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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES: A FIXED POINT APPROACH

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Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right. \\ &\qquad \left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\| \\ &\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\ &\qquad + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\|, \end{aligned} \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{8}$, and

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) \\ &\qquad + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right.\right. \\ &\qquad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right) \right\|, \end{aligned} \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 4$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [20] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen type quadratic equation*.

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \le \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

See also [19]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [16] proved the Hyers-Ulam stability of additive functional inequalities.

We recall a fundamental result in fixed point theory.

THEOREM 1.1. [2,6] *Let (X,d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [12] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 14, 15, 18]).

This paper is organized as follows: In Section 2, we investigate quadratic functional equations.

In Section 3, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in complex Banach spaces.

In Section 4, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Quadratic functional equation

THEOREM 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{2.1}$$

if and only if the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Sufficiency. Assume that $f : X \to Y$ satisfies (2.1)
Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.
Letting $y = z = 0$ in (2.1), we get

$$\begin{aligned} 2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) &= f(x), \qquad (2.2) \\ 2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) &= f(-x) \end{aligned}$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain

$$\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$$

for all $x, y \in X$, which means that $f : X \to Y$ is a quadratic mapping.

Necessity. Assume that $f : X \to Y$ is quadratic.

By $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, one can easily get $f(0) = 0$, $f(x) = f(-x)$ and $f(2x) = 4f(x)$ for all $x \in X$. So

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big) &+ f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ &= \Big[2f\Big(\frac{x}{2}\Big) + 2f\Big(\frac{y+z}{2}\Big)\Big] + \Big[2f\Big(-\frac{x}{2}\Big) + 2f\Big(\frac{y-z}{2}\Big)\Big] \\ &= 4f\left(\frac{x}{2}\right) + f\Big(\frac{y+z+y-z}{2}\Big) + f\Big(\frac{y+z-y+z}{2}\Big) \\ &= f(x) + f(y) + f(z) \end{aligned}$$

for all $x, y, z \in X$, which is the functional equation (2.1) and the proof is complete. □

Corollary 2.2. *Let X and Y be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ = 4f(x) + 4f(y) + 4f(z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Assume that $f : X \to Y$ satisfies (2.3)

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get

$$2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

3. Quadratic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{8}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.1) in complex normed spaces.

LEMMA 3.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\\
&\qquad\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\|\\
&\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z)\\
&\qquad + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\|
\end{aligned} \tag{3.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get

$$\|4f\left(\frac{x}{2}\right) - f(x)\| \leq 0$$

and so $4f(\frac{x}{2}) = f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\\
&\qquad\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\|\\
&\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z)\\
&\qquad + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\|\\
&= |\rho| \left\| 4f\left(\frac{x+y+z}{2}\right) + 4f\left(\frac{x-y-z}{2}\right) + 4f\left(\frac{y-x-z}{2}\right)\right.\\
&\qquad\left. + 4f\left(\frac{z-x-y}{2}\right) - 4f(x) - 4f(y) - 4f(z)\right\|
\end{aligned}$$

$$\leq 4|\rho| \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right.$$
$$\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\|$$

and so

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)$$
$$= f(x) + f(y) + f(z)$$

for all $x, y, z \in X$, since $|\rho| < \frac{1}{8} < \frac{1}{4}$.

The converse is obviously true. $\square$

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0,0,0) = 0$ such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{4} \varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping such that

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right.$$
$$\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\|$$
$$\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \tag{3.4}$$
$$+ f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| + \varphi(x, y, z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L} \varphi(x, 0, 0) \tag{3.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.4), we get $\|f(0)\| \leq |\rho| \|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \varphi(x, 0, 0) \tag{3.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi\left(x, 0, 0\right), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [13]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi\left(x, 0, 0\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq 4\varepsilon\varphi\left(\frac{x}{2}, 0, 0\right) \\ &\leq 4\varepsilon\frac{L}{4}\varphi\left(x, 0, 0\right) \leq L\varepsilon\varphi\left(x, 0, 0\right) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.6) that $d(f, Jf) \leq 1$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{3.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leq \mu\varphi\left(x, 0, 0\right)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{1}{1-L}.$$

So

$$\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x, 0, 0)$$

for all $x \in X$.

It follows from (3.3) and (3.4) that

$$\begin{aligned}
&\left\|Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right)\right.\\
&\qquad\qquad \left.+Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z)\right\|\\
&= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right)\right.\\
&\qquad\qquad \left.+f\left(\frac{z-x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|\\
&\le \lim_{n\to\infty} 4^n |\rho| \left\|f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right)\right.\\
&\qquad\qquad \left.+f\left(\frac{z-x-y}{2^n}\right) - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\|\\
&\qquad\qquad + \lim_{n\to\infty} \frac{1}{4^n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)\\
&= \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y)\\
&\qquad\qquad -4Q(x) - 4Q(y) - 4Q(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&\left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right)\right. \\
&\qquad\qquad \left. + Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z)\right\| \\
&\le \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\
&\qquad\qquad -4Q(x) - 4Q(y) - 4Q(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.5). Then we have

$$\begin{aligned}
\|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \max\left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \right\} \\
&\le \lim_{n\to\infty} \frac{1}{4^n} \varphi\left(\frac{x}{2^n}, 0, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (3.5). $\square$

COROLLARY 3.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
&\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right. \\
&\qquad\qquad \left. - f(x) - f(y) - f(z)\right\| \qquad (3.8) \\
&\le \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\
&\qquad\qquad -4f(x) - 4f(y) - 4f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^r \theta}{2^r - 4} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. $\square$

Theorem 3.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x,y,z \in X$ Let $f : X \to Y$ be an even mapping satisfying (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x,0,0)$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(2x,0,0) \le L\varphi(x,0,0) \tag{3.9}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (3.9) that $d(f, Jf) \le L$. So

$$d(f,Q) \le \frac{L}{1-L}.$$

So

$$\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x,0,0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.8). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{4-2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. $\square$

REMARK 3.6. If ρ is a real number such that $-\frac{1}{8} < \rho < \frac{1}{8}$ and Y is a real Banach space, then all the assertions in this section remain valid.

4. Quadratic ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 4$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.2) in complex normed spaces.

LEMMA 4.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad -4f(x) - 4f(y) - 4f(z)\| \\ &\leq \left\|\rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \end{aligned} \tag{4.1}$$

for all $x,y,z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$\|2f(2x) - 8f(x)\| \leq 0 \tag{4.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (4.1) and (4.2) that

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\
&\qquad -4f(x)-4f(y)-4f(z)\|\\
&\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right.\\
&\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|\\
&= \left\|\rho\left(\frac{1}{4}f(x+y+z)+\frac{1}{4}f(x-y-z)+\frac{1}{4}f(y-x-z)\right.\right.\\
&\qquad \left.\left.+\frac{1}{4}f(z-x-y)-f(x)-f(y)-f(z)\right)\right\|\\
&= \frac{|\rho|}{4}\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\
&\qquad -4f(x)-4f(y)-4f(z)\|
\end{aligned}$$

and so

$$f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)=4f(x)+4f(y)+4f(z)$$

for all $x,y,z\in X$, since $|\rho|<4$. So f is quadratic.

The converse is obviously true. $\square$

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (4.1) in complex Banach spaces.

THEOREM 4.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0)=0$ such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)\le \frac{L}{4}\varphi(x,y,z) \tag{4.3}$$

for all $x,y,z\in X$. Let $f : X\to Y$ be an even mapping satisfying

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\
&\qquad -4f(x)-4f(y)-4f(z)\|\\
&\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right.\\
&\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|+\varphi(x,y,z)
\end{aligned} \tag{4.4}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{4(1-L)}\varphi(x, x, 0) \tag{4.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.4), we get $\|8f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \le \frac{L}{4}\varphi(x, x, 0) \tag{4.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x, 0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [13]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le L\varepsilon\varphi(x, x, 0)$$

for all $a \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (4.6) that $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{4.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (4.7) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \leq \mu\varphi\left(x,x,0\right)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{4(1-L)}.$$

So

$$\|f(x) - Q(x)\| \leq \frac{L}{4(1-L)}\varphi(x,x,0)$$

for all $x \in X$.

It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\left\|Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right)\right.\\
&\qquad\qquad \left. + Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z)\right\|\\
&= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right)\right.\\
&\qquad\qquad \left. + f\left(\frac{z-x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|
\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty} 4^n|\rho| \left\| f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) \right.\\
&\qquad \left. + f\left(\frac{z-x-y}{2^n}\right) - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right) \right\| \\
&\qquad + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&= \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\
&\qquad -4Q(x) - 4Q(y) - 4Q(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&\left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) \right.\\
&\qquad \left. + Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z) \right\| \\
&\le \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\
&\qquad -4Q(x) - 4Q(y) - 4Q(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $Q : X \to Y$ is quadratic.

The rest of the proof is similar to the proof of Theorem 3.2. □

COROLLARY 4.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
&\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\
&\qquad -4f(x) - 4f(y) - 4f(z)\| \\
&\le \left\| \rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right.\right. \\
&\qquad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \\
&\qquad + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{4.8}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. $\square$

THEOREM 4.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x,y,z \in X$ Let $f : X \to Y$ be an even mapping satisfying (4.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x,x,0)$$

for all $x \in X$.

Proof. It follows from (4.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x,x,0) \tag{4.9}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (4.9) that $d(f,Jf) \le \frac{1}{4}$. So $d(f,Q) \le \frac{1}{4(1-L)}d(f,Jf)$, which implies the inequality

$$d(f,Q) \le \frac{1}{1-L}.$$

So

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x,x,0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. $\square$

Corollary 4.5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.8)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 4.6. If ρ is a real number such that $-4 < \rho < 4$ and Y is a real Banach space, then all the assertions in this section remain valid.

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