# LOCAL VOLATILITY FOR QUANTO OPTION PRICES WITH STOCHASTIC INTEREST RATES

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ABSTRACT. This paper is about the local volatility for the price of a European quanto call option. We derive the explicit formula of the local volatility with constant foreign and domestic interest rates by adapting the methods of Dupire and Derman & Kani. Furthermore, we obtain the Dupire equation for the local volatility with stochastic interest rates.

## 1. Introduction

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable, which allows that investors are to obtain exposure to foreign assets without the corresponding foreign exchange risk. A quanto option has both the strike price and the underlying asset price denominated in foreign currency. At exercise, the value of the option is calculated as the option's intrinsic value in the foreign currency, which is then converted to the domestic currency at the fixed exchange rate.

Pricing options based on the classical Black-Scholes(1973) [1] model, on which most of the research on quanto options has focused, has a problem of assuming a constant volatility which leads to smiles and skews in the implied volatility for the underlying asset price. One way to

overcome such handicaps of constant volatility is using a local volatility model which treats the volatility as a deterministic function of the underlying asset price, current time, maturity and the strike price.

Indeed, local volatility models were introduced and developed by B. Dupire(1994) [3] and E. Derman & I. Kani(1998) [2] as they found that there is a unique diffusion process consistent with the risk-neutral densities derived from the market prices of European options. The main advantage of local volatility models is that the only source of randomness is the price of underlying asset, making local volatility easy to calibrate.

In this paper, we modify and adjust the methods of [3] and [2] to obtain the explicit formula of local volatility for the quanto option price with constant foreign and domestic riskless rates. And then we derive an equation of local volatility for the quanto option price under the stochastic foreign and domestic riskless rates.

We derive the risk-neutral dynamics of the process for the underlying asset with respect to different currency in Section 2. Then, in Section 3, under the model specified in Section 2, we adapt the method of [2] to find the explicit formula of local volatility for the quanto option price with constant foreign and domestic riskless rates. Finally, in Section 4, we derive the analogue of Dupire equation for the local volatility for the quanto option price with constant foreign and domestic riskless rates, and extend this equation to the case of stochastic foreign and domestic riskless rates.

## 2. A risk-neutral dynamics in the quanto framework

Given a complete probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$, let $S_t$ be the asset price on a non-dividend paying asset in foreign currency and $V_t$ be the foreign exchange rate in domestic currency of one unit of the foreign currency with constant volatilities $\sigma_S$ and $\sigma_V$, respectively, which have the following dynamics:

$$\begin{cases} dS_t = \mu_S S_t dt + \sigma_S S_t dB_t, \\ dV_t = \mu_V V_t dt + \sigma_V V_t dW_t, \end{cases}$$

where $\mu_S$ and $\mu_V$ are constants. Also, $B_t$ and $W_t$ are two standard Brownian motions with the correlation $\rho$.

Now, we will find the risk-neutral dynamics of the asset price $S_t$ in domestic currency on a non-dividend paying asset. By the no-arbitrage

condition and the risk-neutral valuation method, under the risk-neutral probability measure $\mathbb{Q}$, it holds that

$$\mathbb{E}_{\mathbb{Q}}\left[V_T|\mathcal{F}_t\right] = V_t e^{\left(r^d - r^f\right)(T-t)},$$

where the constants $r^f$ and $r^d$ are the foreign and domestic riskless rates, respectively. Thus, the risk-neutral dynamics of $V_t$ in domestic currency can be represented as

$$(1) \qquad dV_t = \left(r^d - r^f\right) V_t dt + \sigma_V V_t d\tilde{W}_t,$$

where $\tilde{W}_t$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Under the probability measure $\mathbb{P}$, applying $S_t V_t$ to the Itô formula, we have

$$\begin{aligned}
d\left(S_t V_t\right) &= V_t dS_t + S_t dV_t + dS_t dV_t \\
&= V_t \left(\mu_S S_t dt + \sigma_S S_t dB_t\right) + S_t \left(\mu_V V_t dt + \sigma_V V_t dW_t\right) + \rho \sigma_S \sigma_V S_t V_t dt \\
&= S_t V_t \left(\mu_S + \mu_V + \rho \sigma_S \sigma_V\right) dt + S_t V_t \left(\sigma_S dB_t + \sigma_V dW_t\right),
\end{aligned}$$

and hence, under the risk-neutral probability measure $\mathbb{Q}$, it follows that

$$(2) \qquad d\left(S_t V_t\right) = r^f S_t V_t dt + S_t V_t \left(\sigma_S d\tilde{B}_t + \sigma_V d\tilde{W}_t\right)$$

in domestic currency, where $\tilde{B}_t$ is a standard Brownian motion under $\mathbb{Q}$. From (1), using the Itô formula, the risk-neutral dynamics of $\frac{1}{V_t}$ in domestic currency can be also represented as

$$\begin{aligned}
(3) \qquad d\left(\frac{1}{V_t}\right) &= \left(-\frac{1}{V_t^2}\right) dV_t + \frac{1}{2}\frac{2}{V_t^3} \left(dV_t\right)^2 \\
&= \left(-\frac{1}{V_t^2}\right) \left\{\left(r^d - r^f\right) V_t dt + \sigma_V V_t d\tilde{W}_t\right\} + \frac{1}{V_t^3}\sigma_V^2 V_t^2 dt \\
&= \left(r^f - r^d + \sigma_V^2\right) \frac{1}{V_t} dt - \frac{\sigma_V}{V_t} d\tilde{W}_t.
\end{aligned}$$

Finally, using again the Itô formula with (2) and (3), the risk-neutral dynamics of $S_t$ in domestic currency can be obtained as follows:

$$
\begin{aligned}
dS_t &= d\left(S_t V_t \frac{1}{V_t}\right) \\
&= \frac{1}{V_t} d\left(S_t V_t\right) + S_t V_t d\left(\frac{1}{V_t}\right) + d\left(S_t V_t\right) d\left(\frac{1}{V_t}\right) \\
&= \frac{1}{V_t}\left\{ r^f S_t V_t dt + S_t V_t \left(\sigma_S d\tilde{B}_t + \sigma_V d\tilde{W}_t\right)\right\} \\
&\quad + S_t V_t \left\{\left(r^d - r^f + \sigma_V^2\right)\frac{1}{V_t}dt - \frac{\sigma_V}{V_t}d\tilde{W}_t\right\} - S_t\left(\rho\sigma_S\sigma_V + \sigma_V^2\right)dt \\
&= \left(r^f - \rho\sigma_S\sigma_V\right)S_t dt + \sigma_S S_t d\tilde{B}_t.
\end{aligned}
$$

Adapting and modifying the methods of [2], [3], we will derive the local volatility for the quanto option price with constant riskless rates in next sections. Suppose that the asset price $S_t$ in domestic currency on a non-dividend paying asset follows the risk-neutral dynamics given by

$$
(4)\qquad dS_t = \left\{ r^f - \rho\sigma_S\left(t, S_t\right)\sigma_V\right\}S_t dt + \sigma_S\left(t, S_t\right)S_t d\tilde{B}_t,
$$

where $\sigma_S\left(t, S_t\right)$ denotes the local volatility function for this process.

## 3. The local volatility for the standard quanto option price

E. Derman and I. Kani(1998) [2] characterized the local volatility as a risk-neutral expectation of the instantaneous volatility, conditional on the final asset price being equal to the strike price. The following theorem adapts their method to obtain the quanto option framework with constant foreign and domestic riskless rates.

THEOREM 3.1. *Suppose that the asset price in domestic currency is the stochastic process which follows (4). Let $C_q$ be the price of a European quanto call option at time $t$ in domestic currency with foreign strike price $K$ and maturity $T$. Then the local volatility for this process is expressed by*

$$
(5)\quad \sigma_S\left(S_t; K, T\right)
$$

$$
= \frac{\rho\sigma_V\left(C_q - K\frac{\partial C_q}{\partial K}\right) \pm \sqrt{\rho^2\sigma_V^2\left(C_q - K\frac{\partial C_q}{\partial K}\right)^2 + 2K^2\frac{\partial^2 C_q}{\partial K^2}\left\{\frac{\partial C_q}{\partial T} + r^f K\frac{\partial C_q}{\partial K} - \left(r^f - r^d\right)C_q\right\}}}{K^2\frac{\partial^2 C_q}{\partial K^2}}.
$$

*Proof.* We can write the price of a European quanto call option at time $t$ in domestic currency with foreign strike price $K$ and maturity $T$ as

$$(6) \qquad C_q\left(S_t; K, T\right) = \mathbb{E}_{\mathbb{Q}}\left[V_0 e^{-r^d(T-t)} \max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right]$$

under the risk-neutral probability measure $\mathbb{Q}$, where $V_0$ is the some pre-determined fixed exchange rate.

Differentiating (6) with respect to $K$, it gives

$$\frac{\partial C_q}{\partial K} = -\mathbb{E}_{\mathbb{Q}}\left[V_0 e^{-r^d(T-t)} H\left(S_T - K\right)\middle| \mathcal{F}_t\right],$$

where $H\left(\cdot\right)$ denotes the Heaviside function. Differentiating again (6) with respect to $K$, it gives

$$\frac{\partial^2 C}{\partial K^2} = \mathbb{E}_{\mathbb{Q}}\left[V_0 e^{-r^d(T-t)} \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right],$$

where $\delta\left(\cdot\right)$ denotes the Dirac-delta function. Also, differentiating (6) with respect to $T$, it gives

$$\frac{\partial C_q}{\partial T} = -r^d C_q + V_0 e^{-r^d(T-t)} \frac{\partial}{\partial T} \mathbb{E}_{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right].$$

Applying the Itô formula to the option's payoff, we have

$$d\max\left(S_T - K, 0\right)$$

$$= \frac{\partial}{\partial S_T} \max\left(S_T - K, 0\right) dS_T + \frac{1}{2}\frac{\partial^2}{\partial S_T^2} \max\left(S_T - K, 0\right) \left(dS_T\right)^2$$

$$= H\left(S_T - K\right)\left\{\left(r^f - \rho\sigma_S\sigma_V\right) S_T dT + \sigma_S S_T d\tilde{B}_T\right\} + \frac{1}{2}\delta\left(S_T - K\right)\sigma_S^2 S_T^2 dT$$

from (4).

Now, taking the expectation on both sides, it follows that

$$d\mathbb{E}_{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right]$$

$$= \left(r^f - \rho\sigma_S\sigma_V\right)\mathbb{E}_{\mathbb{Q}}\left[S_T H\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT + \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT$$

$$= \left(r^f - \rho\sigma_S\sigma_V\right)\mathbb{E}_{\mathbb{Q}}\left[\left(S_T - K\right) H\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT$$

$$\quad + \left(r^f - \rho\sigma_S\sigma_V\right) K\mathbb{E}_{\mathbb{Q}}\left[H\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT + \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT$$

$$= \left(r^f - \rho\sigma_S\sigma_V\right)\mathbb{E}_{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right] dT$$

$$\quad + \left(r^f - \rho\sigma_S\sigma_V\right) K\mathbb{E}_{\mathbb{Q}}\left[H\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT + \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right] dT,$$

and hence,

$$\frac{\partial}{\partial T}\mathbb{E}_{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right]$$

$$= \left(r^f - \rho\sigma_S\sigma_V\right)\mathbb{E}_{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\middle| \mathcal{F}_t\right]$$

$$+ \left(r^f - \rho\sigma_S\sigma_V\right)K\mathbb{E}_{\mathbb{Q}}\left[H\left(S_T - K\right)\middle| \mathcal{F}_t\right] + \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right].$$

Finally, we obtain

$$\frac{\partial C_q}{\partial T} = -r^d C_q + \left(r^f - \rho\sigma_S\sigma_V\right)C_q - \left(r^f - \rho\sigma_S\sigma_V\right)K\frac{\partial C_q}{\partial K}$$

$$+ \frac{1}{2}V_0 e^{-r^d(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| \mathcal{F}_t\right]$$

$$= -r^d C_q + \left(r^f - \rho\sigma_S\sigma_V\right)C_q - \left(r^f - \rho\sigma_S\sigma_V\right)K\frac{\partial C_q}{\partial K}$$

$$+ \frac{1}{2}V_0 e^{-r^d(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2 S_T^2 \delta\left(S_T - K\right)\middle| S_T = K\right]\middle| \mathcal{F}_t\right]$$

$$= -r^d C_q + \left(r^f - \rho\sigma_S\sigma_V\right)C_q - \left(r^f - \rho\sigma_S\sigma_V\right)K\frac{\partial C_q}{\partial K}$$

$$+ \frac{1}{2}K^2 V_0 e^{-r^d(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2\middle| S_T = K\right]\mathbb{E}_{\mathbb{Q}}\left[\delta\left(S_T - K\right)\middle| \mathcal{F}_t\right]$$

$$= -r^d C_q + \left(r^f - \rho\sigma_S\sigma_V\right)C_q - \left(r^f - \rho\sigma_S\sigma_V\right)K\frac{\partial C_q}{\partial K}$$

$$+ \frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2\middle| S_T = K\right],$$

which follows that

$$\frac{\partial C_q}{\partial T} + \left(r^f - \rho\sigma_S\sigma_V\right)K\frac{\partial C_q}{\partial K} - \frac{1}{2}K^2\frac{\partial^2 C_q}{\partial K^2}\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2\middle| S_T = K\right]$$

$$- \left(r^f - r^d - \rho\sigma_S\sigma_V\right)C_q = 0.$$

Regarding $\sigma_S\left(S_t; K, T\right) = \sqrt{\mathbb{E}_{\mathbb{Q}}\left[\sigma_S^2\middle| S_T = K\right]}$, we get the desired result.

$\square$

## 4. The Dupire's method and local volatility with stochastic interest rates

As another way to the local volatility, we apply the method of B. Dupire(1994) [3] which uses the Fokker-Planck equation (see Chapter 8

of [4]) for the process (4) to get the equation of local volatility for the quanto option price with constant foreign and domestic riskless rates and extend this equation to the case of stochastic foreign and domestic riskless rates. To begin with the case of constant rates, the following theorem gives the equation for the price of a European quanto call option.

THEOREM 4.1. *With the assumptions of Theorem 3.1, $C_q$ satisfies the following equation:*

$$(7) \quad \frac{\partial C_q}{\partial T} + \left( r^f - \rho \sigma_S \sigma_V \right) K \frac{\partial C_q}{\partial K} - \frac{1}{2} \sigma_S^2 K^2 \frac{\partial^2 C_q}{\partial K^2} - \left( r^f - r^d - \rho \sigma_S \sigma_V \right) C_q = 0$$

*for the local volatility $\sigma_S = \sigma_S \left( S_t; K, T \right)$.*

*Proof.* Let $p \left( t, S_t; T, S_T \right)$ be the risk-neutral probability density function of $S_T$. Then we have the following equation:

$$(8) \quad C_q \left( S_t; K, T \right) = \int_{-\infty}^{\infty} V_0 e^{-r^d(T-t)} \max \left( S_T - K, 0 \right) p \left( t, S_t; T, S_T \right) dS_T$$

$$= \int_{K}^{\infty} V_0 e^{-r^d(T-t)} \left( S_T - K \right) p \left( t, S_t; T, S_T \right) dS_T.$$

Since $p \left( t, S_t; T, S_T \right)$ must satisfy the Fokker-Planck equation, we obtain

$$(9) \quad \frac{\partial p}{\partial T} - \frac{1}{2} \frac{\partial^2}{\partial S_T^2} \left( \sigma_S^2 S_T^2 p \right) + \frac{\partial}{\partial S_T} \left\{ \left( r^f - \rho \sigma_S \sigma_V \right) S_T p \right\} = 0.$$

Now, differentiating (8) with respect to $K$, it gives

$$\frac{\partial C_q}{\partial K} = - \int_{K}^{\infty} V_0 e^{-r^d(T-t)} p \left( t, S_t; T, S_T \right) dS_T$$

and

$$\frac{\partial^2 C_q}{\partial K^2} = V_0 e^{-r^d(T-t)} p \left( t, S_t; T, K \right).$$

Also, differentiating (8) with respect to $T$ so that applying (9) and the integration by parts, it gives

$$\frac{\partial C_q}{\partial T} = -r^d C_q + \int_K^\infty V_0 e^{-r^d(T-t)} (S_T - K) \frac{\partial p}{\partial T} dS_T$$

$$= -r^d C_q + \int_K^\infty V_0 e^{-r^d(T-t)} (S_T - K)$$

$$\times \left[ \frac{1}{2} \frac{\partial^2}{\partial S_T^2} (\sigma_S^2 S_T^2 p) - \frac{\partial}{\partial S_T} \left\{ \left( r^f - \rho \sigma_S \sigma_V \right) S_T p \right\} \right] dS_T$$

$$= -r^d C_q + \frac{1}{2} V_0 e^{-r^d(T-t)} \sigma_S^2 K^2 p$$

$$+ \left( r^f - \rho \sigma_S \sigma_V \right) \int_K^\infty V_0 e^{-r^d(T-t)} (S_T - K) p \, dS_T$$

$$+ \left( r^f - \rho \sigma_S \sigma_V \right) \int_K^\infty V_0 e^{-r^d(T-t)} K p \, dS_T$$

$$= -r^d C_q + \frac{1}{2} \sigma_S^2 K^2 \frac{\partial^2 C_q}{\partial K^2} + \left( r^f - \rho \sigma_S \sigma_V \right) C_q - \left( r^f - \rho \sigma_S \sigma_V \right) K \frac{\partial C_q}{\partial K}.$$

Thus, the proof is complete. $\qquad\square$

We refer (7) to the Dupire equation for the price of a European quanto call option. This also gives us the Dupire formula for the local volatility, which is equally expressed by (5).

We now assume more general case that riskless rates are stochastic. Then the risk-neutral dynamics of $S_t$ in domestic currency can be written as

$$(10) \qquad dS_t = \left\{ r_t^f - \rho \sigma_S (t, S_t) \sigma_V \right\} S_t dt + \sigma_S (t, S_t) S_t d\tilde{B}_t,$$

where $r_t^f$ is the foreign riskless rate which follows some stochastic process. We also assume that the domestic riskless rate $r^d = r_t^d$ in the previous section also follows some stochastic process. The following theorem gives the Dupire equation for the price of a European quanto call option. However, to obtain the usable local volatility from the equation, we may need some numerical procedure.

THEOREM 4.2. *Suppose that the asset price in domestic currency is the stochastic process which follows* (10). *Let* $C_q \left( S_t, r_t^f, r_t^d; K, T \right)$ *be the price of a European quanto call option at time $t$ in domestic currency $Y$ with foreign strike price $K$ and maturity $T$, and let* $p \left( t, S_t, r_t^f, r_t^d; T, S_T, r_T^f, r_T^d \right)$ *be the risk-neutral joint probability density*

function of $S_T$, $r_T^f$ and $r_T^d$. Then $C_q\left(S_t, r_t^f, r_t^d; K, T\right)$ satisfies the following Dupire equation:

$$\frac{\partial C_q}{\partial T} = \frac{1}{2}\sigma_S^2 K^2 \frac{\partial^2 C_q}{\partial K^2} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{K}^{\infty} V_0 e^{-z(T-t)}\left\{(x-K)z + y - \rho\sigma_S\sigma_V\right\}$$
$$\times\, xp\left(t, S_t, r_t^f, r_t^d; T, x, y, z\right)dxdydz$$

for the local volatility $\sigma_S = \sigma_S\left(S_t, r_t^f, r_t^d; K, T\right)$.

   *Proof.* As before, the price of a European quanto call option is

(11)
$$C_q\left(S_t, r_t^f, r_t^d; K, T\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{K}^{\infty} V_0 e^{-z(T-t)}(x-K)\, p\left(t, S_t, r_t^f, r_t^d; T, x, y, z\right)dxdydz.$$

Now, differentiating (11) with respect to $K$, it gives

$$\frac{\partial C_q}{\partial K} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{K}^{\infty} V_0 e^{-z(T-t)} p\left(t, S_t, r_t^f, r_t^d; T, x, y, z\right)dxdydz$$

and

$$\frac{\partial^2 C_q}{\partial K^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_0 e^{-z(T-t)} p\left(t, S_t, r_t^f, r_t^d; T, K, y, z\right)dydz.$$

Also, differentiating (11) with respect to $T$ so that applying the Fokker-Planck equation for $p\left(t, S_t, r_t^f, r_t^d; T, S_T, r_T^f, r_T^d\right)$ and the integration by

parts, it gives

$$\frac{\partial C_q}{\partial T} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_K^{\infty} V_0 e^{-z(T-t)} (x - K) \left(-z + \frac{\partial p}{\partial T}\right) dxdydz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_K^{\infty} V_0 e^{-z(T-t)} (x - K)$$

$$\times \left[-z + \left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma_S^2 x^2 p\right) - \frac{\partial}{\partial x}\left\{(y - \rho\sigma_S\sigma_V) xp\right\}\right]\right] dxdydz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{-\int_K^{\infty} V_0 z e^{-z(T-t)} (x - K) p dx\right.$$

$$\left. +\frac{1}{2}V_0 e^{-z(T-t)}\sigma_S^2 K^2 p + \int_K^{\infty} V_0 e^{-z(T-t)} (y - \rho\sigma_S\sigma_V) xp dx\right\} dydz$$

$$= \frac{1}{2}\sigma_S^2 K^2 \frac{\partial^2 C_q}{\partial K^2}$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_K^{\infty} V_0 e^{-z(T-t)} \left\{(x - K) z + y - \rho\sigma_S\sigma_V\right\} xp dxdydz.$$

$\square$

## References

[1] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, The Journal of Political Economy **81** (3) (1973), 637–654.

[2] E. Derman and I. Kani, *Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility*, International Journal of Theoretical and Applied Finance **1** (1) (1998), 61–110.

[3] B. Dupire, *Pricing with a Smile*, Risk Magazine **7** (1) (1994), 18–20.

[4] M. Overhaus, A. Lamnouar, A. Bermúdez, H. Bueshler, A. Ferraris and C. Jordinson, *Equity Hybrid Derivatives*, John Wiley & Sons. Hoboken, NJ (2007).

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