Korean J. Math. **23** (2015), No. 3, pp. 379–391
http://dx.doi.org/10.11568/kjm.2015.23.3.379

NONLINEAR BIHARMONIC EQUATION WITH POLYNOMIAL GROWTH NONLINEAR TERM

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Abstract. We investigate the existence of solutions of the nonlinear biharmonic equation with variable coefficient polynomial growth nonlinear term and Dirichlet boundary condition. We get a theorem which shows that there exists a bounded solution and a large norm solution depending on the variable coefficient. We obtain this result by variational method, generalized mountain pass geometry and critical point theory.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. Choi and Jung [3] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$).

Received July 12, 2015. Revised September 1, 2015. Accepted September 2, 2015.
2010 Mathematics Subject Classification: 35J35, 35J40.
Key words and phrases: Biharmonic boundary value problem, polynomial growth, variational method, generalized mountain pass geometry, critical point theory, ($P.S.$) condition.

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∗ This work was financially supported by the Kunsan National University's research program for faculty member in the year 2014.

We obtained these results by using variational reduction method. Jung and Choi [5] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.1) has at least three nontrivial solutions by using degree theory. Tarantello [10] also studied

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1), \tag{1.2}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

She showed that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.4) has a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [8] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$ then (1.2) has at least four solutions by variational linking theorem and Leray-Schauder degree theory.

In this paper we consider the following nonlinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = a(x)g(u) \qquad \text{in } \Omega, \tag{1.3}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where we assume that $c \in R$ is not an eigenvalue of $-\Delta$ and that $a : \overline{\Omega} \to R$ is a continuous function which changes sign in Ω.

We assume that g satisfies the following conditions:
$(g1)$ $g \in C(R, R)$,
$(g2)$ there are constants a_1, $a_2 \geq 0$ such that

$$|g(u)| \leq a_1 + a_2|u|^{\mu-1},$$

where $2 < \mu < \frac{2n}{n-2}$ if $n \geq 3$.
$(g3)$ there exists a constant $r_0 \geq 0$ such that

$$0 < \mu G(\xi) = \mu \int_0^\xi g(t)dt \leq \xi g(\xi) \qquad \text{for } |\xi| \geq r_0.$$

$(g4)$ $g(u) = o(|u|)$ as $u \to 0$.
We note that $(g3)$ implies the existence of the positive constants a_3, a_4, a_5 such that

$$\frac{1}{\mu}(\xi g(\xi) + a_3) \geq G(\xi) + a_4 \geq a_5|\xi|^\mu \qquad \text{for } \xi \in R. \tag{1.4}$$

Khanfir and Lassoued [6] showed the existence of at least one solution for the nonlinear elliptic boundary problem when g is locally *Hölder* continuous on R_+.

We are trying to find the weak solutions of (1.3), that is,

$$\int_\Omega ((\Delta^2 u + c\Delta u - a(x)g(u))v dx = 0 \qquad \text{for } v \in H,$$

where the space H is introduced in section 2. Let us set

$$\Omega^+ = \{x \in \Omega | a(x) > 0\}, \qquad \Omega^- = \{x \in \Omega |\ a(x) < 0\}$$

and let

$$a^+ = a \cdot \chi_{\Omega^+}, a^- = -a \cdot \chi_{\Omega^-}.$$

Since $a(x)$ changes sign, the open subsets Ω^+ and Ω^- are nonempty. Now we can write $a = a^+ - a^-$. Our main results are as follows:

THEOREM **A** . *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies $(g1)$-$(g4)$ and $g(u)u - \mu G(u)$ is bounded. Then (1.3) has at least one bounded solution.*

THEOREM **B.** *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies $(g1)$-$(g4)$, $g(u)u - \mu G(u)$ is not bounded and there exists a small $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$. Then (1.3) has at least two solutions, (i) one of which is bounded and (ii) the other solution of which is large norm such that*

$$\max_{x\in\Omega}|u(x)| > M \qquad \textit{for some} \quad M > 0.$$

In Section 2, we prove that $I(u)$ is continuous and *Fréchet* differentiable and satisfies the (P.S.) condition. In Section 3, we prove Theorem **A** . In Section 4, we prove Theorem **B** by variational method, generalized mountain pass geometry and critical point theory.

2. Eigenspaces and Palais-Smale condition

The eigenvalue problem with Dirichlet boundary condition

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many eigenvalues λ_k, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$, the suitably normalized with respect to $L^2(\Omega)$ inner product, where each eigenvalue λ_k is repeated as often as its multiplicity. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \Lambda u \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has also infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that $\lambda_1(\lambda_1 - c) \leq \lambda_2(\lambda_2 - c) \leq \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \ \sum |\lambda_k(\lambda_k - c)| < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
which is proved in [2].
Let

$$H_+ = \{u \in H | \ h_k = 0 \text{ if } \ \lambda_k(\lambda_k - c) < 0\},$$

$$H_- = \{u \in H | \ h_k = 0 \text{ if } \ \lambda_k(\lambda_k - c) > 0\}.$$

Then $H = H_- \oplus H_+$, for $u \in H$, $u = u^- + u^+ \in H_- \oplus H_+$. Let P_+ be the orthogonal projection on H_+ and P_- be the orthogonal projection on H_-. We can wtite $P_+ u = u^+$, $P_- u = u^-$, for $u \in H$.

We are looking for the weak solutions of (1.1). The weak solutions of (1.1) coincide with the critical points of the associated functional

$$I(u) \in C^1(H, R),$$

$$\begin{aligned} I(u) &= \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2] dx - \int_\Omega a(x) G(u) dx \qquad (2.1) \\ &= \frac{1}{2}(\|P_+ u\|^2 - \|P_- u\|^2) - \int_\Omega a(x) G(u) dx. \end{aligned}$$

By $(g1)$ and $(g2)$, I is well defined. By the following Proposition 2.1, $I \in C^1(H, R)$ and I is *Fréchet* differentiable in H:

PROPOSITION 2.1. *Assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, and g satisfies $(g1)-(g4)$. Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - a(x)g(u)h]dx. \qquad (2.2)$$

If we set

$$K(u) = \int_\Omega a(x)G(u)dx,$$

then $K'(u)$ is continuous with respect to weak convergence, $K'(u)$ is compact, and

$$K'(u)h = \int_\Omega a(x)g(u)hdx \qquad \text{for all } h \in H.$$

This implies that $I \in C^1(H,R)$ and $K(u)$ is weakly continuous.

The proof of Proposition 2.1 has the same process as that of the proof in Appendix B in [9].

PROPOSITION 2.2. *(Palais-Smale condition)*
Assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, g satisfies $(g1)-(g4)$ and $f \in L^2(\Omega)$. We also assume that $g(u)u - \mu G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Then $I(u)$ satisfies the Palais-Smale condition.

Proof. We assume that $g(u)u - \mu G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Suppose that (u_m) is a sequence with $I(u_m) \leq M$ and $I'(u_m) \to 0$ as $m \to \infty$. Then by $(g2)$, $(g3)$, and *Hölder* inequality and Sobolev Embedding Theorem, for large m and $\mu > 2$ with

$u = u_m$, we have

$$\begin{aligned}
M + \frac{1}{2}\|u\| &\geq I(u) - \frac{1}{2}I'(u)u = \int_\Omega [\frac{1}{2}a(x)g(u)u - a(x)G(u)]dx \\
&= \int_\Omega a^+(x)[\frac{1}{2}g(u)u - G(u)]dx - \int_\Omega a^-(x)[\frac{1}{2}g(u)u - G(u)]dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x)\cdot G(u)dx \\
&\quad - \max_\Omega |\frac{1}{2}g(u)u - G(u)| \int_{\Omega^-} a^-(x)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x)\cdot (a_3|u|^\mu - a_4)\, dx \\
&\quad - \max_\Omega |\frac{1}{2}g(u)u - G(u)| \int_{\Omega^-} a^-(x)dx.
\end{aligned}$$

Thus if $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$, then we have

$$1 + \|u\| \geq M_1 \int_\Omega |u|^\mu \geq M_2 \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}\cdot\mu}. \tag{2.3}$$

Moreover since

$$|I'(u_m)\varphi| \leq \|\varphi\| \tag{2.4}$$

for large m and all $\varphi \in H$, choosing $\varphi = u_m^+ \in H_+$ gives

$$\begin{aligned}
\|u_m^+\|^2 &= \int_\Omega \left(\Delta^2 u_m + c\Delta u_m\right)\cdot u_m^+ \\
&= \int_\Omega a(x)g(u_m)u_m^+ \\
&\leq \int_\Omega |a(x)||g(u_m)||u_m| \\
&\leq \|a\|_\infty \int_\Omega (a_1|u_m|^\mu + a_2|u_m|) \\
&\leq C_1 \int_\Omega |u_m|^\mu + C_2\|u_m\|_{L^2(\Omega)} \\
&\leq C_1 \int_\Omega |u_m|^\mu + C_2'\|u_m\|.
\end{aligned}$$

Taking $\varphi = -u_m^-$ in (2.4) yields

$$\begin{aligned}
\|u_m^-\|^2 &= \int_\Omega \left(\Delta^2 u_m + c\Delta u_m\right)\cdot(-u_m^-)\\
&= \int_\Omega a(x)g(u_m)\cdot(-u_m^-)\\
&\le \int_\Omega |a(x)||g(u_m)||u_m|\\
&\le \|a\|_\infty \int_\Omega (a_1|u_m|^\mu + a_2|u_m|)\\
&\le C_3\int_\Omega |u_m|^\mu + C_4\|u_m\|_{L^2(\Omega)}\\
&\le C_3\int_\Omega |u_m|^\mu + C_4'\|u_m\|.
\end{aligned}$$

Thus, by (2.3), we have

$$\begin{aligned}
\|u_m\|^2 = \|u_m^+\|^2 + \|u_m^-\|^2 &\le M_3\int_\Omega |u_m|^\mu + M_4\|u_m\|\\
&\le M_5\left(1+\|u_m\|\right) + M_4\|u_m\| \le M_6\left(1+\|u_m\|\right),
\end{aligned}$$

from which the boundedness of (u_m) follows. Thus (u_m) converges weakly in H. Since $P_\pm I'(u_m) = \pm P_\pm u_m + P_\pm\tilde{\mathcal{P}}(u_m)$ with $\tilde{\mathcal{P}}$ compact and the weak convergence of $P_\pm u_m$ imply the strong convergence of $P_\pm u_m$ and hence (PS) condition holds. □

3. At least one bounded solution

We shall show that $I(u)$ satisfies generalized mountain pass geometrical assumptions.
We recall generalized mountain pass geometry:
Let $H = V \oplus X$, where H is a real Banach space and $V \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(H,R)$, satisfies $(P.S.)$ condiion, and
(i) there are constants ρ, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho\cap X} \ge \alpha$,
(ii) there is an $e \in \partial B_1\cap X$ and $R > \rho$ such that if $Q = (\bar{B_R}\cap V)\oplus\{re|\, 0 < r < R\}$, then $I|_{\partial Q} \le 0$.

Then I possesses a critical value $b \geq \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) |\ \gamma = id \text{ on } \ \partial Q\}.$$

Let $H_k = \text{span}\{\phi_1, \ldots, \phi_k\}$. Then H_k is a subspace of H such that

$$H = \oplus_{k \in N} H_k \qquad \text{and} \quad H = H_k \oplus H_k^{\perp}.$$

Let

$$B_r = \{u \in H |\ \|u\| \leq r\},$$

$$Q = (\bar{B}_R \cap H_k) \oplus \{re |\ 0 < r < R\}.$$

We have the following generalized mountain pass geometrical assumptions:

Lemma 3.1. *Assume that $\lambda_k < c < \lambda_{k+1}$ and g satisfies $(g1)-(g4)$. Then*
(i) there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_k^{\perp}} \geq \alpha$, and
(ii) there is an $e \in \partial B_1 \cap H_k^{\perp}$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap H_k) \oplus \{re |\ 0 < r < R\}$, then $I|_{\partial Q} \leq 0$, and
(iii) there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$.

Proof. (i) Let $u \in H_k^{\perp}$. We note that

$$\text{if } \ u \in H_k^{\perp}, \int_\Omega (\Delta^2 u + c\Delta u) u dx \geq \lambda_{k+1}(\lambda_{k+1} - c)\|u\|^2_{L^2(\Omega)} > 0.$$

Thus by $(g3)$, (1.2) and the *Hölder* inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \int_\Omega a(x) G(u) \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty \int_\Omega C_1 |u|^\mu \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty C_1' \|u\|^\mu \end{aligned}$$

for $C_1, C_1' > 0$. Since $\mu > 2$, there exist $\rho > 0$ and $\alpha > 0$ such that if $u \in \partial B_\rho$, then $I(u) \geq \alpha$.

(ii) Let $u \in (\bar{B}_r \cap H_k) \oplus \{re|\ 0 < r\}$. Then $u = v + w$, $v \in B_r \cap H_k$, $w = re$. We note that

$$\text{if } \ v \in H_k, \int_\Omega (\Delta^2 v + c\Delta v) v dx \leq \lambda_k(\lambda_k - c)\|v\|^2_{L^2(\Omega)} < 0.$$

Thus we have

$$\begin{aligned} I(u) &= \frac{1}{2}r^2 - \frac{1}{2}\|P_- v\|^2 - \int_\Omega a(x)G(v + re) \\ &\leq \frac{1}{2}r^2 + \frac{1}{2}(\lambda_k(\lambda_k - c))\|v\|^2_{L^2(\Omega)} - \int_{\Omega^+} a(x)(a_5|v + re|^\mu - a_4) \end{aligned}$$

Since $\mu > 2$, there exists $R > 0$ such that if $u \in Q = (\bar{B}_R \cap H_k) \oplus \{re|\ 0 < r < R\}$, then $I(u) < 0$

(iii) If we choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\text{supp}(\psi) \subset \Omega^+$, then we have

$$\begin{aligned} I(t\psi) &\leq \frac{1}{2}\|P_+(t\psi)\|^2 - \frac{1}{2}\|P_-(t\psi)\|^2 - \int_{\Omega^+} a(x)\,(a_3 t^\mu \psi^\mu - a_4) \\ &\leq \frac{1}{2}\|t\psi\|^2 - \int_{\Omega^+} a(x)\,(a_3 t^\mu \psi^\mu - a_4) \\ &= \frac{1}{2}t^2 - \int_{\Omega^+} a(x)\,(a_3 t^\mu \psi^\mu - a_4) \end{aligned}$$

for all $t > 0$. Since $\mu > 2$, for t_0 great enough, $u_0 = t_0\psi$ is such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. □

THEOREM **A** . *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies (g1)-(g4) and $g(u)u - \mu G(u)$ is bounded. Then (1.3) has at least one bounded solution.*

Proof. By Proposition 2.1 and Proposition 2.2, $I(u) \in C^1(H,\mathrm{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1, there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_m^\perp} \geq \alpha$, and there is an $e \in \partial B_1 \cap H_k^\perp$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap H_k) \oplus \{re|\ 0 < r < R\}$, then $I|_{\partial Q} \leq 0$, and there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. By the generalized mountain pass theorem, $I(u)$ has a critical value $b \geq \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) |\ \gamma = id \text{ on } \ \partial Q\}.$$

We denote by $\tilde{u}$ a critical point of I such that $I(\tilde{u}) = b$. We claim that there exists a constant $C > 0$ such that

$$\|a^+(x)^{\frac{1}{\mu}}\tilde{u}\|_{L^2(\Omega)} \le C\left(1 + L\int_{\Omega^-} a^-(x)dx\right)^{\frac{1}{\mu}},$$

where $L = \max_\Omega |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$.
In fact, we have

$$b \le \max I(tu_0), \qquad 0 \le t \le 1,$$

and

$$\begin{aligned} I(tu_0) &= t^2\left(\frac{1}{2}\|P_+u_0\|^2 - \frac{1}{2}\|P_-u_0\|^2\right) - \int_\Omega a(x)G(tu_0)dx \\ &\le t^2\|u_0\|^2 - \int_\Omega a^+(x)G(tu_0)dx + \int_\Omega a^-(x)G(tu_0)dx \\ &\le t^2\|u_0\|^2 - a_3t^\mu\int_\Omega a^+(x)u_0^\mu + a_4\int_\Omega a^+(x) + a_5t^\mu\int_\Omega a^-(x)u_0^\mu \\ &= Ct^2 - Ct^\mu + C + C't^\mu. \end{aligned}$$

Since $0 \le t \le 1$, b is bounded: $b < \tilde{C}$.
We can write

$$\begin{aligned} b &= I(\tilde{u}) - \frac{1}{2}I'(\tilde{u})\tilde{u} \\ &= \int_\Omega a(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \\ &= \int_\Omega a^+(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx - \int_\Omega a^-(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega a^+(x)g(\tilde{u})\tilde{u} - \max_\Omega|\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|\int_{\Omega^-} a^-(x)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu\int_\Omega a^+(x)\left(a_3|\tilde{u}|^\mu - a_4\right) - L\int_{\Omega^-} a^-(x)dx, \end{aligned}$$

where $L = \max_\Omega |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$. Thus we have

$$\begin{aligned} C\left(1 + L\int_{\Omega^-} a^-(x)dx\right) &\ge \int_\Omega a^+(x)|\tilde{u}|^\mu \\ &\ge \left[\int_\Omega \left(a^+(x)^{\frac{1}{\mu}}|\tilde{u}|\right)^2\right]^{\frac{\mu}{2}}, \end{aligned} \tag{3.1}$$

from which we can conclude that $\tilde{u}$ is bounded. In fact, suppose that $\tilde{u}$ is not bounded. Then for any $R > 0$, $|\tilde{u}| \geq R$. Thus we have

$$\int_{\Omega} a^+(x)|\tilde{u}|^{\mu} \geq R^{\mu} \int_{\Omega} a^+(x)dx$$

for any R, which contradicts to the fact (3.1) and the proof of theorem is complete. □

4. At least two solutions

THEOREM **B.** *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies $(g1)$-$(g4)$, $g(u)u - \mu G(u)$ is not bounded and there exists a small $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$. Then (1.3) has at least two solutions, (i) one of which is bounded and (ii) the other solution of which is large norm such that*

$$\max_{x\in\Omega}|u(x)| > M \qquad \text{for some} \quad M > 0.$$

Proof. Assume that $\frac{1}{2}g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H,\mathrm{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1 and generalized mountain pass theorem, $I(u)$ has a critical value b with critical point $\tilde{u}$ such that $I(\tilde{u}) = b$. If $\int_{\Omega^-} a^-(x)dx$ is sufficiently small, by (3.1), we have

$$\int_{\Omega} a^+(x)|\tilde{u}|^{\mu} \leq C$$

for $C > 0$, from which we can conclude that $\tilde{u}$ is bounded and the proof of (i) is complete. □

Next we shall prove (ii). We may assume that $R_n < R_{n+1}$ for all $n \in N$. Let us set $D_n = B_{R_n} \cap H_n$, $\partial D_n = \partial B_{R_n} \cap H_n$.

LEMMA 4.1. *Assume that g satisfies $(g1)$-$(g4)$. Then there exists an $R_n > 0$ such that*

$$I(u) \leq 0 \qquad \text{for} \;\; u \in H_n \backslash B_{R_n}, \tag{4.1}$$

where $B_{R_n} = \{u \in H| \; \|u\| \leq R_n\}$.

Proof. Let us choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\text{supp}(\psi) \subset \Omega^+$. Then, by $(g3)$, (1.2) and the *Hölder* inequality, we have

$$\begin{aligned} I(t\psi) &= \frac{1}{2}\|P_+ t\psi\|^2 - \frac{1}{2}\|P_- t\psi\|^2 - \int_\Omega a(x)G(t\psi) \\ &\leq \frac{1}{2}t^2 - \|a\|_\infty \int_\Omega C_1 t^\mu \psi^\mu + \|a\|_\infty a_1 t \\ &\leq \frac{1}{2}t^2 - t^\mu \|a\|_\infty C_1' \psi^\mu + \|a\|_\infty a_1 t \end{aligned}$$

for $C_1, C_1' > 0$. Since $\mu > 2$, there exist t_n great enough for each n and an $R_n > 0$ such that $u_n = t_n\psi$ and $I(u_n) < 0$ if $u_n \in H_n \backslash B_{R_n}$ and $\|u_n\| > R_n$, so the lemma is proved □

Let us set

$$\Gamma_n = \{\gamma \in C([0,1], H) | \ \gamma(0) = 0 \text{ and } \ \gamma(1) = u_n\}$$

and

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{[0,1]} I(\gamma(u)) \qquad n \in N.$$

Proof of Theorem **B** (ii).

We assume that $g(u)u - \mu G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H, R)$ and satisfies the Palais-Smale condition. By Lemma 4.1,there exists an $R_n > 0$ such that $I(u_m) \leq 0$ for $u_n \in H_n \backslash B_{R_n}$. We note that $I(0) = 0$. By Lemma 4.1 and the generalized mountain pass theorem, for n large enough $b_n > 0$ is a critical value of I and $\lim_{n\to\infty} b_n = +\infty$. Let $\tilde{u_n}$ be a critical point of I such that $I(\tilde{u_n}) = b_n$. Then for each real number M, $\max_\Omega |\tilde{u_n}(x)| \geq M$. In fact, by contradiction , $\Delta^2 u + c\Delta u = a(x)g(u)$ and $\max_\Omega |\tilde{u_n}(x)| \leq K$ imply that

$$I(\tilde{u_n}) \leq \max_{|\tilde{u_n}| \leq K} (\frac{1}{2} g(\tilde{u_n})\tilde{u_n} - G(\tilde{u_n})) \int_\Omega |a(x)|,$$

which means that b_n is bounded. This is absurd to the fact that $\lim_{n\to\infty} b_n = +\infty$. Thus we complete the proof.

References

[1] Chang, K. C., *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser, (1993).

[2] Choi, Q. H. and Jung, T., *Multiplicity of solutions and source terms in a fourth order nonlinear elliptic equation,* Acta Mathematica Scientia **19** (4) (1999), 361–374.

[3] Choi, Q. H. and Jung, T., *Multiplicity results on nonlinear biharmonic operator,* Rocky Mountain J. Math. **29** (1) (1999), 141–164.

[4] Choi, Q. H. and Jung, T., *An application of a variational reduction method to a nonlinear wave equation,* J. Differential Equations **7** (1995), 390–410.

[5] Jung, T. S. and Choi, Q. H., *Multiplicity results on a nonlinear biharmonic equation*, Nonlinear Analysis, Theory, Methods and Applications, **30** (8) (1997), 5083–5092.

[6] Khanfir, S. and Lassoued, L., *On the existence of positive solutions of a semilinear elliptic equation with change of sign,* Nonlinear Analysis, TMA, **22**, No.11, 1309-1314(1994).

[7] Lazer, A. C. and Mckenna, J. P., *Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems,* J. Math. Anal. Appl. **107** (1985), 371–395.

[8] Micheletti, A. M. and Pistoia, A., *Multiplicity results for a fourth-order semilinear elliptic problem,* Nonlinear Analysis, TMA, **31** (7) (1998), 895–908.

[9] Rabinowitz, P. H., *Minimax methods in critical point theory with applications to differential equations,* CBMS. Regional conf. Ser. Math. **65**, Amer. Math. Soc., Providence, Rhode Island (1986).

[10] Tarantello, *A note on a semilinear elliptic problem,* Diff. Integ.Equat., **5** (3) (1992), 561–565.

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