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APPROXIMATE ADDITIVE MAPPINGS IN 2-BANACH SPACES AND RELATED TOPICS: REVISITED

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Abstract. W. Park [J. Math. Anal. Appl. **376** (2011) 193–202] proved the Hyers-Ulam stability of the Cauchy functional equation, the Jensen functional equation and the quadratic functional equation in 2-Banach spaces. But there are serious problems in the control functions given in all theorems of the paper.

In this paper, we correct the statements of these results and prove the corrected theorems.

Moreover, we prove the superstability of the Cauchy functional equation, the Jensen functional equation and the quadratic functional equation in 2-Banach spaces under the original given conditions.

1. Introduction and preliminaries

In 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta = \delta(\varepsilon) > 0$ *such that if a mapping* $f : \mathcal{G} \to \mathcal{H}$ *satisfies the inequality*

$$d(f(x \circ y), f(x) \star f(y)) < \delta$$

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for all $x, y \in \mathcal{G}$, then a homomorphism $F : \mathcal{G} \to \mathcal{H}$ exits with

$$d(f(x), F(x)) < \varepsilon$$

for all $x \in \mathcal{G}$?

In 1941, Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces. Thereafter, we call that type the Hyers-Ulam stability.

In the 1960's, Gähler [3,4] introduced the concept of linear 2-normed spaces.

Definition 1.1. Let $\mathcal{X}$ be a real linear space with $\dim \mathcal{X} > 1$ and let $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a function satisfying the following properties:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$
for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a *2-norm* on $\mathcal{X}$ and the pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a *linear 2-normed space.* Sometimes the condition (d) called the *triangle inequality.*

We introduce a basic property of linear 2-normed spaces.

Lemma 1.2. ([5]) *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $x \in \mathcal{X}$ and $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*

In the 1960's, Gähler and White [6–8] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if

$$\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$$

for all $y \in \mathcal{X}$.

Definition 1.4. *A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a convergent sequence if there is an $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Triangle inequality implies the following lemma.

LEMMA 1.5. ([5]) *For a convergent sequence* $\{x_n\}$ *in a linear* 2*-normed space* $\mathcal{X}$,

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for all $y \in \mathcal{X}$.

DEFINITION 1.6. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space*.

DEFINITION 1.7. *A* 2*-Banach space* X *is called a normed* 2*-Banach space if* X *is a normed space with norm* $\|\cdot\|$.

REMARK 1.8. The left sides and the right sides of the inequalities (2.1), (2.4), (3.1), (3.8), (4.1) and (4.4) in [5] are not well-defined, since the left elements of the left sides are elements of $\mathcal{Y}$ and the right elements of the left sides are elements of $\mathcal{X}$, and the real numbers $\|z\|^r$ of the right sides are not defined in $\mathcal{Y}$, which is not a normed space. So to define the right sides, the space $\mathcal{Y}$ must be a normed space.

In this paper, we correct the statements of the results given in [5] and prove the corrected theorems. We moreover prove the superstability of the Cauchy functional equation, the Jensen functional equation and the quadratic functional equation in 2-Banach spaces under the original given conditions.

Throughout this paper, let $\mathcal{X}$ be a normed linear space and let $\mathcal{Y}$ be a normed 2-Banach space.

2. Approximate additive mappings

In this section, we prove the Hyers-Ulam stability of the Cauchy functional equation in 2-Banach spaces.

THEOREM 2.1. *Let* $\theta \in [0,\infty)$, $p, q \in (0,\infty)$ *with* $p+q<1$ *and let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \le \theta \|x\|^p \|y\|^q \|z\|$$

for all $x, y \in \mathcal{X}$ *and all* $z \in \mathcal{Y}$. *Then there is a unique additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - A(x), z\| \le \frac{1}{2 - 2^{p+q}} \theta \|x\|^{p+q} \|z\|$$

for all $x \in \mathcal{X}$ *and all* $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 2.1] for the case $r = 1$. □

THEOREM 2.2. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q > 1$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x), z\| \leq \frac{1}{2^{p+q} - 2} \theta \|x\|^{p+q} \|z\|$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 2.2] for the case $r = 1$. □

Now we prove the superstability of the Cauchy functional equation in 2-Banach spaces.

THEOREM 2.3. *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ with $r \neq 1$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \tag{2.1}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then $f : \mathcal{X} \to \mathcal{Y}$ is an additive mapping.

Proof. Replacing z by sz in (2.1) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\|f(x+y) - f(x) - f(y), sz\| \leq \theta \|x\|^p \|y\|^q \|sz\|^r$$

and so

$$\|f(x+y) - f(x) - f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \frac{|s|^r}{|s|} \tag{2.2}$$

for all $x, y \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all $s \in \mathbb{R} \setminus \{0\}$.

If $r > 1$, then the right side of (2.2) tends to zero as $s \to 0$.

If $r < 1$, then the right side of (2.2) tends to zero as $s \to +\infty$.

Thus

$$\|f(x+y) - f(x) - f(y), z\| = 0$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By Lemma 1.2, $f(x+y) - f(x) - f(y) = 0$ for all $x, y \in \mathcal{X}$, i.e., $f : \mathcal{X} \to \mathcal{Y}$ is additive. □

3. Approximate Jensen mappings

In this section, we prove the Hyers-Ulam stability of the Jensen functional equation in 2-Banach spaces.

THEOREM 3.1. *Let $\theta \in [0,\infty)$, $p,q \in (0,\infty)$ with $p+q<1$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0)=0$ and*

$$\left\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), z\right\| \le \theta\|x\|^p\|y\|^q\|z\|$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - J(x), z\| \le \frac{1+3^q}{3-3^{p+q}}\theta\|x\|^{p+q}\|z\|$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 3.1] for the case $r = 1$. □

THEOREM 3.2. *Let $\theta \in [0,\infty)$, $p,q \in (0,\infty)$ with $p+q>1$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0)=0$ and*

$$\left\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), z\right\| \le \theta\|x\|^p\|y\|^q\|z\|$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x), z\| \le \frac{1+3^q}{3^{p+q}-3}\theta\|x\|^{p+q}\|z\|$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 3.2] for the case $r = 1$. □

Now we prove the superstability of the Jensen functional equation in 2-Banach spaces.

THEOREM 3.3. *Let $\theta \in [0,\infty)$, $p,q,r \in (0,\infty)$ with $r \ne 1$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0)=0$ and*

$$\left\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), z\right\| \le \theta\|x\|^p\|y\|^q\|z\|^r \tag{3.1}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then $f : \mathcal{X} \to \mathcal{Y}$ is an additive mapping.

Proof. Replacing z by sz in (3.1) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\left\| 2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), sz \right\| \leq \theta \|x\|^p \|y\|^q \|sz\|^r$$

and so

$$\left\| 2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), z \right\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \frac{|s|^r}{|s|}$$

for all $x, y \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all $s \in \mathbb{R} \setminus \{0\}$.

The rest of the proof is similar to the proof of Theorem 2.3. □

4. Approximate quadratic mappings

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation in 2-Banach spaces.

THEOREM 4.1. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q < 2$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x), z\| \leq \frac{1}{4 - 2^{p+q}} \theta \|x\|^{p+q} \|z\|$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 4.1] for the case $r = 1$. □

THEOREM 4.2. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q > 2$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Then there is a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x), z\| \leq \frac{1}{2^{p+q} - 4} \theta \|x\|^{p+q} \|z\|$$

for all $x \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. The proof is the same as in the proof of [5, Theorem 4.2] for the case $r = 1$. □

Now we prove the superstability of the quadratic functional equation in 2-Banach spaces.

THEOREM 4.3. *Let* $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ *with* $r \neq 1$ *and let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \tag{4.1}$$

for all $x, y \in \mathcal{X}$ *and all* $z \in \mathcal{Y}$. *Then* $f : \mathcal{X} \to \mathcal{Y}$ *is a quadratic mapping.*

Proof. Replacing z by sz in (4.1) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), sz\| \leq \theta \|x\|^p \|y\|^q \|sz\|^r$$

and so

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y), z\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \frac{|s|^r}{|s|}$$

for all $x, y \in \mathcal{X}$, all $z \in \mathcal{Y}$ and all $s \in \mathbb{R} \setminus \{0\}$.

The rest of the proof is similar to the proof of Theorem 2.3. $\square$

References

[1] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.

[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941) 222–224.

[3] S. Gähler, 2-*metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963) 115–148.

[4] S. Gähler, *Lineare* 2-*normierte Räumen*, Math. Nachr. **28** (1964) 1–43.

[5] W.-G. Park, *Approximate additive mappings in 2-Banach spaces and related topics*, J. Math. Anal. Appl. **376** (2011) 193–202.

[6] S. Gähler, *Über* 2-*Banach-Räume*, Math. Nachr. **42** (1969) 335–347.

[7] A. White, 2-*Banach spaces*, Doctorial Diss., St. Louis Univ., 1968.

[8] A. White, 2-*Banach spaces*, Math. Nachr. **42** (1969) 43–60.

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