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VALUE FUNCTION AND OPTIMALITY CONDITIONS

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Abstract. In the optimal control problem, at first we search the expected optimal solution by using Pontryagin type's necessary conditions called the maximum principle. Next we use the sufficient conditions to conclude that the searched solution is optimal. In this article the sufficient conditions are studied. The value function is used for sufficient conditions.

1. Introduction

Let Z be a complete metric space. Consider the controlled system:

$$\min \psi(x(T))$$

subject to

$$\begin{aligned} x'(t) &= f(t, x(t), u(t)) \quad \text{a.e.} \quad \text{in } [0,T], \\ u(t) &\in U(t) \quad \text{a.e.} \quad \text{in } [0,T], \\ x(0) &= \xi_0, \\ g(t, x(t)) &\leq 0 \quad \forall t \in [0,T], \end{aligned}$$

where

$$\begin{aligned} f &: [0,T] \times \mathbb{R}^n \times Z \to \mathbb{R}^n, \\ U &: [0,T] \Rightarrow Z, \end{aligned}$$

(the symbol '$\Rightarrow$' means that the related function is a set valued function).

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To solve this problem, at first we should find the $x(\cdot)$ which can be optimal. At this time we use the necessary conditions for optimality called the maximum principle. See [4] and [6] for this subject. Secondly, we should verify by the sufficient conditions that the searched solution is really optimal. In this article we study the sufficient conditions for optimality by using the value function.

We set

$$F(t,x) = f(t,x,u(t)),$$

and assume that

(i) $\forall (t,x) \in [0,T] \times \mathbb{R}^n, F(t,x)$ is nonempty, convex and compact,
(ii) $\forall x \in \mathbb{R}^n, F(\cdot,x)$ is measurable,
(iii) $\exists m \in L^1(0,T)$ such that for almost all $t \in [0,T]$, $\forall x \in \mathbb{R}^n$,

$$\sup_{v \in F(t,x)} \|v\| \le m(t)(1+\|x\|),$$

(iv) $\exists k \in L^1(0,T)$ such that $F(t,\cdot)$ is $k(t)-$Lipschitz a.e. in $[0,T]$,
(v) g and ψ are continuous.

Some notations are needed:

$$B_R(x) = \{y \in \mathbb{R}^n \quad | \quad \|y-x\| \le R\}$$

$$\Omega = \{(t,x) \in [0,T] \times \mathbb{R}^n \quad | \quad g(t,x) \le 0\}$$

$$S^g_{[t_0,T]}(x_0) = \left\{ x(\cdot) \in AC(t_0,T) \quad | \quad \begin{array}{c} x'(t) \in F(t,x(t)) \quad \text{a.e.} \quad [t_0,T], \\ g(t,x(t)) \le 0 \quad \forall t \in [t_0,T], \\ x(t_0) = x_0 \end{array} \right\}.$$

where $AC(t_0,T)$ is the set of absolutely continuous functions from $[t_0,T]$ to $\mathbb{R}^n$. The value function associated to the above problem is defined by: for all $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$ with $g(t_0,x_0) \le 0$,

$$V(t_0,x_0) = \inf \left\{ \psi\left(x(T)\right) \quad | \quad x(\cdot) \in S^g_{[t_0,T]}(x_0) \right\}.$$

We set

$$V(t,x) = +\infty \quad \forall\ (t,x) \notin \Omega,$$

and define

$$\mathrm{Dom}(V) = \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^n \quad | \quad V(t,x) \neq \pm\infty \right\}.$$

See [3] for the properties of the value function. In general, the value function is not differentiable. Therefore we need to define the more generalized derivatives and differentials for our purpose.

DEFINITION 1.1. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$ be an extended function, $v \in \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) \neq \infty$. We define:

$$\partial_+\varphi(x_0) = \left\{ p \in \mathbb{R}^n | \limsup_{x\to x_0} \frac{\varphi(x) - \varphi(x_0) - < p, x - x_0 >}{\|x - x_0\|} \leq 0 \right\},$$

$$D_\uparrow\varphi(x_0)(v) = \liminf_{h\to 0^+, v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h},$$

$$D_\downarrow\varphi(x_0)(v) = \limsup_{h\to 0^+, v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}.$$

2. Some basic results

Let $z \in S^g_{[t_0,T]}(x_0)$ and set

$$\varphi(t) = V(t, z(t)) \quad \forall\, t \in [t_0, T].$$

LEMMA 2.1. *Assume that f is continuous and $U(\cdot) = U$ is compact. If for a constant $C > 0$ and $\rho(t) \in \mathbb{R}^n$, we have*

$$\rho(t) \in \partial_+ V(t, z(t)), \qquad \forall\, t \in [t_0, T],$$

and

$$\|\rho(t)\| \leq C, \qquad \forall\, t \in [t_0, T],$$

then there exists a constant M such that

$$D_\uparrow\varphi(t)(1) \leq M, \qquad \forall\, t \in [t_0, T]$$

Proof. We have

$$\begin{aligned} D_\uparrow\varphi(t)(1) &= \liminf_{h\to 0^+, v\to 1} \frac{V(t+hv, z(t+hv)) - V(t, z(t))}{h} \\ &= \liminf_{h\to 0^+, v\to 1} \frac{V\left(t + hv, z(t) + h\frac{z(t+hv)-z(t)}{h}\right) - V(t, z(t))}{h}. \qquad (1) \end{aligned}$$

But, since f is continuous and $U(\cdot) = U$ is compact, there exists M_1 such that for all $t \in [t_0, T]$,

$$\|f(t, z(t), u(t))\| \leq M_1.$$

Therefore we have

$$\begin{aligned}\frac{z(t+h)-z(t)}{h} &= \frac{1}{h}\int_t^{t+h} f(s,z(s),u(s))ds \\ &\in \frac{1}{h}\int_t^{t+h} M_1B_1\ ds \\ &\in M_1B_1.\end{aligned}$$

This implies that there exists a sequence $h_n \to 0^+$ and $\xi \in M_1B_1$ such that

$$\frac{z(t+h_n)}{h_n} \quad \to \quad \xi. \tag{2}$$

Therefore (1) and (2) imply that

$$\begin{aligned}D_\uparrow\varphi(t)(1) &= \liminf_{h\to 0^+, v\to\xi} \frac{V(t+hv, z(t)+hv)-V(t,z(t))}{h} \\ &= D_\downarrow V(t,z(t))(1,\xi) \\ &\le \ <\rho(t),(1,\xi)> \\ &\le C(1+M_1) \\ &= M.\end{aligned}$$

□

Lemma 2.2. *Suppose that V is lower semi-continuous. Under the same hypotheses with Lemma 2.1, $\varphi(\cdot)$ is Lipschitz continuous in $[t_0,T]$.*

Proof. Consider the set valued function $\bar{F}:\mathbb{R}^2 \Rightarrow \mathbb{R}^2$ such that $\bar{F}(\tau,y)=\{(1,M)\}$ where M is the constant of Lemma 2.1. Set $K = Ep(\varphi)$ (see [2] for the definition of Ep). Note that K is closed. We fix $s \ge t_0$. Now we consider the following differential inclusion:

$$\begin{aligned}(\tau,y) &\in \bar{F}(\tau,y) \\ (\tau,y)(0) &= (s,\varphi(s)) \in K.\end{aligned} \tag{3}$$

By Lemma 2.1 and the fact that

$$T_K(\tau,y) \supset T_K(\tau,\varphi(\tau)) \quad \forall\, y \ge \varphi(\tau)$$

(see [2] for the definition of T_K), we have for all $(\tau, y) \in K$,

$$\begin{aligned} (1, M) &\in Ep(D_{\uparrow}\varphi(\tau)) \\ &= T_K(\tau, \varphi(\tau)) \\ &\subset T_K(\tau, y), \end{aligned}$$

in other words, for all $(\tau, y) \in K$,

$$\bar{F}(\tau, y) \cap T_K(\tau, y) = \{(1, M)\} \neq \emptyset.$$

See [2] for the definition of T_K. By the viability theorem, there exists a solution of (3) such that $(\tau, y)(r) \in K$ for all $0 \leq r \leq T - s$. But (3) has only one solution. Therefore

$$(\tau, y) = (s + r, \varphi(s) + Mr) \in K,$$

in other words,

$$0 \leq \varphi(s + r) - \varphi(s) \leq Mr.$$

□

LEMMA 2.3. *Suppose (i) $\sim$ (v). Then for all $R \geq 0$ and for all $(t_0, x_0) \in \Omega$, with $\|x_0\| \leq R$, there exists $L_R \geq R$ such that for all $x \in S^g_{[t_0,T]}(x_0)$ and for all $t \in [t_0, T]$,*

$$\|x(t)\| \leq L_R.$$

Proof. Let $x \in S^g_{[t_0,T]}(x_0)$. Then for almost all $t \in [t_0, T]$,

$$x'(t) \in F(t, x(t)) \subset F(t, 0) + k(t)\|x(t)\|B_1.$$

Therefore for all $t \in [t_0, T]$,

$$\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} m(s)ds + \int_{t_0}^{t} k(s)\|x(s)\|ds.$$

We can apply the Gronwall's Lemma for the conclusion. □

PROPOSITION 2.4. *Assume (i) $\sim$ (v). Furthermore, suppose that $V(t, \cdot)$ is L_R-Lipschitz on $B_R(0) \cap \mathrm{Dom}(V(t, \cdot))$ for all $t \in [t_0, T]$. Then for all $(t_0, x_0) \in \mathrm{Dom}(V)$, for all $x \in S^g_{[t_0,T]}(x_0)$, the function*

$$[t_0, T] \ni t \ \rightarrow V(t, x(t))$$

is absolutely continuous.

Proof. Let $x_1 \in S^g_{[t_0,T]}(x_0)$ and $t_0 \le t_1 \le t_2 \le T$. Since

$$V(t_0, x_0) = \inf \left\{ V(t, x(t)) \quad | \quad x(\cdot) \in S^g_{[t_0,T]}(x_0) \right\},$$

there exists $x_2 \in S^g_{[t_0,T]}(x_1(t_1))$ such that

$$V(t_2, x_2(t_2)) \le V(t_1, x_1(t_1)) + t_2 - t_1.$$

By the proof of Lemma 2.3, for all $i = 1, 2$, we have,

$$\begin{aligned} \|x_i(t_2) - x_i(t_1)\| &\le \int_{t_1}^{t_2} m(s)ds + \int_{t_1}^{t_2} k(s)\|x_i(s)\|ds \\ &\le \int_{t_1}^{t_2} m(s)ds + L_{\|x_0\|} \int_{t_1}^{t_2} k(s)\|x_i(s)\|ds. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\le V(t_2, x_1(t_2)) - V(t_1, x_1(t_1)) \\ &\le V(t_2, x_1(t_2)) - V(t_2, x_2(t_2)) + |t_2 - t_1| \\ &\le L_R\|x_1(t_2) - x_2(t_2)\| + |t_2 - t_1| \\ &\le L_R \left(\|x_1(t_2) - x_1(t_1)\| + \|x_2(t_2) - x_1(t_1)\|\right) + |t_2 - t_1| \\ &\le 2L_R \left(\int_{t_1}^{t_2} m(s)ds + L_{\|x_0\|} \int_{t_1}^{t_2} k(s)ds \right) + |t_2 - t_1|. \end{aligned}$$

By the definition of absolutely continuity, this implies that the function

$$t \mapsto V(t, x(t))$$

is absolutely continuous. □

3. Sufficient conditions

We define the map G for all $(t, x) \in \Omega$,

$$G(t, x) = \{v \in F(t, x) \quad | \quad D_{\uparrow}V(t, x)(1, v) \le 0\}.$$

THEOREM 3.1. *Under the hypothesis of Proposition 2.4 if*

$$\begin{cases} x'(t) \in G(t, x(t)) \quad \text{a.e.} \quad \text{in } [t_0, T] \\ x(t_0) = x_0 \end{cases}, \tag{4}$$

then

$$V(t_0, x_0) = \psi(x(T)).$$

Therefore $x(\cdot)$ is optimal.

Proof. By the definition of G, we have

$$x'(t) \in F(t, x(t)) \quad \text{a.e.} \quad \text{in } [t_0, T],$$

and

$$g(t, x(t)) \leq 0 \quad \text{in } [t_0, T].$$

Set

$$\varphi(t) = V(t,x(t)).$$

Proposition 2.4 implies that φ is absolutely continuous. Hence φ is differentiable almost everywhere. On the other hand φ is nondecreasing, therefore $\varphi' \geq 0$ a.e. Hence to end the proof, it is sufficient to prove that

$$\varphi'(t) \leq 0 \quad \text{a.e.} \quad \text{in } [t_0, T].$$

The condition (4) implies that there exist $h_i \to 0^+$ and $v_i \to x'(t)$ such that for almost all $t \in [t_0, T]$,

$$\begin{aligned} 0 &\geq D_\uparrow V(t, x(t))(1, x'(t)) \\ &\geq \liminf_{h\to 0^+, v\to x'(t)} \frac{V(t+h, x(t)+hv) - V(t, x(t))}{h} \\ &\geq \lim_{i\to\infty} \frac{V(t+h_i, x(t)+h_i v_i) - V(t, x(t))}{h_i}. \end{aligned} \tag{5}$$

Since for all $(t, x) \notin \mathrm{Dom}(V)$, $V(t, x) = \infty$, we have for sufficiently large i,

$$(t + h_i, x(t) + h_i v_i) \in \mathrm{Dom}(V)$$

Fix $t \in [t_0, T]$ such that $\varphi'(t)$ exists. Then

$$\varphi'(t) = \lim_{h\to 0^+} \frac{V(t+h, x(t)+hv) - V(t, x(t))}{h} \leq 0$$

by (5) and Lipschitz continuity of $V(t, \cdot)$. □

Consider the set valued function: $\forall (t, x) \in \Omega$,

$$\bar{G}(t, x) = \{f(t, x, u) \quad | \quad u \in U(t),\, \exists (p_t, p_x) \in \partial_+ V(t, x), \\ p_t + < p_x, f(t, x, u) > \geq 0\}.$$

Definition 3.2. Let $y : [t_0, T] \to \mathbb{R}^n$. For all $t \in [t_0, T)$, we define

$$Dy(t) = \mathrm{Limsup}_{s\to t^+} \frac{y(s) - y(t)}{s - t}$$

(see [2] for the definition of Limsup).

Definition 3.3. We say that a continuous function $y : [t_0, T] \to \mathbb{R}^n$ is a contingent solution of the system:

$$\begin{array}{rcl} x'(t) & \in & \bar{G}(t, x(t)), \\ x(t_0) & = & x_0, \end{array} \tag{6}$$

if

$$Dy(t) \cap \bar{G}(t, y(t)) \neq \emptyset, \qquad \forall\, t \in [t_0, T).$$

Proposition 3.4. *Suppose that $U(\cdot) = U$ is compact and f is continuous. If z is a contingent solution of (6), then z is Lipschitz continuous.*

Proof. Let $R = \sup\{|f(t, z(t), u)| \quad | \quad u \in U, \quad t \in [t_0, T]\}$. Consider the viability problem:

$$\begin{array}{rcll} s'(t) & = & 1, & s(t_0) = t_0, \\ y'(t) & = & \bar{B}_R, & y(t_0) = x_0, \\ (s(t), y(t)) & \in & \text{Graph}(z) & \forall\, t \in [t_0, T]. \end{array}$$

By the viability theorem and the fact that z is a contingent solution of (6), the above problem has a solution. We have $s(t) = t$ and y is R-Lipschitz continuous. Since $z(t) = y(t)$, z is also Lipschitz continuous. □

Theorem 3.5. *Under the assumptions of Proposition 3.4 if z is a contingent solution of the system (6), then z is optimal.*

Proof. By Proposition 3.4, z is Lipschitz continuous and therefore z is differentiable almost everywhere. Therefore we have

$$z'(t) \in \bar{G}(t, z(t)) \subset f(t, z(t), U) \quad \text{a.e.} \quad \text{in} \quad [t_0, T].$$

By applying Theorem 8.2.9 of [2], z is also a trajectory of the dynamical system. Set $\varphi(t) = V(t, z(t))$. Since z is a contingent solution, $\partial_+ V(t, z(t)) \neq \emptyset$ for all $t \in [t_0, T)$. Therefore by the definition of $\bar{G}$, for all $t \in [t_0, T)$, for all $v \in Dz(t) \cap \bar{G}(t, z(t))$, for all $(p_t, p_x) \in \partial_+ V(t, z(t))$,

$$D_\uparrow \varphi(t)(1) \leq D_\downarrow V(t, z(t))(1, v) \leq < (p_t, p_x), (1, v) >= 0.$$

By Lemma 2.2, $\varphi(\cdot)$ is 0-Lipschitz continuous. Therefore

$$V(t, z(t)) = \varphi(t) = \varphi(t_0) = V(t_0, x_0), \quad \forall\, t \in [t_0, T].$$

In other words, z is optimal. □

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusion*, Springer-Verlag, Gründlehren der Math. Wiss. (1984).

[2] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, Basel, Berlin, (1990).

[3] P. Cannarsa and H. Frankowska, *Some characterizations of optimal trajectories in optimal control theory*, SIAM J. Control Optim. **29** (1991), 1322–1347.

[4] F. H. Clarke, *A general theorem on necessary conditions in optimal control*, J. Discret. Contin. Dyn. Syst. **29** (2011), 485–503.

[5] H. Frankowska, *Optimal Trajectories Associated to a Solution of Contingent Hamilton-Jacobi Equation*, Applied Mathematics and Optimization **19** (1993), 291–311.

[6] R. Vinter *Optimal Control*, Birkhäuser, Boston, (2000).

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