l^2}}\frac{\cos\left(\sqrt{2/l^2}\xi\right)\pm1}{\sin\left(\sqrt{2/l^2}\xi\right)}+a_1r\sqrt{2l^2}\frac{\sin\left(\sqrt{2/l^2}\xi\right)}{\cos\left(\sqrt{2/l^2}\xi\right)\pm1}, \\
v_{23}(x,t) &= \frac{\omega}{2a_1r\sqrt{2l^2}}\frac{\cos\left(\sqrt{2/l^2}\xi\right)\pm1}{\sin\left(\sqrt{2/l^2}\xi\right)}; \\
u_{24}(x,t) &= \frac{a_{-1}}{r\sqrt{2l^2}}\frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{1/2l^2}\xi\right)}+a_1r\sqrt{2l^2}\frac{\sin\left(\sqrt{1/2l^2}\xi\right)}{\cos\left(\sqrt{2/l^2}\xi\right)}, \\
v_{24}(x,t) &= \frac{\omega}{2a_1r\sqrt{2l^2}}\frac{\cos\left(\sqrt{2/l^2}\xi\right)}{\sin\left(\sqrt{1/2l^2}\xi\right)}.
\end{aligned}
$$

In order to obtain exact traveling wave solutions to the surface wave Eq.(3), by applying the wave transformation defined as in Eq.(5) into the Eq.(3), we can obtain an ordinary differential equation in the following

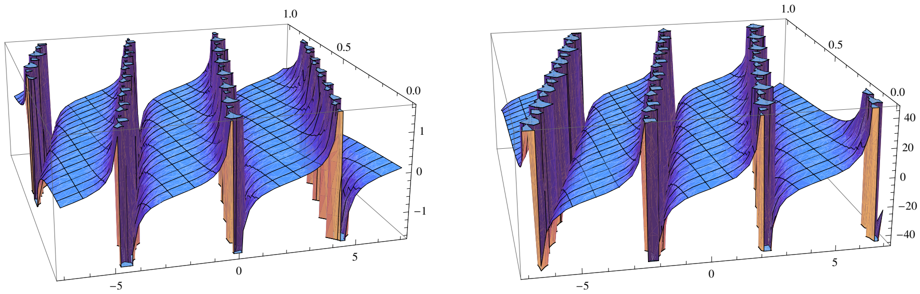

FIGURE 2. Figures represent the solutions u_{19} and v_{19} for the Geng-Xue coupled equations (2) under the given parameters $\omega = 3.5, l = 0.1, q = -1, b_1 = 1, A = -10, B = 1$.

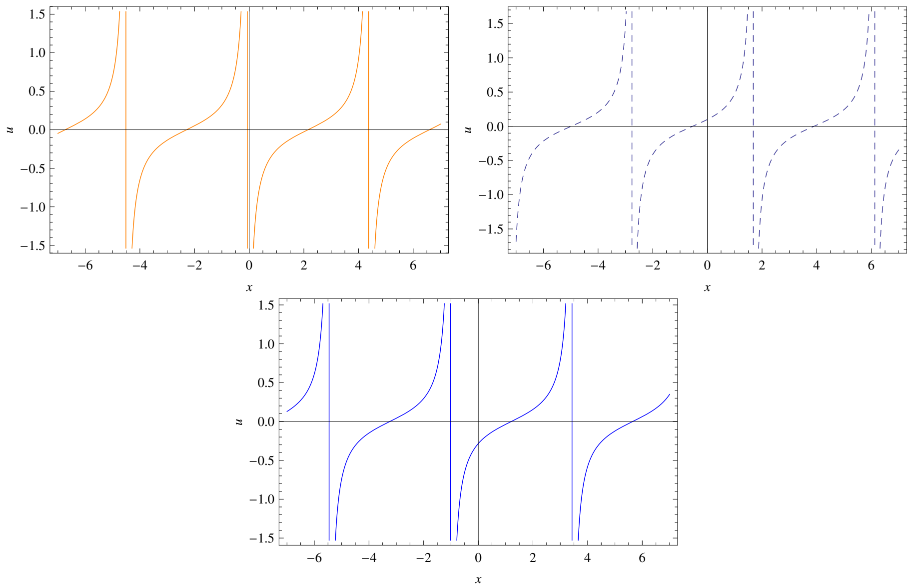

FIGURE 3. Figures represent the solutions u_{19} for the Geng-Xue coupled equations (2) under $t = 0, 0.5, 1$.

form

$$l(1-\omega)u' + 6l(u - u^2 + 2u^3)u' + l(1+\omega)u''' + 14l(uu''' + 2u'u'') = 0. \tag{17}$$

Now, we employ the improved Riccati equation mapping method, to solve the ODE Eq.(17) and as a result we can obtain exact traveling wave

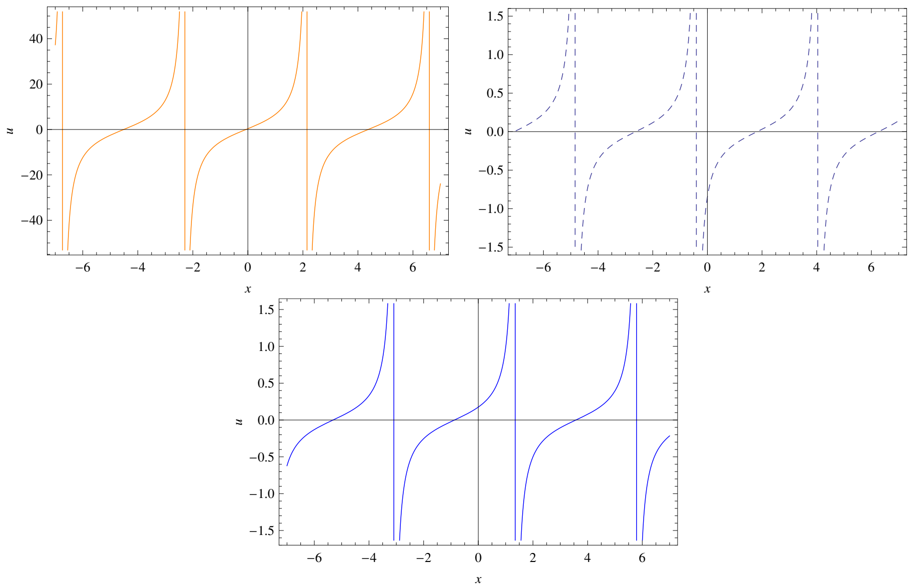

FIGURE 4. Figures represent the solutions v_{19} for the Geng-Xue coupled equations (2) under $t = 0, 0.5, 1$.

solutions of the surface waves equation Eq.(3). To determine parameters m of u, we balance the linear terms of highest order in Eq.(17) with the highest order nonlinear terms. The balancing procedure provides $3m + 1 = m + 3 \Rightarrow m = 1$, so the solution of the ordinary differential equation Eq.(17) can be written in form

$$u(\eta) = \frac{a_{-1}}{\psi(\xi)} + a_0 + a_1\psi(\xi). \tag{18}$$

By substituting Eq.(18) into (17) and making use of Eq.(8), we obtain the system of algebraic equations for $a_{-1}, a_0, a_1, \omega, r, p, q$ and l by equating all coefficients of the functions $\psi(\xi)$ to zero. Solving the system of algebraic equations with the aid of MAPLE, two possible sets of solutions obtained;

Case 1.

$$(19)\quad \begin{cases} l = \pm\sqrt{\frac{3228.0345}{2849.8270p^2-51399.3079qr}}, r=r, p=p, q=q, \\ \omega = -3.333\left(263 - \frac{52.7546p^2}{p^4-4qr} + \frac{211.0183qr}{p^2-4qr}\right) \\ \qquad \times\left(-\frac{52.7546p^2}{p^4-4qr} + \frac{211.0183qr}{p^2-4qr} - 395\right)^{-1}, \\ a_{-1} = 0, \\ a_0 = 0.5\Big(\frac{-17.5849p^2+70.3394qr}{p^2-4qr} + \frac{212.5123p}{(-0.0988p^2+0.3950qr)^{1/2}} \\ \qquad \pm\frac{-2.8031p^3+11.2123pqr}{(-0.0988p^2+0.3950qr)^{3/2}} - 210\Big) \\ \qquad \times\left(-\frac{52.7546p^2}{p^2-4qr} + \frac{211.0183qr}{p^2-4qr} - 395\right)^{-1}, \\ a_1 = \mp\frac{5.3801q}{\sqrt{-0.0988p^2+0.3950qr}}. \end{cases}$$

Case 2.

$$(20)\quad \begin{cases} l = \pm\sqrt{\frac{3228.0345}{2849.8270p^2-51399.3079qr}}, r=r, p=p, q=q, \\ \omega = -3.333\left(263 - \frac{52.7546p^2}{p^4-4qr} + \frac{211.0183qr}{p^2-4qr}\right) \\ \qquad \times\left(-\frac{52.7546p^2}{p^4-4qr} + \frac{211.0183qr}{p^2-4qr} - 395\right)^{-1}, \\ a_{-1} = \mp\frac{5.3801q}{\sqrt{-0.0988p^2+0.3950qr}}, \\ a_0 = 0.5\Big(\frac{-17.5849p^2+70.3394qr}{p^2-4qr} + \frac{212.5123p}{(-0.0988p^2+0.3950qr)^{1/2}} \\ \qquad \pm\frac{-2.8031p^3+11.2123pqr}{(-0.0988p^2+0.3950qr)^{3/2}} - 210\Big) \\ \qquad \times\left(-\frac{52.7546p^2}{p^2-4qr} + \frac{211.0183qr}{p^2-4qr} - 395\right)^{-1}, \quad a_1 = 0. \end{cases}$$

For Case 1, we obtained many solutions of Eq.(3) as follows: When $\Delta = p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$, we have the following exact traveling wave solutions for Eq.(3) with $\xi = l(x-\omega)t$:

$$u_1(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p + \sqrt{\Delta}\tanh(\tfrac{\sqrt{\Delta}}{2}\xi)\right] + a_0,$$

$$u_2(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p + \sqrt{\Delta}\coth(\tfrac{\sqrt{\Delta}}{2}\xi)\right] + a_0,$$

$$u_3(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p + \sqrt{\Delta}\left(\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi)\right)\right] + a_0,$$

$$u_4(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p + \sqrt{\Delta}\left(\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi)\right)\right] + a_0,$$

$$u_5(x,t) = \pm\frac{5.3801}{4(\sqrt{-0.0988p^2+0.3950qr}}\left[p+\sqrt{\Delta}\left(\tanh(\tfrac{\sqrt{\Delta}}{4}\xi)\pm\coth(\tfrac{\sqrt{\Delta}}{4}\xi)\right)\right]+a_0,$$
$$u_6(x,t) = \mp\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p+\frac{\sqrt{\Delta(A^2+B^2)}-A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi)+B}\right]+a_0,$$
$$u_7(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p-\frac{\sqrt{\Delta(B^2-A^2)}+A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi)+B}\right]+a_0,$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$,

$$u_8(x,t) = \mp\frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}\left[\frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-p\cosh(\frac{\sqrt{\Delta}}{2}\xi)}\right]+a_0,$$
$$u_9(x,t) = \pm\frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}\left[\frac{2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{p\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)}\right]+a_0,$$
$$u_{10}(x,t) = \mp\frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}\left[\frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-p\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}}\right]+a_0,$$
$$u_{11}(x,t) = \mp\frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}\left[\frac{2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{-p\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}}\right]+a_0,$$
$$u_{12}(x,t) = \mp\frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}\left[\frac{4r\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{-2p\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}}\right]$$
$$+a_0.$$

When $\Delta = p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$, we have the following exact traveling wave solutions for Eq.(3) with $\xi = l(x-\omega)t$:

$$u_{13}(x,t) = \mp\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p+\sqrt{-\Delta}\tan(\tfrac{\sqrt{-\Delta}}{2}\xi)\right]+a_0,$$
$$u_{14}(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p+\sqrt{-\Delta}\cot(\tfrac{\sqrt{-\Delta}}{2}\xi)\right]+a_0,$$
$$u_{15}(x,t) = \mp\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p+\sqrt{-\Delta}\left(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi)\right)\right]$$
$$+a_0,$$
$$u_{16}(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[p+\sqrt{-\Delta}\left(\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi)\right)\right]$$
$$+a_0,$$
$$u_{17}(x,t) = \pm\frac{5.3801}{4\sqrt{-0.0988p^2+0.3950qr}}\left[-2p+\sqrt{-\Delta}\left(\tan(\tfrac{\sqrt{-\Delta}}{4}\xi)-\cot(\tfrac{\sqrt{-\Delta}}{4}\xi)\right)\right]$$
$$+a_0,$$
$$u_{18}(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p+\frac{\pm\sqrt{-\Delta(A^2-B^2)}-A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi)+B}\right]+a_0,$$
$$u_{19}(x,t) = \pm\frac{5.3801}{2\sqrt{-0.0988p^2+0.3950qr}}\left[-p-\frac{\pm\sqrt{-\Delta(A^2-B^2)}+A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi)+B}\right]+a_0,$$

where A and B are two non-zero real constants satisfying $A^2 - B^2 > 0$,

$$u_{20}(x,t) = \pm \frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}} \left[\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+p\cos(\frac{\sqrt{-\Delta}}{2}\xi)} \right] + a_0,$$

$$u_{21}(x,t) = \mp \frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}} \left[\frac{2r\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)} \right] + a_0,$$

$$u_{22}(x,t) = \mp \frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}} \left[\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+p\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}} \right] + a_0,$$

$$u_{23}(x,t) = \mp \frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}} \left[\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}} \right] + a_0,$$

$$u_{24}(x,t) = \mp \frac{5.3801q}{2\sqrt{-0.0988p^2+0.3950qr}}$$
$$\left[\frac{4r\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi}{-2p\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{-\Delta})} \right] + a_0.$$

When $r = 0$ and $pq \neq 0$, we have the following exact traveling wave solutions for Eq.(3) with $\xi = l(x-\omega)t$:

$$u_{25}(x,t) = \pm \frac{5.3801p}{2\sqrt{-0.0988p^2}} \left[\frac{d}{d+\cosh(p\xi)-\sinh(p\xi)} \right],$$

$$u_{26}(x,t) = \pm \frac{5.3801p}{2\sqrt{-0.0988p^2}} \left[\frac{\cosh(p\xi)+\sinh(p\xi)}{d+\cosh(p\xi)+\sinh(p\xi)} \right] + a_0,$$

where d is an arbitrary constant.

4. Conclusion

In this paper, the generalized Riccati equation mapping technique is implemented to obtain exact traveling wave solutions of three important nonlinear partial differential equations. Also, many number of closed-form exact traveling wave solutions to the considered equations are presented. In particular, with the aid of symbolic computation such as Maple, we obtain exact traveling wave solutions of the considered equations. Also, graphs of some solution structure are provided in order to understand those new solutions and understand physical phenomena of considered equations.

It should be noted that the obtained solutions contain some arbitrary constants. Also, the arbitrary constants provide the enough freedom to construct exact travelling wave solutions that may be used to study real structure of the considered physical problem. The graphical descriptions of some obtained solutions are represented in Figs. 1-4. Fig.1 shows the profile of travelling wave solutions u_5 of Eq.(1) with

$\omega = 3.5, a_0 = -2, p = -10$ and also the two dimensional plot of the solution for different time parameters $t = 0, 1, 2$. Fig.2 and 3 present the behaviour of three and two dimensional plot of the obtained solutions u_{19} and v_{19} of Eq.(2) under the given parameters $\omega = 3.5, l = 0.1, q = -1, b_1 = 1, A = -10, B = 1$. It should be mentioned that the obtained solutions are depicted in terms of the hyperbolic, the trigonometric and the rational functional form. It is noted that the Novikov and Geng-Xue equations both admit peakon solutions in a weak sense.

The results reveal that the generalized Riccati equation mapping method can be more suitable to obtain exact solutions for the nonlinear PDEs with higher order nonlinearity. Also, all the obtained solutions are verified by putting them back into the original equations.

References

[1] AH.Bhrawy, MA.Abdelkawy and A.Biswas, *Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi's elliptic function method*, Commun. Nonlinear Sci. Numer. Simul. **18** (2013), 915–925.

[2] AH.Bhrawy, A.Biswas, M.Javidi, WX.Ma, Z.Pinar and A.Yildirim, *New solutions for (1+1)-dimensional and (2+1)-dimensional Kaup-Kupershmidt equations*, Results Math. **63** (2013) 675–686.

[3] A.-M.Wazwaz, *Multiple soliton solutions and rational solutions for the (2+1)-dimensional dispersive long water-wave system*, Ocean Engineering **60** (2013), 95–98.

[4] R.Rach, J.-S.Duan and A.-M.Wazwaz, *Solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method*, J. Math. Chem. **52** (2014), 255–267.

[5] H.Triki and A.M.Wazwaz, *A variety of exact periodic wave and solitary wave solutions for the coupled higgs equation*, Z.Naturforsch.A, (Journal of Physical Sciences) **67** (2012), 545–549.

[6] H.Triki and A.M.Wazwaz, *Traveling wave solutions for fifth-order KdV type equations with time-dependent coefficients*, Commun. Nonlinear Sci. Numer. Simulat. **19** (2014), 404–408.

[7] H.Naher and F.A.Abdullah, *New traveling wave solutions by the extended generalized riccati equation mapping method of the (2+1) dimensional evolution equation*, J. Appl. Math. **2012** (2012), 1–18.

[8] M.Dehghan and A.Nikpour, *The solitary wave solution of coupled Klein-Gordon-Zakharov equations via two different numerical methods*, Comput. Phys. Comm. **184** (2013), 2145–2158.

[9] M.K.Elboree, *Hyperbolic and trigonometric solutions for some nonlinear evolution equation*, Commun. Nonlinear Sci. Numer. Simulat. **17** (2010) 4085–4096.

[10] B.Salim Bahrami, H.Abdollahzadeh, I.M.Berijani, D.D.Ganji and M.Abdollahazdeh, *Exact travelling solutions for some nonlinear physical*

models by (G'/G)-expansion method, Pramana Journal of Physics **77** (2011), 263–275.

[11] A.Biswas, G.Ebadi, H.Triki, A.Yildirim and N.Yousefzadeh, *Topological soliton and other exact solutions to KdV-Cauch-Dodd-Gibbon equation*, Results Math. **63** (2013), 687–703.

[12] A.J.M.Jawad, M.D.Petkovic and A.Biswas, *Soliton solutions to a few coupled nonlinear wave equations by tanh method*, Iran. J. Sci. Technol. **37** (2013), 109–115.

[13] S.Zhou and C.Mu, *Global conservative solutions for a model equation for shallow water waves of moderate amplitude*, J.Differental Equations **256** (2014), 1793–1816.

[14] A.Biswas, A.J.M.Jawad, W.N.Manrakhan, A.K.Sarma and K.R.Khan, *Optical solitons and complexitons of the Schrodinger-Hirota equation*, Optics and Laser Technology **44** (2012), 2265–2269.

[15] H.Kim and R.Sakthivel, *New exact travelling wave solutions of some nonlinear higher dimensional physical models*, Rep. Math. Phys. **70** (2012), 39–50.

[16] H.Kim, J.-H.Bae and R.Sakthivel, *Exact travelling wave solutions of two important nonlinear partial differential equations*, Z.Naturforsch.A 69a (2014), 155–162.

[17] S.D.Zhu, *The generalized Riccati equation mapping method in nonlinear evolution equation: application to (2+1)-dimensional Boitilion-Pempinelle equation*, Chaos Solitons Fractals **37** (2008), 1335–1342.

[18] E.M.E.Zayed and A.H.Arnous, *Many exact solutions for nonlinear dynamics of DNA model using the generalized Riccati equation mapping method*, Sci. Res. Ess **8** (2013), 340–346.

[19] V.Novikov, *Generalizations of the Cmassa-Holm equation*, J.Phys. A **42** (2009) 342002.

[20] X.Geng and B.Xue, *An extension of integrable peakon equations with cubic nonlinearity*, Nonolinearity **22** (2009), 1847–1856.

Hyunsoo Kim
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Republic of Korea
E-mail: hskiminkorea@gmail.com

Jin Hyuk Choi
Humanitas College
Kyung Hee University
Yongin 446-701, Republic of Korea
E-mail: jinhchoi@khu.ac.kr