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STRUCTURAL AND SPECTRAL PROPERTIES OF k-QUASI-$*$-PARANORMAL OPERATORS

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Abstract. For a positive integer k, an operator T is said to be k-quasi-$*$-paranormal if $||T^{k+2}x||||T^kx|| \geq ||T^*T^kx||^2$ for all $x \in H$, which is a generalization of $*$-paranormal operator. In this paper, we give a necessary and sufficient condition for T to be a k-quasi-$*$-paranormal operator. We also prove that the spectrum is continuous on the class of all k-quasi-$*$-paranormal operators.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H. In paper [10] authors introduced the class of k-quasi-$*$-paranormal operators defined as follows:

Definition 1.1. T is a k-quasi-$*$-paranormal operator if

$$||T^{k+2}x||||T^kx|| \geq ||T^*T^kx||^2$$

for every $x \in H$, where k is a natural number.

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A k-quasi-$*$-paranormal operator for a positive integer k is an extension of $*$-paranormal operator, i.e., $||T^2x|| \geq ||T^*x||^2$ for unit vector x. A 1-quasi-$*$-paranormal operator is called a quasi-$*$-paranormal operator and it is normaloid [10], i.e., $||T^n|| = ||T||^n$, for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of T). $*$-paranormal operator and quasi-$*$-paranormal operator have been studied by many authors and it is known that they have many interesting properties similar to those of hyponormal operators (see [5, 9, 11, 14]).

It is clear that

$$*\text{-paranormal} \Rightarrow \text{quasi-}*\text{-paranormal} \Rightarrow \text{normaloid}$$

and

$$\begin{aligned}\text{quasi-}*\text{-paranormal} &\Rightarrow k\text{-quasi-}*\text{-paranormal}\\ &\Rightarrow (k+1)\text{-quasi-}*\text{-paranormal}.\end{aligned}$$

In [14], the authors give a example to show that a quasi-$*$-paranormal operator need not be a $*$-paranormal operator.

Example 1.2. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers n. Consider the operator $T_{A,B}$ on $\oplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $T_{A,B}$ is a quasi-$*$-paranormal operator, but not a $*$-paranormal operator.

We give the following example to show that there also exists a $(k+1)$-quasi-$*$-paranormal operator, but not a k-quasi-$*$-paranormal operator.

Example 1.3. Given a bounded sequence of positive numbers α : $\alpha_1, \alpha_2, \alpha_3, \dots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on l_2 defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthogonal basis for l_2. Straightforward calculations show that W_α is a k-quasi-$*$-paranormal operator if and only if

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$\alpha_{i+1}\alpha_{i+2} \geq \alpha_i^2 \ (i = k, k+1, k+2, \cdots).$$

So, if $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \cdots$ and $\alpha_k > \alpha_{k+2}$, then W_α is a $(k+1)$-quasi-$*$-paranormal operator, but not a k-quasi-$*$-paranormal operator.

Now it is natural to ask whether k-quasi-$*$-paranormal operators are normaloid or not. For $k > 1$, an answer has been given: there exists a nilpotent operator which is a k-quasi-$*$-paranormal operator. But it need not be normaloid.

In section 2, we give a necessary and sufficient condition for T to be a k-quasi-$*$-paranormal operator. In section 3, we prove that the spectrum is continuous on the class of all k-quasi-$*$-paranormal operators.

2. k-quasi-$*$-paranormal operators

In the sequel, we shall write $N(T)$ and $R(T)$ for the null space and range space of T, respectively.

LEMMA 2.1. [10] *T is a k-quasi-$*$-paranormal operator* $\Leftrightarrow T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ *for all* $\lambda > 0$.

THEOREM 2.2. *If T does not have a dense range, then the following statements are equivalent:*

(1) *T is a k-quasi-$*$-paranormal operator;*

(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on* $H = \overline{R(T^k)} \oplus N(T^{*k})$, *where* $T_1^{*2}T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$ *for all* $\lambda > 0$ *and* $T_3^k = 0$. *Furthermore,* $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T^k)}$. Since T is a k-quasi-$*$-paranormal operator, we have

$$P(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0.$$

Therefore

$$T_1^{*2}T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

On the other hand, for any $x = (x_1, x_2) \in H$, we have

$$(T_3^k x_2, x_2) = (T^k(I-P)x, (I-P)x) = ((I-P)x, T^{*k}(I-P)x) = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where M is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [8], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point and T_3 is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T_1^{*2}T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$ for all $\lambda > 0$ and $T_3^k = 0$. Since

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} T^kT^{*k} &= \begin{pmatrix} T_1^kT_1^{*k} + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} (\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where $A = A^* = T_1^kT_1^{*k} + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} (\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^*$. Hence, for all $\lambda > 0$,

$$T^kT^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^kT^{*k}$$

$$= \begin{pmatrix} A(T_1^{*2}T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2)A & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

It follows that $T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ on $H = \overline{R(T^{*k})} \oplus N(T^k)$. Thus T is a k-quasi-$*$-paranormal operator. □

Corollary 2.3. [10] *Let T be a k-quasi-$*$-paranormal operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad H = \overline{R(T^k)} \oplus \; N(T^{*k}).$$

Then T_1 is a $*$-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Corollary 2.4. [11] *If T is a quasi-$*$-paranormal operator and $R(T)$ is not dense, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} on \ H = \overline{R(T)} \oplus N(T^*)$$

where T_1 is a $*$-paranormal operator on $\overline{R(T)}$.

Corollary 2.5. *Let T be a k-quasi-$*$-paranormal operator and $0 \neq \mu \in \sigma_p(T)$. If T is of the form $T = \begin{pmatrix} \mu & B \\ 0 & C \end{pmatrix}$ on $H = N(T-\mu) \oplus N(T-\mu)^\perp$, then $B = 0$.*

Proof. Let P be the projection onto $N(T-\mu)$ and $x \in N(T-\mu)$. Since T is a k-quasi-$*$-paranormal operator and $x = \frac{1}{\mu^k}T^k x \in R(T^k)$, we have

$$P(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0,$$

then

$$\mu^4 - 2\lambda(\mu^2 + BB^*) + \lambda^2 \geq 0 \text{ for all } \lambda > 0,$$

which yields that

$$\mu^4 - 2\lambda\mu^2 + \lambda^2 \geq 2\lambda BB^* \text{ for all } \lambda > 0.$$

Hence $B = 0$. □

3. Spectral properties of k-quasi-$*$-paranormal operators

For every $T \in B(H)$, $\sigma(T)$ is a compact subset of $\mathbb{C}$. The function σ viewed as a function from $B(H)$ into the set of all compact subsets of $\mathbb{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [2] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [6, 12]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [7]. And this result has been extended to quasi-hyponormal operators by Djordjević in [3], to p-hyponormal operators by Hwang and Lee in [13], and to (p, k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators by Duggal, Jeon and Kim in [4]. In this section we extend this result to k-quasi-$*$-paranormal operators.

Lemma 3.1. *Let T be a k-quasi-$*$-paranormal operator. Then the following assertions hold:*
(1) *If T is quasinilpotent, then $T^{k+1} = 0$.*
(2) *For every non-zero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = N(T - \lambda) \oplus (N(T - \lambda))^\perp$ is: $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.*

Proof. (1) Suppose T is a k-quasi-$*$-paranormal operator. If the range of T^k is dense, then T is a $*$-paranormal operator, which leads to that T is normaloid, hence $T = 0$. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k})$$

where T_1 is a $*$-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Theorem 2.2. Since $\sigma(T_1) = \{0\}$, $T_1 = 0$. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

.

(2) If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $N(T - \lambda)$ reduces T by Corollary 2.5. So we have that $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $H = N(T - \lambda) \oplus$

$(N(T-\lambda))^{\perp}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$. □

LEMMA 3.2. [1] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*
(1) *φ is a faithful $*$-representation of the algebra $B(H)$ on K;*
(2) *$\varphi(A) \geq 0$ for any $A \geq 0$ in $B(H)$;*
(3) *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.*

THEOREM 3.3. *The spectrum σ is continuous on the set of k-quasi-$*$-paranormal operators.*

Proof. Suppose T is a k-quasi-$*$-paranormal operator. Let φ: $B(H) \to B(K)$ be Berberian's faithful $*$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a k-quasi-$*$-paranormal operator. In fact, since T is a k-quasi-$*$-paranormal operator, we have

$$T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0 \quad \text{for all } \lambda > 0.$$

Hence we have

$$\begin{aligned}&(\varphi(T))^{*k}((\varphi(T))^{*2}(\varphi(T))^2 - 2\lambda\varphi(T)(\varphi(T))^* + \lambda^2)(\varphi(T))^k \\ &= \varphi(T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k) \text{ by Lemma 3.2} \\ &\geq 0 \text{ by Lemma 3.2,}\end{aligned}$$

so $\varphi(T)$ is also a k-quasi-$*$-paranormal operator. By Lemma 3.1, we have T belongs to the set $C(i)$ (see definition in [4]). Therefore, we have that the spectrum σ is continuous on the set of k-quasi-$*$-paranormal operators by [4, Theorem 1.1]. □

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in H$ such that $(T-\lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. If T is hyponormal, then $\sigma_{jp}(T) = \sigma_p(T)$. Here we show that if T is a k-quasi-$*$-paranormal operator, then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

LEMMA 3.4. *Let T be a k-quasi-$*$-paranormal operator and $\lambda \neq 0$. Then $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.*

Proof. It is obvious from Corollary 2.5. □

The following example provides an operator T which is a k-quasi-$*$-paranormal operator, however, the relation $N(T) \subseteq N(T^*)$ does not hold.

EXAMPLE 3.5. [14] Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers n. Consider the operator $T_{A,B}$ on $\oplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $T_{A,B}$ is a quasi-$*$-paranormal operator, hence $T_{A,B}$ is a k-quasi-$*$-paranormal operator, however for the vector $x = (0, 0, 1, -1, 0, 0, \cdots)$, $T_{A,B}(x) = 0$, but $T^*_{A,B}(x) \neq 0$. Therefore, the relation $N(T_{A,B}) \subseteq N(T^*_{A,B})$ does not always hold.

THEOREM 3.6. *Let T be a k-quasi-$*$-paranormal operator. Then* $\sigma_{jp}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$.

Proof. It is clearly by Lemma 3.4. □

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