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THE BASKET NUMBERS OF KNOTS

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Abstract. Plumbing surfaces of links were introduced to study the geometry of the complement of the links. A basket surface is one of these plumbing surfaces and it can be presented by two sequential presentations, the first sequence is the flat plumbing basket code found by Furihata, Hirasawa and Kobayashi and the second sequence presents the number of the full twists for each of annuli. The minimum number of plumbings to obtain a basket surface of a knot is defined to be *the basket number* of the given knot. In present article, we first find a classification theorem about the basket number of knots. We use these sequential presentations and the classification theorem to find the basket number of all prime knots whose crossing number is 7 or less except two knots 7_1 and 7_5.

1. Introduction

Orientable surfaces whose boundary is the given link, known as *Seifert surfaces* have been studied for many interesting invariants of links such as Seifert pairings, Alexander polynomials, signatures and etc. A *plumbing surface* obtained from a 2-dimensional disc by plumbings annuli found by Rudolph [20] used to study extensively for the fibreness of links and surfaces [3–5, 8, 16, 19, 23]. Among these plumbing surfaces the main focus of the present article is the basket surfaces, a precise definition can be found in Definition 2.1.

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The third author's first preprint about these plumbing surfaces from a canonical Seifert surface had a critical mistake. In the process of resolving this mistake, the third author have made a few results about the banded surfaces and flat banded surfaces [15] and dipole graphs which is also known as a braidzel surface [17], and a complete bipartite graph $K_{2,n}$ [12]. The mistake was finally fixed in [11].

The present work is one of this series of articles presenting links as a boundary of the surface obtained in a embedding of certain graphs as described in [7]. One might consider these plumbing surfaces as special embeddings of the bouquets of circles [6].

A sequence of articles by Hayashi and Wada [10], Furihata, Hirasawa and Kobayashi [3] and the third author [11] deal with the flat plumbing basket surface of a given link L. The work of Furihata *et al.* [3] provided not only the existence theorem using a very tangible alternating definition of the flat plumbing basket surface but also a coding algorithm, the resulting code is called *flat plumbing basket code*, to present links as the boundaries of flat plumbing basket surfaces from a special closed braid presentation of the link.

We generalize this presentation to find a new presentation for the basket surfaces as follow. These plumbing basket surface can be presented by two sequential presentations, the first sequence is is the flat plumbing basket code first found by Furihata *et al.* and the second sequence presents the number of the full twists.

In present article, we use these sequential codes to find the basket number of all prime knots with 7 crossings or less except 7_1 and 7_5 by applying DT-notation and a computer program "knotfinder" of `Knotscape` [24].

The outline of this paper is as follows. We first provide some preliminary definitions and results in Section 2. We explained how two sequential presentations present the basket surfaces. Also we provide two classification theorems of the flat plumbing basket number of 0 and 2 with a explanation how we find DT-notation and use the computer program "knotfinder" of `Knotscape` in Section 3. We conclude with a remark on further research in Section 4.

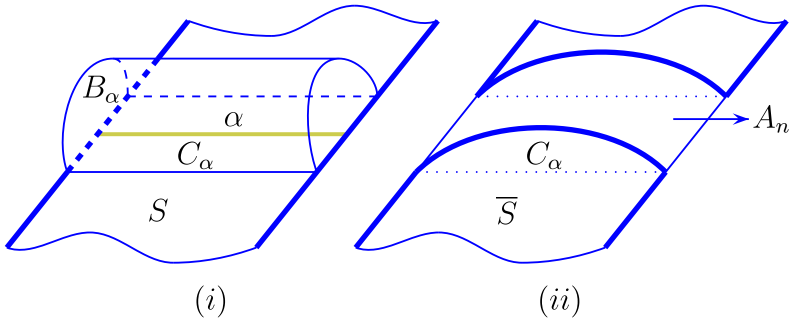


FIGURE 1. (i) A geometric shape of α, B_α and C_α on a Seifert surface S and (ii) a new Seifert surface $\overline{S}$ obtained from S by a top A_n plumbing along the path α.

2. Preliminaries

A compact orientable surface $\mathcal{F}$ is called a *Seifert surface* of a link L if the boundary of $\mathcal{F}$ is isotopic to the given link L. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of L, this algorithm was named after him as *Seifert's algorithm* [22]. A Seifert surface $\mathcal{F}_L$ of an oriented link L produced by applying Seifert's algorithm to a link diagram is called a *canonical Seifert surface*.

The main topic of the article is the basket surfaces. Rudolph first defined the top plumbing as follows. Let α be a proper arc on a Seifert surface S. Let B_α be a 3-cell on top of S along a tubular neighborhood C_α of α on S. Let $A_n \subset B_\alpha$ be an n times full twisted annulus such that $A_n \cap \partial B_\alpha = C_\alpha$. The *top plumbing* on S along a path α is the new surface $S' = S \cup C_\alpha$ where A_n, B_α, C_α satisfy the previous conditions as depicted in Figure 1. Thus, two consecutive plumbings are non-commutative in general. Rudolph found a few interesting results with regards to the top and bottom plumbings in [20]. For the rest of article, all plumbings are top plumbing unless state differently.

DEFINITION 2.1. A Seifert surface $\mathcal{F}$ is a *basket surface* if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_0 *_\alpha A_n$ which can be constructed by plumbing A_n to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$ where A_n is an annulus with n full twists. We say that a surface $\mathcal{F}$ is *a basket surface of* a link L if it is a basket surface and $\partial\mathcal{F}$ is equivalent to L. The minimum number of plumbings among all basket surfaces of a link L is called the *basket number* of the link L, denoted by $bk(L)$.

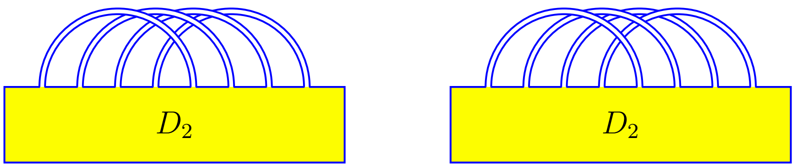


FIGURE 2. (a) A flat plumbing basket surface of the trefoil knot and (b) a flat plumbing basket surface of the figure eight knot.

let us remark that if we only use untwisted annulus A_0 for entire plumbing, the resulting surface is called *flat plumbing basket surface*. An alternative definition of the flat plumbing basket surfaces is given in [3] and it is very easy to follow. The *trivial open book decomposition* of $\mathbb{R}^3$ is a decomposition of $\mathbb{R}^3$ into the half planes in the following form. In a cylindrical coordinate, it can be presented

$$\mathbb{R}^3 = \bigcup_{\theta \in [0,2\pi)} \{(r,\theta,z) | r \geq 0, z \in \mathbb{R}\}$$

where $\{(r,\theta,z)|r \geq 0, z \in \mathbb{R}\}$ is called a *page* for $\theta \in [0, 2\pi)$. Let $\mathcal{O}$ be the *trivial open book decomposition* of the 3-sphere $\mathbb{S}^3$ which is obtained from the trivial open book decomposition of $\mathbb{R}^3$ by the one point compactification. A Seifert surface is said to be a flat plumbing basket surface if it consists of a single page of $\mathcal{O}$ as a 2-disc D_2 and finitely many bands which are embedded in distinct pages [3]. Flat plumbing basket surfaces of (i) the trefoil knot and (ii) the figure eight knot in the trivial open book decomposition are depicted in Figure 2 where D_2 is presented as a shaded rectangular region and the top horizontal line of the rectangle is in the z-axis and the top hemi-spherical annuli are contained in different pages. Furihata, Hirasawa and Kobayashi [3] introduced a sequential presentation of these flat plumbing basket surfaces by giving numbering on the boundary of the disc D_2 where each of flat annulus belongs to the pages in the trivial open book decomposition. For example, the flat plumbing basket surface in Figure 2 (a) is presented by 12341234 and 12431243 for (b).

A previous study by the third author found an upper bound for the basket number of a link using a braid presentation of the link as follow.

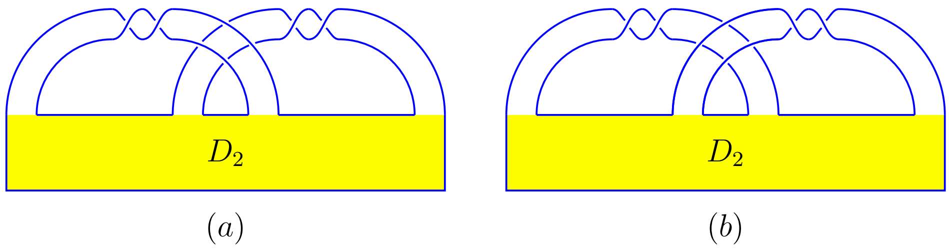


FIGURE 3. Basket surfaces of (a) the trefoil knot and (b) the figure eight knot whose basket numbers are 2 [11].

THEOREM 2.2. ([11]) *Let L be a link which is the closure of a braid $\beta \in B_n$ where the length of the braid β is m. Then the basket number of L is less than or equal to $m - n + 1$, i.e.,*

$$bk(L) \leq m - n + 1.$$

EXAMPLE 2.3. ([11]) The basket number of the trefoil knot and figure eight knot is 2 as illustrated in Figure 3.

There are some articles which find the flat plumbing basket number of knots [2, 9, 11, 13]. These numbers are obvious upper bounds for the basket number of knots because the basket number is always less than or equal to the flat plumbing basket number.

Let us explain rational tangle and rational link since out main result involves with them. Rational knots and links comprise the simplest class of links. The first twenty five knots, except for 8_5, are rational. Furthermore all knots and links up to ten crossings are either rational or are obtained by inserting rational tangles into a small number of planar graphs. Rational links are alternating with one or two unknotted components, and they are also known in the literature as Viergeflechte, four-plats or 2-bridge knots depending on their geometric representation [14].

The notion of a tangle was introduced in 1967 by Conway in his work on enumerating and classifying knots and links, and he defined the rational knots as numerator or denominator closures of the rational tangles [1]. The following formula explain how to find the continued fraction from rational tangle.

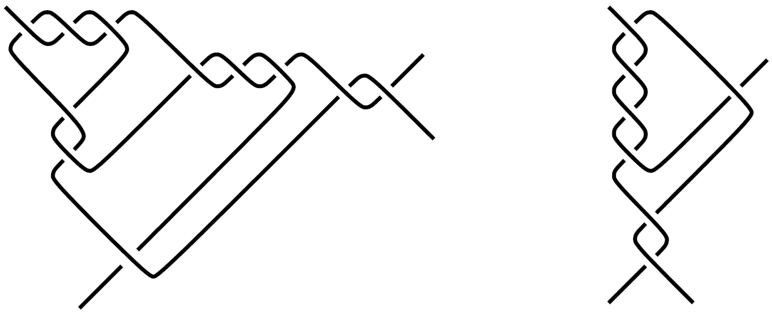


FIGURE 4. Rational tangles $(3,2,3,1,2)$ (left) and $(4,1,2,0)$ (right).

$$(1) \qquad (q_n, q_{n-1}, \ldots, q_1) \sim q = q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_n}}}$$

THEOREM 2.4. ([1]) *Two rational tangles are isotopic if and only if they have the same continued fraction.*

Schubert [21] originally stated the classification of rational knots and links by representing them as 2-bridge links. Theorem 2.5 has hitherto been proved by taking the 2-fold branched covering spaces of S^3 along 2-bridge links, showing that these correspond bijectively to oriented diffeomorphism classes of lens spaces, and invoking the classification of lens spaces [18] [14]. The following statement of Schubert's theorem is a formulation of the Theorem in the language of Conway's tangles.

THEOREM 2.5. ([14,21]) *Suppose that rational tangles with continued fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given (p and q are relatively prime. Similarly for p' and q'.) If $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ denote the corresponding rational knots obtained by taking numerator closures of these tangles, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are isotopic if and only if*

1. $p = p'$ *and*
2. *either* $q \equiv q' \bmod p$ *or* $qq' \equiv 1 \bmod p$.

3. The basket codes and results

Using the definition in the previous section, for a given link L, Furihata *et al.* [3] found an algorithm to find a flat plumbing basket surface from a closed braid $\overline{\beta} = L$. Since we are dealing with basket surfaces, we may choose twisted A_n plumbing instead of flat plumbings used for flat plumbing basket surfaces.

The basket code of a basket surface is two sequences $(a_1 a_2 \ldots a_{2m} : b_1, b_2, \ldots, b_m)$ where the first part $a_1 a_2 \ldots a_{2m}$ is the flat plumbing basket code defined in [3] and modified in [2] and the second part $b_1, b_2, \ldots, b_m$ presents the number of full twists for annulus connecting two i which appear exactly twice in the flat plumbing basket code. In fact, we perform A_{b_i}-plumbing which appears in i-th page of the trivial open book decomposition.

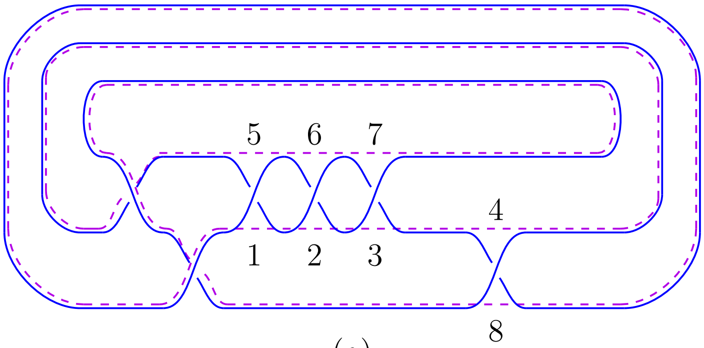
`Algorithm`

- *Step* 1. For a give link L, we find its braid representation β, the closed braid $\overline{\beta} = L$.
- *Step* 2. Apply the method in [3] to obtain a basket surface $\mathcal{F}$ while allowing either A_2 or A_{-2} plumbings.
- *Step* 3. Find a basket code $(a_1 a_2 \ldots a_{2m} : b_1, b_2, \ldots, b_m)$ of the basket surface.

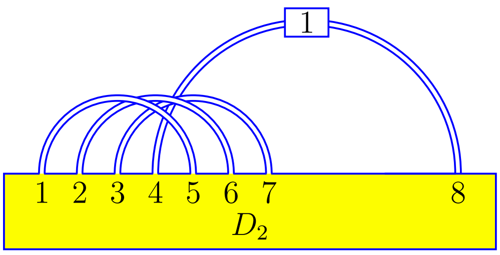
This algorithm can be demonstrated in the following Example 3.1. We recommend the reader to compare this with [2, Example 3.1] which find a flat plumbing basket surface of the knot 5_2.

Example 3.1. A basket code of the knot 5_2 is $(12341234 : 0, 0, 0, 1)$.

Proof. For the knot 5_2, we first present it as a closed braid $\overline{\sigma_2 \sigma_1^{-1} (\sigma_2)^{-3} \sigma_1^{-1}}$ on three strings as illustrated in Figure 5 (a). Although theorem in [3] stated differently, one can choose any two generators of the Artin's braid group B_3 as stated in [11, Theorem 4.10]. We choose the first $\sigma_2 \sigma_1^{-1}$ to have a disc $\mathcal{D}$ which is the union of three discs, bounded by three Seifert circles, joined by two half twisted bands presented by $\sigma_2 \sigma_1^{-1}$ as indicated by the dashed purple line in Figure 5 (a). Since the rest word $(\sigma_2)^{-3} \sigma_1^{-1}$ has the length 4 and $(\sigma_2)^{-3}$ has the different sign to σ_2. we need three flat plumbings. However σ_1^{-1} has the same sign to σ_1^{-1}, so we need A_2-plumbing. Therefore, we find $(12341234 : 0, 0, 0, 1)$



(a)



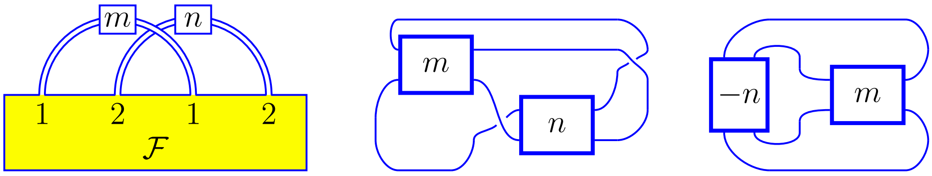
(b)

FIGURE 5. (a) The knot 5_2 as a closed braid, (b) a basket surface of 5_2 for which the basket code is $(12341234 : 0, 0, 0, 1)$.

as the basket code of the basket surface of knot 5_2 as depicted in Figure 5 (b). □

Let us remark that the boundary of basket surfaces with a basket number n has at most $n+1$ components, and the number of components is always congruent to $n+1$ modulo 2. Therefore, the basket numbers of a knot have to be even.

Now, we will provide a classification theorem of knots and links by the basket numbers.



(a) (b) (c)

FIGURE 6. (a) The basket surface $\mathcal{F}$ of the basket code $(1212 : m, n)$, (b) the boundary of the surface $\mathcal{F}$ as a 2-bridge knot and (c) the boundary of the surface $\mathcal{F}$ as a rational knot.

THEOREM 3.2. (1) *A link L has the basket number* 0 *if and only if L is the trivial knot.*

(2) *A knot K has the basket number* 2 *if and only if K is a rational knot which is equivalent by Theorem 2.5 to a rational knot with continued fraction of the form* $\dfrac{4mn-1}{2n}$ *for some integer m, n.*

Proof. The first statement, link L has the basket number 0 if and only if L is the trivial knot is very straightforward. For the case of the basket number 2, there are four cases of the flat plumbing basket code $(1122), (2211), (1212)$ and (2121). The boundaries of the first two cases are links. If we only consider the boundary of the basket surface, we can rotate the first annulus to 360 degree so that the last cases are the same. Thus we will concentrate the basket code of $(1212 : m, n)$ where m, n are integers.

One may easily find the isotopy between each knot diagrams in Figure 6. The knot diagram in Figure 6 (b) is a 2-bridge knot diagram and the knot diagram in Figure 6 (c) is a rational knot, with rational tangle $(-2n)(2m)$ whose continued fraction is $\dfrac{4mn-1}{2n}$. Since the isotopy classes of rational knot are classified by Theorem 2.5, it completes the proof. □

By using algorithm explained in the beginning of the section, we find the following theorem.

THEOREM 3.3. *The basket number of the prime knots up to* 7 *crossings is given in Table 1.*

Name of knot	Itscontinued fraction	Basket number	Basket Code
Unknot		0	
3_1	$3/1$	2	$(1212:1,1)$
4_1	$5/2$	2	$(2121:1,-1)$
5_1	$5/1$	4	$(12341234:-1,-1,0,0)$
5_2	$7/3$	4	$(1212:-1,-2)$
6_1	$9/2$	2	$(2121:-2,1)$
6_2	$11/4$	4	$(12341234:-1,1,1,1)$
6_3	$13/5$	4	$(12431243:-1,-1,1,1)$
7_1	$7/1$	$4-6$	$(123456123456:1,1,1,1,1,1)$
7_2	$11/5$	2	$(2121:3,1)$
7_3	$13/4$	$4-6$	$(123456123456:0,0,0,0,0,1)$
7_4	$15/4$	2	$(2121:2,2)$
7_5	$17/4$	$4-6$	$(123456156234:0,1,1,1,1,1)$
7_6	$19/7$	4	$(12431243:1,-1,0,-1)$
7_7	$21/8$	4	$(12341234:1,1,-1,-1)$

TABLE 1. The basket number of the prime knots up to 7 crossings.

Proof. The knots $3_1, 4_1, 5_2, 6_1, 7_2$ and 7_4 are all nontrivial and we have found a basket code of length 2, thus, their basket number are 2 by Theorem 3.2 (1). One the other hand, these basket code can be found from their continued fractions using Theorem 2.5 as follows :

$$3_1 \ : \quad \frac{3}{1} \cong \frac{3}{-2} = \frac{4(-1)(-1)-1}{2(-1)}$$

$$4_1 \ : \quad \frac{5}{3} \cong \frac{5}{-2} = \frac{-5}{2} = \frac{4(1)(-1)-1}{2(1)}$$

$$5_2 \ : \quad \frac{7}{3} \cong \frac{7}{-4} = \frac{4(-2)(-1)-1}{2(-2)}$$

$$6_1 \ : \quad \frac{9}{2} \cong \frac{9}{-7} \cong \frac{-9}{7} \cong \frac{-9}{4} = \frac{4(2)(-1)-1}{2(2)}$$

$$7_2 \ : \quad \frac{11}{5} \cong \frac{11}{-6} = \frac{4(-3)(-1)-1}{2(-3)}$$

$$7_4 \ : \quad \frac{15}{4} = \frac{4(2)(2)-1}{2(2)}$$

where $\cong$ presents one of two equivalent moves in Theorem 2.5.

One may observe that to change a given fraction $\frac{p}{q}$ to $\frac{4mn-1}{2n}$ for some integer m and n using a finite sequence of two equivalent moves in Theorem 2.5, we have to obtain $p \equiv 3 \mod 4$ or $-p \equiv 3 \mod 4$. Once we fix the numerator of $\frac{4mn-1}{2n}$ to be one of p or $-p$, we add 1 and factor the resulting integer into $4mn$ for some integer m and n. If there exists a pair of m, n such that the denominator is of the form $2n$ by a finite sequence of two equivalent moves in Theorem 2.5, then the corresponding knot has the basket number 2 and its basket code is $(1212 : m, n)$. Otherwise, the corresponding knot has the basket number bigger than 2. Since $|4mn-1| > |2n|$ for nonzero m, we only need to consider cases that the absolute value of the denominator to be less than the absolute value of the numerator and to be even. So there are only two possibilities for such denominator once we fix the numerator to be congruent to $3 \mod 4$ unless its multiplicative inverse is itself modulo the numerator. Using this idea we find the reminding knots can not have the basket number 2 as follow :

$$5_1 : \quad \frac{5}{1} \cong \frac{5}{-4} = \frac{-5}{4} \neq \frac{4mn-1}{2n}$$

$$6_2 : \quad \frac{9}{2} \cong \frac{9}{-7} = \frac{-9}{7} \cong \frac{-9}{4} \text{ and } \cong \frac{-9}{-2} \neq \frac{\text{4mn}-1}{\text{2n}}$$

$$6_3 : \quad \frac{11}{4} \text{ and } \cong \frac{11}{-8} \neq \frac{\text{4mn}-1}{\text{2n}}$$

$$7_1 : \quad \frac{7}{1} \cong \frac{7}{-6} = \frac{-7}{6} \neq \frac{4mn-1}{2n}$$

$$7_3 : \quad \frac{13}{4} = \frac{-13}{-4} \text{ and } \cong \frac{-13}{-10} \neq \frac{\text{4mn}-1}{\text{2n}}$$

$$7_5 : \quad \frac{17}{7} \cong \frac{17}{-10} = \frac{-17}{10} \text{ and } \cong \frac{-17}{-6} \neq \frac{\text{4mn}-1}{\text{2n}}$$

$$7_6 : \quad \frac{19}{7} \cong \frac{19}{-12} \text{ and } \cong \frac{19}{-8} \neq \frac{\text{4mn}-1}{\text{2n}}$$

$$7_7 : \quad \frac{21}{8} = \frac{-21}{-8} \neq \frac{4mn-1}{2n}$$

It completes the proof of theorem. □

4. Conclusion

The most of works in the article were done in hand and reminder were help by "knotfinder" of `Knotscape`. However, there is a written program which produces DT-code of the knots which is the boundary of the flat plumbing basket surfaces [2]. Y. Chung, the third author and B. Lee are trying to modify it to make a new computer program which finds DT-notations of the given basket code and will be used to identify its corresponding knot using "knotfinder" of `Knotscape`.

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