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WEAK PROPERTY (β_k)

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Abstract. In this paper, we define the weak property (β_k) and get the following strict implications.

$$(UC) \Rightarrow w-(\beta_1) \Rightarrow w-(\beta_2) \Rightarrow \cdots \Rightarrow w-(\beta_\infty) \Rightarrow (BS).$$

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X, we denote the closed unit ball of X. Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively.

A Banach space is said to be reflexive if the natural embedding $\eta : X \to X^{**}$ is onto. Cui, Hudzik and Płuciennik introduced the notion of weak property (β) [2]. We say that a Banach space X has the weak property (β) if there is a number $\delta > 0$ such that for any $x \in B_X$ and any weakly null sequence (x_n) in B_X there exists $k \in \mathbb{N}$ such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

We note that non-reflexive Banach space l_1 has the weak property (β), since weak convergence is equivalent to norm convergence in l_1. With these notions, we can get the following definition.

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Definition 1. A Banach space X has the weak property (β_k) if it is reflexive and there exists $\delta > 0$ such that for any $x \in B_X$ and any weakly null sequence $(x_n) \in B_X$ there exist $n_i \in \mathbb{N},\ i = 1, 2, \cdots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(x + \sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

We say that X has the weak property (β_∞) if it has the weak property (β_k) for some $k \in \mathbb{N}$.

A Banach space X is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

A Banach space X is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. In similar way, we say that a Banach space X has weak Banach-Saks property (w-BS) if any weakly convergent sequence in the space admits a subsequence whose arithmetic means converges in norm. Since any weakly convergent sequence is norm bounded, it follows that Banach-Saks property implies weak Banach-Saks property. We note that weak Banach-Saks property and Banach-Saks property coincide in the reflexive Banach space.

S. Kakutani [3] showed that unform convexity implies Banach-Saks property. T. Nishiura and D. Waterman [4] proved that Banach-Saks property implies reflexivity in Banach spaces.

In the next section, we get the following strict implications.

$$(UC) \Rightarrow w - (\beta_1) \Rightarrow w - (\beta_2) \Rightarrow \cdots \Rightarrow w - (\beta_\infty) \Rightarrow (BS).$$

J.R. Partington mentioned the same implications in the spaces which satisfy property (A_k) without proofs [5]. We gave the detailed proofs in proving the implications of alternate signs (A_k) properties [1] and can get the above results in the weak property (β_k) by the similar techniques.

2. Main Parts

We begin with lemma.

Lemma 2.1. *Let $x_n,\ x \in X$. Suppose that (x_n) is weakly null and let α be a positive number such that $\|x\| > \alpha$. Then there exists a subsequence (x_{n_i}) of (x_n) such that $\|x - x_{n_i}\| \geq \alpha$ for all $i \in \mathbb{N}$.*

Proof. The proof is done by contradiction. Assume the assertion is false ; $\|x - x_n\| < \alpha$ except finite n. Then

$$\begin{aligned}\alpha < \|x\| &= \sup_{\|x^*\|=1} |x^*(x)| \\ &= \sup_{\|x^*\|=1} \lim_{n\to\infty} |x^*(x - x_n)| \\ &\leq \sup_{\|x^*\|=1} \limsup_{n} \|x^*\|\|x - x_n\| \\ &= \limsup_{n} \|x - x_n\| \leq \alpha.\end{aligned}$$

We get the contradiction. □

Using the above Lemma 2.1, we get the followings.

Proposition 2.2. *Let X be a Banach space.*

(1) *If X is uniformly convex, then it has the weak property (β_1).*
(2) *If X has the weak property (β_k), then it has the weak property (β_{k+1}).*
(3) *If X has the weak property (β_∞), then it has the Banach-Saks property.*

Proof. (1) Since X is uniformly convex, there exists $0 < \delta\left(\frac{1}{2}\right) < 1$ such that for all $x, y \in B_X$, if $\|x - y\| \geq \frac{1}{2}$, then

$$\frac{1}{2}\|x + y\| \leq 1 - \delta\left(\frac{1}{2}\right).$$

Let $\delta = \min\left\{\frac{1}{4}, \delta\left(\frac{1}{2}\right)\right\}$. Suppose that $x \in B_X$ and (x_n) is a weakly null sequence $(x_n) \in B_X$. If $\|x\| \leq \frac{1}{2}$, then $\left\|\frac{1}{2}(x + x_1)\right\| \leq \frac{3}{4} \leq 1 - \delta$.
If $\|x\| > \frac{1}{2}$, then by Lemma 2.1, there exists $n_1 \in \mathbb{N}$ such that $\|x - x_{n_1}\| \geq \frac{1}{2}$, since (x_n) is weakly null. Then $\|\frac{1}{2}(x + x_{n_1})\| \leq 1 - \delta\left(\frac{1}{2}\right) \leq 1 - \delta$. Since uniform convexity implies reflexivity, we get the result.

(2) Since X has the weak property (β_k), there exists $0 < \delta_0 < 1$ such that for any $x \in B_X$ and for any weakly null sequence $(x_n) \in B_X$ there exist $n_i \in \mathbb{N},\ i = 1, 2, \cdots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\|\frac{1}{k+1}\left(x + \sum_{i=1}^{k} x_{n_i}\right)\right\| \leq 1 - \delta_0.$$

Let $\delta = \frac{k+1}{k+2}\delta_0$. Suppose that $y \in B_X$ and (y_n) is a weakly null sequence in B_X. Then there exist $n_i \in \mathbb{N},\ i = 1, 2, \cdots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(y + \sum_{i=1}^{k} y_{n_i} \right) \right\| \leq 1 - \delta_0.$$

Let $y_{n_{k+1}} = y_{n_k+1}$. Then

$$\begin{aligned} \left\| \frac{1}{k+2} \left(y + \sum_{i=1}^{k+1} y_{n_i} \right) \right\| &\leq \frac{k+1}{k+2} \left\| \frac{1}{k+1} \left(y + \sum_{i=1}^{k} y_{n_i} \right) + \frac{1}{k+1} y_{n_{k+1}} \right\| \\ &\leq \frac{k+1}{k+2}(1-\delta_0) + \frac{1}{k+2} \\ &= 1 - \frac{k+1}{k+2}\delta_0 = 1 - \delta. \end{aligned}$$

Since X has the weak property (β_k), it is reflexive and we get the result.

(3) Suppose that X has the weak property (β_k). Then there exists $0 < \delta < 1$ such that for all weakly null sequence (x_n) in B_X and x in B_X, there exist $n_i \in \mathbb{N},\ i = 1, 2, \cdots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(x + \sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

Let (y_n) be a weakly null sequence in B_X and $n_1 = 1$. Then since $(y_n)_{n>n_1}$ is weakly null, there exist $n_i \in \mathbb{N}$,where $i = 2, 3, \cdots, k+1$ with $1 = n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(\sum_{i=1}^{k+1} y_{n_i} \right) \right\| \leq 1 - \delta.$$

Let $n_{k+2} = n_{k+1} + 1$. Then since $(y_n)_{n>n_{k+2}}$ is weakly null, there exist $n_i \in \mathbb{N}$, where $i = k+3, k+4, \cdots, 2k+2$ with $n_{k+2} < n_{k+3} < \cdots < n_{2k+2}$ such that

$$\left\| \frac{1}{k+1} \left(\sum_{i=k+2}^{2k+2} y_{n_i} \right) \right\| \leq 1 - \delta.$$

Continue this process, we obtain a subsequence (y_{n_i}) of (y_n) for which given any $j \in \mathbb{N}$,

$$\left\| \frac{1}{k+1} \left(\sum_{i=(j-1)k+j}^{jk+j} y_{n_i} \right) \right\| \leq 1 - \delta.$$

Now, using Kakutani's method [3], we conclude that that there exists a subsequence (y'_n) of (y_n) such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} y'_n \right\| \to 0 \qquad \text{as } n \to \infty.$$

This means that X has the weak Banach-Saks property. Since the weak Banach-Saks property is equivalent to the Banach-Saks property in a reflexive Banach space, we get the result. □

We now consider the converse of the above proposition.

Example 2. There exists a non-uniformly convex Banach space with the weak property (β_1). Consider $(\mathbb{R}^2, \|\cdot\|_\infty)$. Then it is easy to see that $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not uniformly convex. Since weak convergence is equivalent to norm convergence in finite dimensional Banach space, it has the weak property (β_1).

The following can be found in [5].

Example 3. For $x = (x_n) \in l_2$, we define a norm $\|x\|_{(k)}$ by

$$\|x\|_{(k)} = \left[\sup_{n_1 < n_2 < \cdots < n_k} \left(\sum_{i=1}^{k} |x_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2, \cdots, n_k} |x_n|^2 \right]^{\frac{1}{2}}.$$

Then $\|x\|_2 \leq \|x\|_{(k)} \leq \sqrt{k}\|x\|_2$. Let $X_k = \left(l_2, \|\cdot\|_{(k)}\right)$.

For a finite subset A and subset B of $\mathbb{N}$, we say $A < B$ if $\max A < \min B$. Let supp $x = \{n \in \mathbb{N} : x = \sum_{n=1}^{\infty} a_n e_n, a_n \neq 0\}$, where (e_n) is a Schauder basis in a Banach space X. For $x, y \in X$, we say $x < y$ only if supp $x <$ supp y. We need the following lemma [1].

Lemma 2.3. *Let X_k be the space defined by in Example 3. If $x_1, x_2, \cdots, x_k, x_{k+1} \in B_{X_k}$ with $x_1 < x_2 < \cdots < x_k < x_{k+1}$, then*

$$\left\| \sum_{i=1}^{k+1} x_i \right\|_{(k)} \leq \sqrt{k^2+1}.$$

By the above lemma, we get the following.

Proposition 2.4. *X_k satisfies the weak property (β_k) but does not satisfy the weak property (β_{k-1}), where $k \geq 2$.*

Proof. Since X_k is isomorphic to l_2, unit vector basis $(e_n)_{n\geq 1}$ is weakly null and so is $(e_n)_{n\geq 2}$. But

$$\left\| e_1 + \sum_{i=1}^{k-1} e_{n_i} \right\|_{(k)} = k,$$

for all choice of $n_i \in \mathbb{N}$ where $1 < n_1 < n_2 < \cdots, n_{k-1}$. This means that X_k does not have (β_{k-1}).

Let $\delta = 1 - \frac{\sqrt{k^2+2}}{k+1}$. Then $0 < \delta < 1$, since $k \geq 1$. Suppose that $(x_n) = \left(\sum_{i=1}^{\infty} a_i^{(n)} e_i \right)$ is a weakly null sequence in B_{X_k} and $x = \sum_{i=1}^{\infty} a_i e_i \in B_{X_k}$. Then there exists $p_0 \in \mathbb{N}$ such that

$$\left\| x - \sum_{i=1}^{p_0} a_i e_i \right\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right).$$

Let $u_0 = \sum_{i=1}^{p_0} a_i e_i$. Then $\|x - u_0\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right)$.

Since $x_n \to 0$ weakly, $a_i^{(n)} = e_i^*(x_n) \to 0$ as $n \to 0$ for each i. Then there exists $N_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{p_0} a_i^{(N_1)} e_i \right\|_{(k)} < \frac{1}{2(k+1)} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right).$$

For $x_{N_1} = \sum_{i=1}^{\infty} a_i^{(N_1)} e_i$, there exists $p_1 > p_0$ such that

$$\left\| \sum_{i=p_1+1}^{\infty} a_i^{(N_1)} e_i \right\|_{(k)} < \frac{1}{2(k+1)} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right).$$

Let $u_1 = \sum_{i=p_0+1}^{p_1} a_i^{(N_1)} e_i$. Then $\|u_1 - x_{N_1}\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right)$.

Since $x_n \to 0$ weakly, $a_i^{(n)} = e_i^*(x_n) \to 0$ as $n \to 0$ for each i. Then there exists $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and

$$\left\| \sum_{i=1}^{p_1} a_i^{(N_2)} e_i \right\|_{(k)} < \frac{1}{2(k+1)} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right).$$

For $x_{N_2} = \sum_{i=1}^{\infty} a_i^{(N_2)} e_i$, there exists $p_2 > p_1$ such that

$$\left\| \sum_{i=p_2+1}^{\infty} a_i^{(N_2)} e_i \right\|_{(k)} < \frac{1}{2(k+1)} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right).$$

Let $u_2 = \sum_{i=p_1+1}^{p_2} a_i^{(N_1)} e_i$. Then $\|u_2 - x_{N_2}\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right)$. Continue this process, we get a block sequence $(u_i)_{i \geq 0}$ such that

$$\|x - u_0\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right)$$

and

$$\|u_i - x_{N_i}\|_{(k)} < \frac{1}{k+1} \left(\sqrt{k^2+2} - \sqrt{k^2+1} \right)$$

where $i \geq 1$.

Since unit vector basis $(e_n)_{n \geq 1}$ in X_k is 1-unconditional, $\|u_i\|_{(k)} \leq 1$, for $i \in \mathbb{N} \cup \{0\}$.

We note that

$$\left\| \sum_{i=0}^{k} u_i \right\|_{(k)} \leq \sqrt{k^2+1},$$

by Lemma 2.3. Then we have

$$\begin{aligned} \left\| x + \sum_{j=1}^{k} x_{N_i} \right\|_{(k)} &\leq \|x - u_0\|_{(k)} + \sum_{i=1}^{k} \|x_{N_i} - u_i\|_{(k)} + \left\| \sum_{i=0}^{k} u_i \right\|_{(k)} \\ &\leq \sqrt{k^2+2}. \end{aligned}$$

Thus,

$$\frac{1}{k+1} \left\| x + \sum_{j=1}^{k} x_{N_i} \right\|_{(k)} \leq 1 - \delta.$$

This completes our proof. □

We finally consider the following.

PROPOSITION 2.5. *$\left(\prod_{s\geq 2} X_s\right)_{l_2}$ satisfies the Banach-Saks property but does not satisfy the weak property (β_∞).*

Proof. $\left(\prod_{s\geq 2} X_s\right)_{l_2}$ has the Banach-Saks property [5]. Let $k \in \mathbb{N}$. If $x^{(n)} = (0, 0, \cdots, 0, e_n, 0, \cdots)$ where usual unit vector e_n in k-th coordinate is only nonzero element of $x^{(n)}$, then $x^{(n)} \in S_{\left(\prod_{s\geq 2} X_s\right)_{l_2}}$. We note that $(x^{(n)})_{n\geq 2}$ is weakly null in $\left(\prod_{s\geq 2} X_s\right)_{l_2}$. But for any $n_i \geq 2,\ i = 1, 2, \cdots, k$,

$$\left\| x^{(1)} + \sum_{i=1}^{k} x^{(n_i)} \right\|_{\left(\prod_{s\geq 2} X_s\right)_{l_2}} = \left\| e_1 + \sum_{i=1}^{k} e_{n_i} \right\|_{(k+1)} = k+1$$

This means that $\left(\prod_{s\geq 2} X_s\right)_{l_2}$ has no the weak property (β_∞). □

By Proposition 2.2, Example 2, Proposition 2.4 and Proposition 2.5, we can get the following strict implication.

$$(UC) \Rightarrow w-(\beta_1) \Rightarrow w-(\beta_2) \Rightarrow \cdots \Rightarrow w-(\beta_\infty) \Rightarrow (BS).$$

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