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ON SEMI-IFP RINGS

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Abstract. We in this note introduce the concept of semi-IFP rings which is a generalization of IFP rings. We study the basic structure of semi-IFP rings, and construct suitable examples to the situations raised naturally in the process. We also show that the semi-IFP does not go up to polynomial rings.

1. Semi-IFP rings

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $N_*(R)$, $N^*(R)$, and $N(R)$ denote the lower nilradical (i.e., the prime radical), the upper nilradical (i.e., the sum of nil ideals), and the set of all nilpotent elements in R, respectively. Note that $N_*(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}$ (resp., $\mathbb{Z}_n$) denotes the ring of integers (resp., modulo n).

It is well-known that the set of all nilpotent elements in a commutative ring coincides with the prime radical. This fact is also possessed by certain sorts of noncommutative rings, and such rings are called 2-*primal*

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by Birkenmeier et al. [3]. Shin [13, Proposition 1.11] proved that given a ring R, $N_*(R) = N(R)$ if and only if every minimal prime ideal P of R is completely prime (i.e., R/P is a domain).

A well-known property between "commutative" and "2-primal" is the *insertion-of-factors-property* (simply, *IFP*), introduced by Bell [2]. A right (or left) ideal I of a ring R is said to have the *IFP* if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$, and we will call a ring *IFP* if the zero ideal has the IFP. Narbonne [12] and Shin [13] used the terms *semicommutative* and *SI* for the IFP, respectively; while, IFP rings were also studied under the name *zero insertive* by Habeb [7]. IFP rings are 2-primal [13, Theorem 1.5].

A ring is called *reduced* if it has no nonzero nilpotent elements. It is trivial to check that reduced rings are IFP, whence the IFP condition is also between "reduced" and "2-primal". It is trivial that subrings of IFP rings are also IFP, so we use this fact freely in this note. A ring is called *Abelian* if every idempotent is central. IFP rings are Abelian by a simple computation.

Following the literature, the *index* (of nilpotency) of a nilpotent element a in a ring R is the least positive integer n such that $a^n = 0$, write $i(a)$ for n; the *index* (of nilpotency) of a subset S of R is the supremum of the indices (of nilpotency) of all nilpotent elements in S, write $i(S)$; and if such a supremum is finite, then S is said to be *of bounded index* (of nilpotency).

We now introduce the concept of semi-IFP rings as a generalization of IFP, and study relationships between semi-IFP rings and near related ring theoretic properties..

DEFINITION 1.1. A ring R is called *semi-IFP* if $a^2 = 0$ for $a \in R$ implies $aRa = 0$.

It is obvious that a ring R is semi-IFP if and only if $a^2 = 0$ for $a \in R$ implies $(RaR)^2 = 0$. Clearly, the class of semi-IFP rings is closed under subrings. We will use this fact freely.

Following [10] a ring R is said to be near-IFP if $\sum_{i=0}^{n} Ra_iR$ contains a nonzero nilpotent ideal whenever a nonzero polynomial $\sum_{i=0}^{n} a_i x^i$ over a ring R is nilpotent.

$U_2(\mathbb{Z}_4)$ is near-IFP by [10, Proposition 1.10(1)], but $U_2(\mathbb{Z}_4)$ is not semi-IFP by Example 1.6 to follow. Let R be a semi-IFP ring and R is of bounded index 2(of nilpotency). Then R is near-IFP by [10, Proposition 1.2]. However we do not know whether semi-IFP rings are near-IFP when given rings are of bounded index(of nilpotency)$\geq$3.

We see in the following that $Mat_n(R)$ cannot be semi-IFP for any ring R and $n \geq 2$, and that the class of semi-IFP rings is not closed under homomorphic images.

EXAMPLE 1.2. Consider the ring $Mat_2(R)$ over any ring R. For $e_{12} \in Mat_2(R)$, we have $e_{12}^2 = 0$ but

$$0 \neq e_{12} = e_{12}e_{21}e_{12} \in e_{12}Mat_2(R)e_{12},$$

showing that $Mat_2(R)$ is not semi-IFP. Consequently, $Mat_n(R)$ for $n \geq 2$ cannot be semi-IFP.

This result also illuminates that the class of semi-IFP rings is not closed under homomorphic images. In fact, let R be the ring of quaternions with integer coefficients. Then R is a domain, and so semi-IFP. However, for any odd prime integer q, the ring $R/qR \cong Mat_n(\mathbb{Z}_q)$, by the argument in [6, Exercise 2A]. Since $Mat_n(\mathbb{Z}_q)$ is not semi-IFP by above, and thus R/qR cannot be semi-IFP.

However, we have the following.

PROPOSITION 1.3. *Let R be a reduced ring. Then $U_n(R)$ is a semi-IFP ring for $n = 2, 3$.*

Proof. It is enough to show that $U_3(R)$ over a reduced ring R is semi-IFP. Let $M^2 = 0$ for

$$M = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in U_3(R).$$

Then $a = d = f = 0$ and $be = 0$. Here $be = 0$ implies $bRe = 0$ since reduced rings are IFP. So for any

$$\begin{pmatrix} x & y & z \\ 0 & u & v \\ 0 & 0 & w \end{pmatrix} \in U_3(R),$$

we have

$$\begin{pmatrix} 0 & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x & y & z \\ 0 & u & v \\ 0 & 0 & w \end{pmatrix} \begin{pmatrix} 0 & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bue \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

entailing that $MU_3(R)M = 0$. Therefore $U_3(R)$ is semi-IFP. $\square$

Recall that for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

By Proposition 1.3, we have the following since $T(R, R)$ is a subring of $U_2(R)$.

Corollary 1.4. *If R is a reduced ring, then $T(R, R)$ is a semi-IFP ring.*

Based on Proposition 1.3, one may suspect that $U_n(R)$ over a reduced ring R may be also a semi-IFP ring for $n \geq 4$. But the following example erases the possibility.

Example 1.5. Let R be any ring and consider $U_4(R)$. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in U_4(R).$$

Then $M^2 = 0$, but

$$0 \neq \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} M \in MU_4(R)M,$$

showing that $U_4(R)$ is not semi-IFP. Therefore $U_n(R)$ cannot be semi-IFP for $n \geq 5$.

$U_n(R)$ is 2-primal over a 2-primal ring R by [3]. So Example 1.5 shows that 2-primal rings need not be semi-IFP.

IFP rings are clearly semi-IFP but the converse need not hold. For example, the ring $U_2(R)$ is non-Abelian over any ring R, so $U_2(R)$ cannot be IFP. But $U_2(A)$ is semi-IFP over a reduced ring A by Proposition 1.3, and thus this also says that semi-IFP rings need not be Abelian.

The condition "A is a reduced ring" in Proposition 1.3 cannot be weakened by the condition "A is a semi-IFP ring" by the following.

Example 1.6. Consider the ring $U_2(\mathbb{Z}_4)$. Note that $\mathbb{Z}_4$ is a semi-IFP ring. For

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in U_2(\mathbb{Z}_4),$$

$M^2 = 0$, but

$$0 \neq \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M \in MU_2(\mathbb{Z}_4)M.$$

So $U_2(\mathbb{Z}_4)$ is not semi-IFP.

Let R be a ring and I be a nilpotent ideal of R with $I^3 = 0$. Then I is a semi-IFP ring without identity. So it is natural to ask whether R is a semi-IFP ring if both R/I and I are semi-IFP rings. However we have a negative answer to this situation by Example 1.6. Indeed, let $I = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$. Then I is a semi-IFP ring with $I^2 = 0$, and R/I is a commutative ring. But R is not semi-IFP.

However if we take another condition "I is reduced" then we can have an affirmative answer as follows. The following is a similar result to [9, Theorem 6] for IFP rings.

Theorem 1.7. *For a ring R suppose that R/I is a semi-IFP ring for a proper ideal I of R. If I is a reduced ring without identity then R is semi-IFP.*

Proof. Let $a^2 = 0$ for $a \in R$. Then $aRa \subseteq I$ since R/I is semi-IFP. Moreover $a^2 = 0$ implies $(aRa)^2 = 0$. But since I is reduced, $aRa = 0$. Therefore R is semi-IFP. □

We let $N_0(R)$ be the Wedderburn radical (i.e., the sum of all nilpotent ideals) of a ring R. Note $N_0(R) \subseteq N_*(R)$.

Proposition 1.8. *Let R be a semi-IFP ring with $i(R) = 2$.*

(1) $N_0(R) = N_*(R) = N^*(R) = N(R)$.

(2) If $f(x)^2 = 0$ for $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, then $(R[x]f(x)R[x])^{n+2} = 0$.

Proof. (1) Let R be a semi-IFP ring with $i(R) = 2$. Take $a \in N(R)$. Then $a^2 = 0$, and since R is semi-IFP we have $(RaR)^2 = 0$. This implies $a \in N_0(R)$.

(2) Let $f(x)^2 = 0$ for $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. By (1), $N_0(R) = N(R)$ and so $R/N_0(R)$ is a reduced ring. It then follows from $f(x)^2 = 0$ that $a_i \in N_0(R)$ for all i. Since R is semi-IFP, we get $(Ra_iR)^2 = 0$. Consider any sum-factor of any coefficient of polynomials in $(R[x]f(x)R[x])^{n+2}$, c say. Then two or more a_j's occur in c for some j, so c is contained in $(Ra_jR)^2$. Thus $c = 0$ and this yields $(R[x]f(x)R[x])^{n+2} = 0$. □

It is well-known that a ring R is reduced if and only if R is 2-primal and semiprime. Moreover, we have the following.

Proposition 1.9. *For a semiprime ring R, then the following conditions are equivalent:*
(1) *R is reduced;*
(2) *R is IFP;*
(3) *R is 2-primal;*
(4) *R is semi-IFP;*
(5) *R is near-IFP.*

Proof. It suffices to show (4)⇒(1) and (5)⇒(1). Assume that the condition (4) holds. If $a^2 = 0$ for $a \in R$, then $aRa = 0$ by assumption. Since R is semiprime, $a = 0$ and so R is reduced.

Let R be a near-IFP and assume that $a^2 = 0$ for $a \in R$. Here assume $a \neq 0$. Then by [10, Proposition 1.2], $RaR(\neq 0)$ contains a nonzero nilpotent ideal of R, I say. But R is semiprime, and so $I = 0$, a contradiction. so a must be zero, entailing that R is reduced. □

Following [5], a ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Regular rings are semiprimitive (hence semiprime) by [5]. So we get the following from Proposition 1.9.

Corollary 1.10. *For a regular ring R, then the following conditions are equivalent:*
(1) *R is reduced;*
(2) *R is IFP;*
(3) *R is 2-primal;*
(4) *R is semi-IFP;*
(5) *R is near-IFP.*

A ring R is called *right Ore* if for $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists the classical right quotient ring of R, and that R is a semiprime right Goldie ring if and only if there exists the classical right quotient ring of R which is semisimple Artinian.

Combing this fact with Proposition 1.9, we have the following which is an extension of [9, Corollary 13].

Proposition 1.11. *Let R be a semiprime right Goldie ring and Q be the classical right quotient ring of R. Then the following conditions are equivalent:*
(1) *R is a reduced ring;*
(2) *R is an IFP ring;*
(3) *R is a semi-IFP ring;*
(4) *R is near-IFP;*
(5) *Q is a reduced ring;*
(6) *Q is an IFP ring;*
(7) *Q is a semi-IFP ring;*
(8) *Q is a finite direct product of division rings.*

Proof. By Proposition 1.9 and the proof of [9, Corollary 13]. □

2. Polynomial rings over semi-IFP rings

Concerning polynomial rings over reduced rings and 2-primal rings, we have the following useful facts:
(1) A ring R is reduced if and only if $R[x]$ is reduced obviously.
(2) A ring R is 2-primal if and only if $R[x]$ is 2-primal by [3].

Based on these results one may naturally conjecture that a ring R is semi-IFP if and only if $R[x]$ is semi-IFP. However the following example erases the possibility.

Example 2.1. The construction and computation are similar to [9, Example 2] for IFP rings. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, c\rangle$ be the free algebra with noncommuting indeterminates a_0, a_1, a_2, c over $\mathbb{Z}_2$. Let B be the set of all polynomials in A with zero constant term. Let I be the ideal of A generated by

$$a_0a_0, a_1a_2 + a_2a_1, a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, a_2a_2,$$

and

$$a_0ra_0, a_2ra_2, (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), r_1r_2r_3r_4$$

for $r, r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \subseteq I$. Set $R = A/I$. Notice that $(a_0 + a_1x + a_2x^2)(a_0 + a_1x + a_2x^2) \in I[x]$, but $(a_0 + a_1x + a_2x^2)c(a_0 + a_1x + a_2x^2) \notin I[x]$ because $a_0ca_1 + a_1ca_0 \notin I$; hence $R[x]$ is not semi-IFP.

Next we claim that R is semi-IFP. Each product of indeterminates a_0, a_1, a_2, c is called a monomial and we say that α is a monomial of degree n if it is a product of exactly n generators. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Observe that H_n is finite for any n and that the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$ then every r_i is in I).

Claim 1. If $f_1^2 \in I$ with $f_1 \in H_1$ then $f_1rf_1 \in I$ for any $r \in A$.

Proof. By the definition of I we obtain the following cases:

$$(f_1 = a_0), (f_1 = a_2), \text{or } \ (f_1 = a_0 + a_1 + a_2).$$

So we complete the proof, using the definition of I again. □

Claim 2. If $f^2 \in I$ with $f \in B$ then $frf \in I$ for all $r \in B$.

Proof. Observe that $f = f_1 + f_2 + f_3 + f_4$ and $r = r_1 + r_2 + r_3 + r_4$ for some $f_1, r_1 \in H_1, f_2, r_2 \in H_2, f_3, r_3 \in H_3$ and some $f_4, r_4 \in I$. Note that $H_i \subseteq I$ for $i \geq 4$. So $frf = f_1r_1f_1 + h$ for some $h \in I$. $f^2 \in I$ implies $f_1f_1 \in I$ since I is homogeneous; hence $f_1r_1f_1 \in I$ by Claim 1. Consequently $frf \in I$. □

To see that R is semi-IFP, it suffices to show that $yry \in I$ for all $r \in A$ if $y^2 \in I$ with $y \in A$. By help of Claim 2, we can obtain the following computations. First write $y = \alpha + z$ for some $\alpha \in \mathbb{Z}_2$ and some $z \in B$. So $\alpha^2 + \alpha z + z\alpha + z^2 = y^2 \in I$; hence $\alpha = 0$. Then $z^2 \in I$; hence $yry = zrz \in I$ for all $r \in A$. Therefore R is a semi-IFP ring.

Proposition 2.2. *Suppose that a ring R is semiprime. Then the following conditions are equivalent:*

(1) *R is semi-IFP;*

(2) *$R[x]$ is semi-IFP.*

Proof. If R is semiprime and semi-IFP then R is reduced by Proposition 1.9, entailing that $R[x]$ is reduced. □

Given a ring R, an endomorphism σ of R, and a σ-derivation δ of R, the *Ore extension* $R[x; \sigma, \delta]$ of R is the ring obtained by giving the polynomial ring $R[x]$ with the new multiplication $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$ then we write $R[x;\sigma]$ and call it a *skew* polynomial ring. If $\sigma = 1$ then we write $R[x;\delta]$ and call it a *differential* polynomial ring. It is also natural to ask whether the class of semi-IFP rings is closed under these two kinds of extensions. But the following provides negative answers.

Example 2.3. There exists a semi-IFP ring over which the skew polynomial ring is not semi-IFP. The argument is essentially due to [4, Example 3.1(1)]. For a division ring D let $R = D \oplus D$, then R is semi-IFP obviously. Define $\sigma : R \to R$ by $\sigma(s,t) = (t,s)$. Then σ is an automorphism of R. Let $S = R[x;\sigma]$ be the skew polynomial ring over R by σ. We `claim` that S is semiprime. Let I be a nonzero ideal of S. Then we pick a nonzero $f(x)$ in I which is of the smallest degree in I. Say $f(x) = a + bx + \cdots + cx^n$ with $a, b, \ldots, c \in R$ and $c \neq 0$. If n is even, then $f(x)^2 = a^2 + \cdots + c\sigma^n(c)x^{2n} = a^2 + \cdots + c^2x^{2n} \in I^2 \subseteq I$ is nonzero because c is nonzero and σ is of order 2. Next if n is odd then $f(x)x = ax + bx^2 + \cdots + cx^{n+1} \in I$; hence $[f(x)x]^2 \in I^2 \subseteq I$ is also nonzero by the same computation. Thus I^2 is nonzero and so S is semiprime. But $N(S) \neq 0$ as can be seen by $((1,0)x)((1,0)x) = 0$; hence S is not reduced. By Proposition 1.9, S is not semi-IFP.

Example 2.4. There exists a semi-IFP ring over which the differential polynomial ring is not semi-IFP. The argument is essentially due to [1, Example 11], [6, Proposition 1.14], and [8, Example 2.1]. Let $R = F[x]/(x^2)$ and define $\delta : R \to R$ by $\delta(x + (x^2)) = 1 + (x^2)$, where F is a field of characteristic 2 and $(x^2) = F[x]x^2$. Then R is semi-IFP since R is commutative and $\dfrac{R}{F[x]x} \cong F$. Next let $S = R[x;\delta]$. Then $[x + (x^2)]^2 = 0$ but $[x + (x^2)]S[x + (x^2)] \neq 0$ so R is not semi-IFP.

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