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CHANGE OF SCALE FORMULAS FOR FUNCTION SPACE INTEGRALS RELATED WITH FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION ON $C_{a,b}[0,T]$

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Abstract. We express generalized Fourier-Feynman transform and convolution product of functionals in a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ as limits of function space integrals on $C_{a,b}[0,T]$. Moreover we obtain change of scale formulas for function space integrals related with generalized Fourier-Feynman transform and convolution product of these functionals.

1. Introduction

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [2] and under translations [1]. Cameron and Storvick [6] expressed the analytic Feynman integral on classical Wiener space as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener

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integrals on classical Wiener space $(C_0[0,1], m_w)$ [5]. These results extended to an abstract Wiener space [18, 19] and for some unbounded functionals [20, 21].

Let $a(t)$ be an absolutely continuous real valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ be a strictly increasing continuous differentiable real valued function with $b(0) = 0$. The generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 in [16], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). For more details on the space $C_{a,b}[0,T]$, see [8, 9]. Let $W(C_{a,b}[0,T])$ denote the class of all μ measurable subsets of $C_{a,b}[0,T]$.

Chang, Choi and Skoug [7–9] studied a generalized analytic Feynman integral on the function space $C_{a,b}[0,T]$. Recently the authors [17] obtained a change of scale formula for a function space integral on a generalized Wiener space $C_{a,b}[0,T]$.

A subset E of $C_{a,b}[0,T]$ is said to be scale-invariant measurable [13] provided ρE is in $W(C_{a,b}[0,T])$ for every $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Let $L^2_{a,b}[0,T]$ be the Hilbert space of continuous functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(t)\, d|a|(t) < \infty \text{ and } \int_0^T v^2(t)\, db(t) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of a on the interval $[0,t]$.

Inner product on $L^2_{a,b}[0,T]$ is defined by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)\, d[|a|(t) + b(t)]$$

and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

We denote function space integral of a $W(C_{a,b}[0,T])$-measurable functional F by

$$\int_{C_{a,b}[0,T]} F(x)\,d\mu(x)$$

whenever the integral exists.

Let $\mathbb{C}_+$ denotes the set of complex numbers with positive real part. Let F be a complex valued measurable functional on $C_{a,b}[0,T]$ such that the function space integral

$$J_F(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)\,d\mu(x) \tag{1.1}$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)\,d\mu(x) = J_F^*(\lambda). \tag{1.2}$$

If the following limit exists for nonzero real q, then we call it a generalized analytic Feynman integral of F over $C_{a,b}[0,T]$ with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)\,d\mu(x) = \lim_{\lambda\to -iq} \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)\,d\mu(x) \tag{1.3}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Next we will introduce the class of functionals that we work with in this paper. The Banach algebra $\mathcal{S}(L_{a,b}^2[0,T])$ is the space of all s-equivalence classes of functionals F on $C_{a,b}[0,T]$ which have the form

$$F(x) = \int_{L_{a,b}^2[0,T]} \exp\{i\langle v, x\rangle\}\,df(v) \tag{1.4}$$

where f is a complex Borel measure on $L_{a,b}^2[0,T]$ and $\langle v, x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral $\int_0^T v(t)\,dx(t)$.

Note that by choosing $a(t) = 0$ and $b(t) = t$ on $[0,T]$, function space $C_{a,b}[0,T]$ reduces to Wiener space $C_0[0,T]$, $\mathcal{S}(L_{a,b}^2[0,T])$ reduces to the Banach algebra $\mathcal{S}$ introduced in [4] by Cameron and Storvick and generalized analytic Feynman integral $\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)\,d\mu(x)$ reduces to analytic Feynman integral $\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\,dm_w(x)$.

The following notations are used throughout this paper:

$$(u, a') = \int_0^T u(t)a'(t)\,dt \quad \text{and} \quad (u^2, b') = \int_0^T u^2(t)b'(t)\,dt$$

for $u \in L^2_{a,b}[0,T]$.

Now we state a very fundamental integration formula on function space $C_{a,b}[0,T]$. It is a generalization of Wiener integration formula on Wiener space. Suppose that $|a(t)| = cb(t)$ on $[0,T]$ for some constant $c \geq 0$ throughout this paper. Let $\{\phi_1, \ldots, \phi_n\}$ be an orthonormal set in $L^2_{a,b}[0,T]$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. Then

(1.5)
$$\begin{aligned} &\int_{C_{a,b}[0,T]} f(\langle\phi_1, x\rangle, \ldots, \langle\phi_n, x\rangle)\,d\mu(x) \\ &= \prod_{k=1}^{n}(2\pi B_k)^{-1/2}\int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\exp\Big\{-\sum_{k=1}^{n}\frac{(u_k - A_k)^2}{2B_k}\Big\}\,du_1\cdots du_n \end{aligned}$$

in the sense that if either side exists, both sides exist and equality holds, where $A_k = (\phi_k, a')$ and $B_k = (\phi_k^2, b')$ for $k = 1, \ldots, n$. Concerning the condition $|a(t)| = cb(t)$, see Example 3.4 in [17].

In this paper, we express generalized Fourier-Feynman transform and convolution product of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ as limits of function space integrals on $C_{a,b}[0,T]$. Moreover we obtain change of scale formulas for function space integrals related with generalized Fourier-Feynman transform and convolution product of these functionals.

To obtain a change of scale formula for function space integral on generalized Wiener space, the authors [17] expressed generalized analytic Feynman integral as a limit of function space integrals on generalized Wiener space. We develop similar properties for generalized Fourier-Feynman transform and convolution product in Sections 2 and 3 below. But one of the main differences between generalized analytic Feynman integral and generalized Fourier-Feynman transform is that the latter involves the concept of limit in the mean but the former does not. Since, as we will see in (2.4) below, generalized analytic Feynman integral is a special case of generalized Fourier-Feynman transform, the results in [17] can be obtained as corollaries of our results.

2. Fourier Feynman transform and a change of scale formula

In this section we give a realtionship between function space integral and generalized Fourier-Feynman transform on $C_{a,b}[0,T]$ for functionals in Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$, that is, we express generalized Fourier-Feynman transform of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ as a limit of function space integrals on $C_{a,b}[0,T]$. We begin this section by introducing the definition of generalized analytic Fourier-Feynman transform for functionals defined on $C_{a,b}[0,T]$.

Let $1 \le p < \infty$, let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \to -iq$ throughout this paper.

Definition 2.1. Let F be a functional on $C_{a,b}[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x+y)\,dm(x). \tag{2.1}$$

For $1 < p < \infty$, we define generalized L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F on $C_{a,b}[0,T]$ by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = \underset{\lambda\to -iq}{\mathrm{l.i.m.}}\, T_\lambda(F)(y), \tag{2.2}$$

whenever this limit exists; that is, for each $\rho > 0$,

$$\lim_{\lambda\to -iq} \int_{C_{a,b}[0,T]} |T_\lambda(F)(\rho x) - T_q^{(p)}(F)(\rho x)|^{p'}\,dm(x) = 0$$

where $1/p+1/p' = 1$. We define generalized L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq} T_\lambda(F)(y), \tag{2.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$, whenever this limit exists [3, 10–12].

By the definitions of generalized analytic Feynman integral (1.3) and generalized L_1 analytic Fourier-Feynman transform (2.3), it is easy to see that

$$T_q^{(1)}(F)(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x+y)\,dm(x). \tag{2.4}$$

Chang and Skoug [9] established existence theorem of generalized analytic Fourier-Feynman integral on $C_{a,b}[0,T]$ for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. Actually, Chang and Skoug only stated generalized Fourier-Feynman transform (2.8) but not (2.6) below in [9]. But one has to obtain (2.6) if he or she wants to get (2.8). In Theorem 2.2 below, we will sketch the proofs of (2.6) and (2.8) for the reader.

To ensure the existence of the Fourier-Feynman transform, Chang and Skoug required some conditions, for example, the condition

$$\int_{L^2_{a,b}[0,T]} \exp\Big\{|2q|^{-1/2}\int_0^T |v(t)|\,d|a|(t)\Big\}\,|df(v)| < \infty$$

is needed to get (2.8) below. We introduce the results under weaker conditions (2.5) and (2.7) in Theorem 2.2 than in [9].

Theorem 2.2. *Let $\lambda \in \mathbb{C}_+$. Let $F \in \mathcal{S}(L^2_{a,b}[0,T])$ be given by* (1.4) *and suppose that the associated measure f satisfies the condition*

$$\int_{L^2_{a,b}[0,T]} \exp\{-\beta|(v,a')|\}\,|df(v)| < \infty, \tag{2.5}$$

where $\beta = \operatorname{Im}(\lambda^{-1/2})$. Then $T_\lambda(F)$ exists and is given by

$$T_\lambda(F)(y) = \int_{L^2_{a,b}[0,T]} \exp\Big\{i\langle v,y\rangle - \frac{1}{2\lambda}(v^2,b') + i\lambda^{-1/2}(v,a')\Big\}\,df(v) \tag{2.6}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, suppose that the associated measure f satisfies the condition

$$\int_{L^2_{a,b}[0,T]} \exp\{-|2q|^{-1/2}|(v,a')|\}\,|df(v)| < \infty. \tag{2.7}$$

Then generalized Fourier-Feynman transform exists and is given by

$$T_q^{(p)}(F)(y) = \int_{L^2_{a,b}[0,T]} \exp\Big\{i\langle v,y\rangle - \frac{i}{2q}(v^2,b') + i\Big(\frac{i}{q}\Big)^{1/2}(v,a')\Big\}\,df(v) \tag{2.8}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. For $\lambda > 0$, since F was given by (1.4), we have

$$T_\lambda(F)(y) = \int_{C_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\{i\lambda^{-1/2}\langle v,x\rangle + i\langle v,y\rangle\}\,df(v)\,d\mu(x)$$

for s-a.e. $y \in C_{a,b}[0,T]$. Now Fubini theorem, the function space integration formula (1.5) and direct calculations show that

$$T_\lambda(F)(y) = \int_{L^2_{a,b}[0,T]} \exp\Big\{ i\langle v, y\rangle - \frac{1}{2\lambda}(v^2, b') + i\lambda^{-1/2}(v, a')\Big\}\, df(v)$$

for s-a.e. $y \in C_{a,b}[0,T]$. But by the condition (2.5), we apply dominated convergence theorem and Morera theorem to show that the last expression, as a function of $\lambda \in \mathbb{C}_+$, is analytic in $\mathbb{C}_+$ and so we obtain (2.6). Also by the condition (2.7), we can apply dominated convergence theorem to obtain (2.8). □

Now we give a relationship between generalized analytic Fourier-Feynman transform and function space integral on $C_{a,b}[0,T]$ for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. In this theorem we express generalized Fourier-Feynman transform of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ as a limit of function space integrals.

Theorem 2.3. *Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2_{a,b}[0,T]$. Let $F \in \mathcal{S}(L^2_{a,b}[0,T])$ be given by* (1.4) *and suppose that the associated measure f satisfies the condition* (2.7). *Then we have*

$$(2.9)\quad \begin{aligned} T_q^{(p)}(F)(y) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_{a,b}[0,T]} \exp\Big\{ &\frac{1-\lambda_n}{2}\sum_{k=1}^n \frac{\langle \phi_k, x\rangle^2}{B_k} \\ &+ (\lambda_n^{1/2}-1)\sum_{k=1}^n \frac{A_k\langle \phi_k, x\rangle}{B_k}\Big\} F(x+y)\, d\mu(x) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. Let $\Gamma(\lambda_n)$ be the function space integral on the right hand side of (2.9). Then by (1.4) and Fubini theorem,

$$\Gamma(\lambda_n) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, y\rangle\} H(v, \lambda_n)\, df(v),$$

where

$$\begin{aligned} H(v, \lambda_n) = \int_{C_{a,b}[0,T]} \exp\Big\{ &\frac{1-\lambda_n}{2}\sum_{k=1}^n \frac{\langle \phi_k, x\rangle^2}{B_k} \\ &+ (\lambda_n^{1/2}-1)\sum_{k=1}^n \frac{A_k\langle \phi_k, x\rangle}{B_k} + i\langle v, x\rangle\Big\}\, d\mu(x). \end{aligned}$$

By Lemma 3.1 of [17], we have

$$H(v,\lambda_n)=\lambda_n^{-n/2}\exp\Big\{-\frac{1}{2}(\psi_n^2,b')+i(\psi_n,a')+\sum_{k=1}^{n}\Big[-\frac{1}{2\lambda_n}B_k(\phi_k,v)_{a,b}^2+i\lambda_n^{-1/2}A_k(\phi_k,v)_{a,b}\Big]\Big\}$$

and $\psi_n=v-\sum_{k=1}^{n}(\phi_k,v)_{a,b}\phi_k$. It was shown in the proof of Theorem 3.2 of [17] that

$$\lim_{n\to\infty}\lambda_n^{n/2}H(v,\lambda_n)=\exp\Big\{-\frac{i}{2q}(v^2,b')+i\Big(\frac{i}{q}\Big)^{1/2}(v,a')\Big\}.$$

Finally, since f satisfies the condition (2.7), we apply dominated convergence theorem to obtain

$$\lim_{n\to\infty}\lambda_n^{n/2}\Gamma(\lambda_n)=\int_{L^2_{a,b}[0,T]}\exp\Big\{i\langle v,y\rangle-\frac{i}{2q}(v^2,b')+i\Big(\frac{i}{q}\Big)^{1/2}(v,a')\Big\}\,df(v).$$

By (2.8) in Theorem 2.2, the proof is completed. □

Let $a(t)=0$ and $b(t)=t$ on $[0,T]$. Then the condition (2.7) is satisfied for any complex Borel measure f. Moreover $A_k=0$ and $B_k=1$ for $k=1,2,\ldots$. Hence we have Theorem 4 of [14] as a corollry of Theorem 2.3, that is, we obtain relationship between analytic Fourier-Feynman transform and Wiener integral on Wiener space for functionals in $\mathcal{S}$.

As we have seen in (2.4) above, if $p=1$, then generalized Fourier-Feynman transform $T_q^{(1)}(F)(y)$ is equal to generalized analytic Feynman integral of $F(x+y)$ with respect to x. Further, taking $y=0$ in Theorem 2.4, we have Theorem 3.2 of [17].

The following is a relationship between $T_\lambda(F)$ and function space integral for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

Theorem 2.4. *Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2_{a,b}[0,T]$. Let $F\in\mathcal{S}(L^2_{a,b}[0,T])$ be given by (1.4) and suppose that the associated measure f satisfies the condition (2.5). Then for all $\lambda\in\mathbb{C}_+$, we have*

$$T_\lambda(F)(y)=\lim_{n\to\infty}\lambda^{n/2}\int_{C_{a,b}[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}\frac{\langle\phi_k,x\rangle^2}{B_k}+(\lambda^{1/2}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k,x\rangle}{B_k}\Big\}F(x+y)\,d\mu(x) \tag{2.10}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. To prove this theorem, we modify the proof of Theorem 2.4 by replacing λ_n by λ whenever it occurs. Then we have

$$\lim_{n\to\infty} \lambda^{n/2} H(v,\lambda) = \exp\Big\{-\frac{1}{2\lambda}(v^2,b') + i\lambda^{-1/2}(v,a')\Big\}.$$

Since f satisfies the condition (2.5), we apply dominated convergence theorem to obtain

$$\lim_{n\to\infty} \lambda^{n/2}\Gamma(\lambda) = \int_{L^2_{a,b}[0,T]} \exp\Big\{i\langle v,y\rangle - \frac{1}{2\lambda}(v^2,b') + i\lambda^{-1/2}(v,a')\Big\}\, df(v).$$

By (2.6) in Theorem 2.2, the proof is completed. □

Our main result in this section, namely a change of scale formula for function space integral related with generalized Fourier-Feynman transform of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ now follows from Theorem 2.4.

THEOREM 2.5. *Let* $\{\phi_n\}$ *be a complete orthonormal set of functionals in* $L^2_{a,b}[0,T]$ *and let* $F \in \mathcal{S}(L^2_{a,b}[0,T])$ *be given by* (1.4). *Then for each* $\rho > 0$

(2.11)

$$\int_{C_{a,b}[0,T]} F(\rho x + y)\, d\mu(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_{a,b}[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\frac{\langle\phi_k,x\rangle^2}{B_k} + (\rho^{-1}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k,x\rangle}{B_k}\Big\} F(x+y)\, d\mu(x)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. Letting $\lambda = \rho^{-2}$ in (2.10) and noting that $T_{\rho^{-2}}(F)(y)$ is equal to the function space integral on the left hand side of (2.11), the proof is completed. Of course, since $\beta = \mathrm{Im}(\lambda^{-1/2}) = \mathrm{Im}\rho = 0$, the conditions (2.5) is satisfied for every complex Borel measure f. □

Letting $y = 0$ in Theorem 2.5, we have change of scale formula for function space integral introduced in Theorem 3.6 of [17].

By choosing $a(t) = 0$ and $b(t) = t$ on $[0,T]$, main results in Section 2 of [14], that is, Theorems 4, 6 and 7 can be obtained as special cases of our Theorems 2.3, 2.4 and 2.5, respectively. Theorem 7 in [14] is a change of scale formula for Wiener integrals related with Fourier-Feynman transform on Wiener space.

In our next example we will explicitly compute function space integral of a functional under a change of scale transformation.

EXAMPLE 2.6. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2_{a,b}[0,T]$. Let

$$F(x) = \exp\{\alpha\langle\phi_1, x\rangle\}$$

for $x \in C_{a,b}[0,T]$ and α is a real or complex number. We evaluate the function space integrals on each side of (2.11). The left hand side of (2.11) can be evaluated as follows.

$$L \equiv \int_{C_{a,b}[0,T]} \exp\{\alpha\rho\langle\phi_1, x\rangle + \alpha\langle\phi_1, y\rangle\}\, d\mu(x).$$

By the integration formula (1.5), we have

$$\begin{aligned} L &= (2\pi B_1)^{-1/2} \exp\{\alpha\langle\phi_1, y\rangle\} \int_{\mathbb{R}} \exp\Big\{\alpha\rho u - \frac{(u-A_1)^2}{2B_1}\Big\}\, du \\ &= \exp\Big\{\alpha\langle\phi_1, y\rangle + \alpha\rho A_1 + \frac{1}{2}\alpha^2\rho^2 B_1\Big\}. \end{aligned}$$

Next we evaluate the function space integral on the right hand side of (2.11). Consider

$$\begin{aligned} R \equiv \int_{C_{a,b}[0,T]} \exp\Big\{&\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\frac{\langle\phi_k, x\rangle^2}{B_k} \\ &+ (\rho^{-1}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k, x\rangle}{B_k} + \alpha\langle\phi_1, x+y\rangle\Big\}\, d\mu(x). \end{aligned}$$

By the integration formula (1.5), we have

$$R = \prod_{k=1}^{n}(2\pi B_k)^{-1/2} \exp\{\alpha\langle\phi_1, y\rangle\} \int_{\mathbb{R}^n} \exp\{I_k(u_1, \ldots, u_n)\}\, du_1 \ldots du_n$$

where

$$\begin{aligned} I_k(u_1, \ldots, u_n) = \frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\frac{u_k^2}{B_k} + (\rho^{-1}-1)\sum_{k=1}^{n}\frac{A_k u_k}{B_k} \\ + \alpha u_1 - \sum_{k=1}^{n}\frac{(u_k - A_k)^2}{2B_k}. \end{aligned}$$

By completing the square and evaluating the integrals we have

$$R = \rho^n \exp\Big\{\alpha\langle\phi_1, y\rangle + \alpha\rho A_1 + \frac{1}{2}\alpha^2\rho^2 B_1\Big\}.$$

Thus we have established that (2.11) is valid for $F(x) = \exp\{\alpha\langle\phi_1, x\rangle\}$. Note that α was a real or complex number. If α is pure imaginary, F belongs to $\mathcal{S}(L^2_{a,b}[0,T])$ and so F is an example of a functional to which Theorem 2.5 applies. On the other hand, if the real part of α is not equal to 0, then F is unbounded. Thus this example shows that the class of functionals for which (2.11) holds is more extensive than $\mathcal{S}(L^2_{a,b}[0,T])$.

3. Convolution and a change of scale formula

In this section we give a relationship between function space integral and convolution product on $C_{a,b}[0,T]$ for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$, that is, we express the convolution product of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ as a limit of function space integrals on $C_{a,b}[0,T]$. We start this section by introducing the definition of convolution product for functionals on $C_{a,b}[0,T]$.

Definition 3.1. Let F and G be functionals on $C_{a,b}[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, we define their convolution product $(F*G)_\lambda$ by

$$(F*G)_\lambda(y) = \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,d\mu(x) \tag{3.1}$$

if it exists. Moreover for a nonzero real number q, convolution product $(F*G)_q$ is defined by

$$(F*G)_q(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,d\mu(x) \tag{3.2}$$

if it exists [10, 11, 15].

Chang and Choi [7] established existence theorem of convolution product for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. Although they only state convolution product (3.6) but not (3.4) below in [7], one has to obtain (3.4) if he or she wants to get (3.6). In Theorem 3.2 below, we will sketch the proofs of (3.4) and (3.6) for the reader.

To ensure the existence of convolution product, Chang and Choi required some conditions, for example, the condition

$$\int_{L^2_{a,b}[0,T]} \exp\Big\{|4q|^{-1/2}\int_0^T |v(t)|\, d|a|(t)\Big\}\,(|df(v)|+|dg(v)|)<\infty$$

is needed to get (3.6) below. We introduce the results under weaker conditions (3.3) and (3.5) in Theorem 3.2 than in [7].

THEOREM 3.2 (Theorem 3.2 of [7]). *Let $F, G \in \mathcal{S}(L^2_{a,b}[0,T])$ with associated finite Borel measures f and g satisfy the condition*

$$\int_{(L^2_{a,b}[0,T])^2} \exp\{-\beta|(v-w,a')|\}\,|df(v)|\,|dg(w)|<\infty, \tag{3.3}$$

where $\beta=\mathrm{Im}((2\lambda)^{-1/2})$. Then for all $\lambda\in\mathbb{C}_+$

$$\begin{aligned}(F*G)_\lambda(y)=\int_{(L^2_{a,b}[0,T])^2} &\exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle-\frac{1}{4\lambda}((v-w)^2,b')\\ &+i(2\lambda)^{-1/2}(v-w,a')\Big\}\,df(v)\,dg(w)\end{aligned} \tag{3.4}$$

for s-a.e. $y\in C_{a,b}[0,T]$. Furthermore, suppose that the associated measures f and g satisfy the condition

$$\int_{(L^2_{a,b}[0,T])^2} \exp\{-|4q|^{-1/2}|(v-w,a')|\}\,|df(v)|\,|dg(w)|<\infty. \tag{3.5}$$

Then convolution product $(F*G)_q$ exists and is given by

$$\begin{aligned}(F*G)_q(y)=\int_{(L^2_{a,b}[0,T])^2} &\exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle-\frac{i}{4q}((v-w)^2,b')\\ &+i\Big(\frac{i}{2q}\Big)^{1/2}(v-w,a')\Big\}\,df(v)\,dg(w)\end{aligned} \tag{3.6}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Proof. For $\lambda>0$, since F and G were given by (1.4), we have

$$\begin{aligned}(F*G)_\lambda(y)=\int_{C_{a,b}[0,T]}\int_{(L^2_{a,b}[0,T])^2} &\exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle\\ &+i(2\lambda)^{-1/2}\langle v-w,x\rangle\Big\}df(v)\,dg(w)\,d\mu(x)\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. By Fubini theorem and the function space integration formula (1.5)

$$(F * G)_\lambda(y) = \int_{(L^2_{a,b}[0,T])^2} \exp\Big\{ \frac{i}{\sqrt{2}} \langle v+w, y\rangle - \frac{1}{4\lambda}((v-w)^2, b') \\ + i(2\lambda)^{-1/2}(v-w, a')\Big\}\, df(v)\, dg(w)$$

for s-a.e. $y \in C_{a,b}[0,T]$. But by the condition (3.3), we apply dominated convergence theorem and Morera theorem to show that the last expression, as a function of $\lambda \in \mathbb{C}_+$, is analytic in $\mathbb{C}_+$ and so we obtain (3.4). Also by the condition (3.5), we can apply dominated convergence theorem to obtain (3.6) and this completes the proof. □

Now we give a relationship between convolution product and function space integral on $C_{a,b}[0,T]$ for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. In this theorem we express convolution product of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ as a limit of function space integrals.

THEOREM 3.3. *Let $F, G \in \mathcal{S}(L^2_{a,b}[0,T])$ with associated measures f and g satisfy the condition (3.5). Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2_{a,b}[0,T]$. Then we have*

(3.7)
$$(F * G)_q(y) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_{a,b}[0,T]} \exp\Big\{ \frac{1-\lambda_n}{2} \sum_{k=1}^n \frac{\langle \phi_k, x\rangle^2}{B_k} \\ + (\lambda_n^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x\rangle}{B_k}\Big\} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, d\mu(x)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. Let $\Gamma^*(\lambda_n)$ be the function space integral on the right hand side of (3.7). Then by (1.4) and Fubini theorem,

$$\Gamma^*(\lambda_n) = \int_{(L^2_{a,b}[0,T])^2} \exp\Big\{ \frac{i}{\sqrt{2}} \langle v+w, y\rangle\Big\} H^*(v, w, \lambda_n)\, df(v)\, dg(w),$$

where

$$H^*(v,w,\lambda_n)=\int_{C_{a,b}[0,T]}\exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{n}\frac{\langle\phi_k,x\rangle^2}{B_k}$$
$$+(\lambda_n^{1/2}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k,x\rangle}{B_k}+\frac{i}{\sqrt{2}}\langle v-w,x\rangle\Big\}\,d\mu(x).$$

By Lemma 3.1 of [17], we have

$$H^*(v,w,\lambda_n)=\lambda_n^{-n/2}\exp\Big\{-\frac{1}{2}(\xi_n^2,b')+i(\xi_n,a')$$
$$+\sum_{k=1}^{n}\Big[-\frac{1}{4\lambda_n}B_k(\phi_k,v-w)_{a,b}^2$$
$$+i(2\lambda_n)^{-1/2}A_k(\phi_k,v-w)_{a,b}\Big]\Big\}$$

with $\xi_n=v-w-\sum_{k=1}^{n}(\phi_k,v-w)_{a,b}\phi_k$. Now we can show as in the proof of Theorem 3.2 of [17] that

$$\lim_{n\to\infty}\lambda_n^{n/2}H^*(v,w,\lambda_n)=\exp\Big\{-\frac{i}{4q}((v-w)^2,b')+i\Big(\frac{i}{2q}\Big)^{1/2}(v-w,a')\Big\}.$$

Since f and g satisfy the condition (3.5) we apply dominated convergence theorem to obtain

$$\lim_{n\to\infty}\lambda_n^{n/2}\Gamma^*(\lambda_n)=\int_{(L_{a,b}^2[0,T])^2}\exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle-\frac{i}{4q}((v-w)^2,b')$$
$$+i\Big(\frac{i}{2q}\Big)^{1/2}(v-w,a')\Big\}\,df(v)\,dg(w).$$

By (3.6) in Theorem 3.2, the proof is completed. □

The following is a relationship between convolution product $(F*G)_\lambda$ and function space integral for functionals in $\mathcal{S}(L_{a,b}^2[0,T])$.

Theorem 3.4. *Let $F,G\in\mathcal{S}(L_{a,b}^2[0,T])$ with associated measures f and g satisfy the condition* (3.3). *Let $\{\phi_n\}$ be a complete orthonormal* *set of functionals in* $L^2_{a,b}[0,T]$*. Then for each* $\lambda \in \mathbb{C}_+$ *we have*

(3.8)

$$(F*G)_\lambda(y) = \lim_{n\to\infty} \lambda^{n/2} \int_{C_{a,b}[0,T]} \exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^n \frac{\langle\phi_k,x\rangle^2}{B_k} + (\lambda^{1/2}-1)\sum_{k=1}^n \frac{A_k\langle\phi_k,x\rangle}{B_k}\Big\} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, d\mu(x)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. To prove this theorem, we modify the proof of Theorem 3.3 by replacing λ_n by λ whenever it occurs. Then we have

$$\lim_{n\to\infty} \lambda^{n/2} H^*(v,w,\lambda) = \exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle - \frac{1}{4\lambda}((v-w)^2,b') + \frac{i}{\sqrt{2\lambda}}(v-w,a')\Big\}.$$

Since f and g satisfy the condition (3.3), we apply dominated convergence theorem to obtain

$$\lim_{n\to\infty} \lambda^{n/2}\Gamma^*(\lambda) = \int_{(L^2_{a,b}[0,T])^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle v+w,y\rangle - \frac{1}{4\lambda}((v-w)^2,b') + \frac{i}{\sqrt{2\lambda}}(v-w,a')\Big\}\, df(v)\, dg(w).$$

By (3.4) in Theorem 3.2, the proof is completed. □

Our main result in this section, namely a change of scale formula for function space integral related with convolution product of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ now follows from Theorem 3.5.

THEOREM 3.5. *Let* $F, G \in \mathcal{S}(L^2_{a,b}[0,T])$ *with associated complex Borel measures* f *and* g*. Let* $\{\phi_n\}$ *be a complete orthonormal set of* *functionals in* $L^2_{a,b}[0,T]$. *Then for each* $\rho > 0$

(3.9)

$$\int_{C_{a,b}[0,T]} F\Big(\frac{y+\rho x}{\sqrt{2}}\Big) G\Big(\frac{y-\rho x}{\sqrt{2}}\Big)\, d\mu(x)$$

$$= \lim_{n\to\infty} \rho^{-n} \int_{C_{a,b}[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\frac{\langle\phi_k, x\rangle}{B_k} + (\rho^{-1}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k, x\rangle}{B_k}\Big\}$$

$$F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, d\mu(x)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. First note that for $\lambda > 0$

$$(F*G)_\lambda(y) = \int_{C_{a,b}[0,T]} F\Big(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\Big) G\Big(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\Big)\, d\mu(x).$$

Letting $\lambda = \rho^{-2}$ in (3.8), we have formula (3.9). Note that since $\beta = \mathrm{Im}((2\lambda)^{-1/2}) = \mathrm{Im}(2^{-1/2}\rho) = 0$, the condition (3.3) is satisfied for every complex Borel measures f and g. □

Taking $y = 0$ in (3.9), we have a change of scale formula for function space integral of product of functional in $\mathcal{S}(L^2_{a,b}[0,T])$.

Corollary 3.6. *Let* $F, G \in \mathcal{S}(L^2_{a,b}[0,T])$ *with associated complex Borel measures* f *and* g. *Let* $\{\phi_n\}$ *be a complete orthonormal set of functionals in* $L^2_{a,b}[0,T]$. *Then we have*

(3.10)

$$\int_{C_{a,b}[0,T]} F\Big(\frac{\rho x}{\sqrt{2}}\Big) G\Big(-\frac{\rho x}{\sqrt{2}}\Big)\, d\mu(x)$$

$$= \lim_{n\to\infty} \rho^{-n} \int_{C_{a,b}[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\frac{\langle\phi_k, x\rangle}{B_k} + (\rho^{-1}-1)\sum_{k=1}^{n}\frac{A_k\langle\phi_k, x\rangle}{B_k}\Big\}$$

$$F\Big(\frac{x}{\sqrt{2}}\Big) G\Big(-\frac{x}{\sqrt{2}}\Big)\, d\mu(x)$$

for each $\rho > 0$.

By choosing $a(t) = 0$ and $b(t) = t$ on $[0,T]$, main results in Section 3 of [14], that is, Theorems 12, 13 and 14 can be obtained as special cases of our Theorems 3.3, 3.4 and 3.5, respectively.

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