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LIPSCHITZ AND ASYMPTOTIC STABILITY OF NONLINEAR SYSTEMS OF PERTURBED DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate Lipschitz and asymptotic stability for perturbed nonlinear differential systems.

1. Introduction

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [9]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asmptotic stability in variation of Brauer[2,4] and uniformly stability in variation of Brauer and Strauss[3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. Furthermore, uniform Lipshitz stability neither implies asymptotic stability nor is it implied by it. Also, Elaydi and Farran[10] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte[15] investigated the stability and

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asymptotic behavior of solutions of the functional differential equation. Gonzalez and Pinto[11] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al.[6,7,8] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo et al.[5,12,13] investigated Lipschitz and asymptotic stability for perturbed differential systems.

In this paper we will obtain some results on ULS and EAS for nonlinear perturbed differential systems. We will employ the theory of integral inequalities to study Lipschitz and asymptotic stability for solutions of the nonlinear differential systems. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

2. Preliminaries

We consider the nonautonomous nonlinear differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds, \; y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel[8].

DEFINITION 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(S)*stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,
(US)*uniformly stable* if the δ in (S) is independent of the time t_0,
(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0) \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t,$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K\,e^{-c(t-t_0)}, 0 \leq t_0 \leq t,$$

provided that $|x_0| < \infty$.

REMARK 2.2. [11] The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t.$$

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \; y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.3. *Let $x(t,t_0,y_0)$ and $y(t,t_0,y_0)$ be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t,t_0,y_0) \in \mathbb{R}^n$,*

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,s,y(s))\, g(s,y(s))\, ds.$$

LEMMA 2.4. [12] *Let $u,p,q,w \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau) w(u(\tau)) d\tau ds,\ t \geq t_0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau) d\tau ds\Big],\ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $u \geq u_0 \geq 0$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau) d\tau ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 2.5. [8]*(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ \ 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],\ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.4 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 2.6. [5] *Let $u, \lambda_1, \lambda_2, \lambda_3, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $u \leq w(u)$. If , for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) w(u(\tau)) d\tau ds,\ \ t \geq t_0 \geq 0,$$

then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\lambda_3(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}\lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 2.7. [5] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \le \quad c \quad &+ \int_{t_0}^{t}\lambda_1(s)u(s)ds + \int_{t_0}^{t}\lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^{t}\lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^{t}\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)w(u(\tau))d\tau ds,\ \ 0 \le t_0 \le t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau \\ +\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds\Big], \end{aligned}$$

$t_0 \le t < b_1$, *where* W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau \\ +\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

We obtain the following corollary from Lemma 2.7.

COROLLARY 2.8. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$

and $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)u(s)ds$$
$$+ \int_{t_0}^{t} \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds\Big],$$

$t_0 \le t < b_1$, *where* W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau)d\tau$$
$$+\lambda_4(s) \textstyle\int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

3. Main Results

In this section, we investigate Lipschitz and asymptotic stability for solutions of the nonlinear perturbed differential systems.

THEOREM 3.1. *For the perturbed (2.2), we assume that*

$$|g(t, y)| \le a(t)w(|y(t)|),$$

where $a \in C(\mathbb{R}^+)$, $a, w \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, $w(u)$ *is nondecreasing in* u, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \int_{t_0}^{s} a(\tau)d\tau ds\Big], \tag{3.1}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS([9],Theorem 3.3). Using the nonlinear variation of constants formula and the ULSV condition of $x = 0$ of (2.1), we obtain

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds$$
$$\le M|y_0| + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} a(\tau) w(\frac{|y(\tau)|}{|y_0|}) d\tau ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Lemma 2.4 yields

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M \int_{t_0}^{t} \int_{t_0}^{s} a(\tau) d\tau ds\Big].$$

Thus, by (3.1), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Letting $w(y(t)) = y(t)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *For the perturbed (2.2), we assume that*

$$|g(t,y)| \le a(t)|y(t)|,$$

where $a \in C(\mathbb{R}^+)$ *and* $a \in L_1(\mathbb{R}^+)$,

$$M(t_0) = \exp(M \int_{t_0}^{\infty} \int_{t_0}^{s} a(\tau) d\tau ds),$$

where $M(t_0) < \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Theorem 3.3. *For the perturbed (2.2), we assume that*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le a(t) w(|y(t)|),$$

where $a \in C(\mathbb{R}^+)$, $a, w \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *is nondecreasing in* u, *and* $\frac{1}{v} w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} a(s) ds\Big], \tag{3.2}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Applying Lemma 2.3, we obtain

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0|a(s)w(\frac{|y(s)|}{|y_0|})ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.5 yields

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t} a(s)ds\Big].$$

Hence, by (3.2), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Letting $w(y(t)) = y(t)$ in Theorem 3.3, we obtain the following corollary.

COROLLARY 3.4. *For the perturbed (2.2), we assume that*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)|y(t)|,$$

where $a \in C(\mathbb{R}^+)$ *and* $a \in L_1(\mathbb{R}^+)$,

$$M(t_0) = \exp(\int_{t_0}^{\infty} Ma(s)ds),$$

where $M(t_0) < \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

THEOREM 3.5. *Let the solution* $x = 0$ *of (2.1) be EASV. Suppose that the perturbing term* $g(t, y)$ *satisfies*

$$|g(t, y(t))| \le e^{-\alpha t}a(t)w(|y(t)|), \tag{3.3}$$

where $\alpha > 0$, $a, w \in C(\mathbb{R}^+)$, $a, w \in L_1(\mathbb{R}^+)$, *and* $w(u)$ *is nondecreasing in u. If*

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty} e^{\alpha s}\int_{t_0}^{s} a(\tau)d\tau ds\Big] < \infty,\ t \ge t_0, \tag{3.4}$$

where $c = |y_0|Me^{\alpha t_0}$, *then all solutions of (2.2) approch zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, by remark 2.2, it is EVS. Using Lemma 2.3 and (3.3), we obtain

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} e^{-\alpha\tau} a(\tau)w(|y(\tau)|)d\tau ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} a(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau ds.\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, an application of Lemma 2.4 and (3.4) obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t} e^{\alpha s}\int_{t_0}^{s} a(\tau)d\tau ds\Big] \le ce^{-\alpha t}M(t_0),\ t \ge t_0,$$

where $c = |y_0|Me^{\alpha t_0}$. Therefore, all solutions of (2.2) approch zero as $t \to \infty$. $\square$

Letting $w(y(t)) = y(t)$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$|g(t, y(t))| \le e^{-\alpha t}a(t)|y(t)|,$$

where $\alpha > 0$, $a \in C(\mathbb{R}^+)$, and $a \in L_1(\mathbb{R}^+)$. If

$$M(t_0) = \exp\int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s} a(\tau)d\tau ds < \infty,\ t \ge t_0,$$

where $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$.

Theorem 3.7. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le e^{-\alpha t}a(t)w(|y(t)|), \tag{3.5}$$

where $\alpha > 0$, $a, w \in C(\mathbb{R}^+)$, $a, w \in L_1(\mathbb{R}^+)$, and $w(u)$ is nondecreasing in u. If

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty} a(s)ds\Big] < \infty, b_1 = \infty,$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, by remark 2.2, it is EVS. Using Lemma 2.3 and (3.5), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \frac{a(s)}{e^{\alpha s}} w(|y(s)|)ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t} a(s)w(|y(s)|e^{\alpha s})ds.\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, an application of Lemma 2.5 obtains

$$|y(t)| \le e^{-\alpha t} W^{-1}\Big[W(c) + M \int_{t_0}^{t} a(s)ds\Big],$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Letting $w(y(t)) = y(t)$ in Theorem 3.7, we obtain the following corollary.

Corollary 3.8. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le e^{-\alpha t} a(t)|y(t)|,$$

where $\alpha > 0$, $a \in C(\mathbb{R}^+)$, and $a \in L_1(\mathbb{R}^+)$. If

$$M(t_0) = \exp(\int_{t_0}^{\infty} Ma(s)ds) < \infty,$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$.

Let us consider the functional differential system

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{3.6}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

We need the lemma to prove the following theorem.

LEMMA 3.9. *Let* $k, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, $u \le w(u)$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c \ge 0$,

(3.7)

$$u(t)\le c+\int_{t_0}^t \lambda_1(s)\Big[\int_{t_0}^s [\lambda_2(\tau)u(\tau)+\lambda_3(\tau)\int_{t_0}^\tau k(r)w(u(r))dr]d\tau+\lambda_4(s)u(s)\Big]ds,$$

for $t \ge t_0 \ge 0$ *and for some* $c \ge 0$. *Then*

(3.8)

$$u(t) \le W^{-1}\Big[W(c)+\int_{t_0}^t \lambda_1(s)(\int_{t_0}^s (\lambda_2(\tau)+\lambda_3(\tau)\int_{t_0}^\tau k(r)dr)d\tau+\lambda_4(s))ds\Big],$$

for $t_0 \le t < b_1$, *where* W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \ge t_0 : \ W(c) + \textstyle\int_{t_0}^t \lambda_1(s)(\int_{t_0}^s(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^\tau k(r)dr)d\tau + \lambda_4(s))ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Define a function $v(t)$ by the right member of (3.7). Then

$$v'(t) = \lambda_1(t)\Big[\int_{t_0}^t (\lambda_2(s)u(s) + \lambda_3(s)\int_{t_0}^s k(\tau)w(u(\tau))d\tau)ds + \lambda_4(t)u(t)\Big],$$

which implies

$$v'(t) \le \lambda_1(t)\Big[\int_{t_0}^t (\lambda_2(s) + \lambda_3(s)\int_{t_0}^s k(\tau)d\tau)ds + \lambda_4(t)\Big]w(v(t)),$$

since v and w are nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

(3.9)

$$v(t) \le c + \int_{t_0}^t \lambda_1(s)\Big[\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^\tau k(r)dr)d\tau + \lambda_4(s)\Big]w(v(s))ds.$$

Then, by the well-known Bihari-type inequality, (3.9) yields the estimate (3.8). □

THEOREM 3.10. *For the perturbed (3.6), we assume that*

$$|g(t,y)| \le a(t)|y(t)| + b(t)\int_{t_0}^t k(s)w(|y(s)|)ds \tag{3.10}$$

and

$$|h(t,y(t),Ty(t))| \le c(t)|y(t)|, \tag{3.11}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, $u \le w(u)$, $w(u)$ is nondecreasing in u, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$,
(3.12)

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}(\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau + c(s))ds\Big],$$

where $M(t_0) < \infty$ and $b_1 = \infty$. Then the zero solution of (3.6) is ULS whenever the zero solution of (2.1) is ULSV.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.6), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS by ([9],Theorem 3.3) . Using the nonlinear variation of constants formula , (3.10), and (3.11), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t}|\Phi(t, s, y(s))|(\int_{t_0}^{s}|g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\ &\le M|y_0| + \int_{t_0}^{t}M|y_0|\Big[\int_{t_0}^{s}[a(\tau)\frac{|y(\tau)|}{|y_0|} + b(\tau)\int_{t_0}^{\tau}k(r)w(\frac{|y(r)|}{|y_0|})dr]d\tau \\ &\quad + c(s)\frac{|y(s)|}{|y_0|}\Big]ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 3.9 yields

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t}(\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau + c(s))ds\Big],$$

Thus, by (3.12), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Remark 3.11. Letting $c(t) = 0$ in Theorem 3.10, we obtain the same result as that of Corollary 3.2.

Theorem 3.12. *For the perturbed (3.6), we assume that*

$$\int_{t_0}^{t}|g(s, y(s))|ds \le a(t)|y(t)| + b(t)\int_{t_0}^{t}k(s)w(|y(s)|)ds, \tag{3.13}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)|y(t)|, \tag{3.14}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, $u \le w(u)$, $w(u)$ is nondecreasing in u, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}(a(s) + c(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big], \tag{3.15}$$

where $M(t_0) < \infty$ and $b_1 = \infty$. Then the zero solution of (3.6) is ULS whenever the zero solution of (2.1) is ULSV.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.6), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS by ([9],Theorem 3.3). Using the nonlinear variation of constants formula , (3.13), and (3.14), we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \Big(\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \Big) ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0| \Big[(a(s) + c(s)) \frac{|y(s)|}{|y_0|} + b(s) \int_{t_0}^{s} k(\tau) w\Big(\frac{|y(\tau)|}{|y_0|}\Big) d\tau \Big] ds \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.6 yields

$$|y(t)| \le |y_0| W^{-1} \Big[W(M) + M \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds \Big],$$

Thus, by (3.15), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

REMARK 3.13. Letting $b(t) = c(t) = 0$ in Theorem 3.12, we obtain the same result as that of Corollary 3.4.

THEOREM 3.14. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbed term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le e^{-\alpha t} \Big(a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s) w(|y(s)|) ds \Big), \tag{3.16}$$

and

$$|h(t, y(t), Ty(t))| \le e^{-\alpha t} c(t) |y(t)|, \tag{3.17}$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L_1(\mathbb{R}^+)$ and $w(u)$ is nondecreasing in u, $u \le w(u)$, and $\frac{1}{v} w(u) \le w(\frac{u}{v})$ for some $v > 0$. If

$$M(t_0) = W^{-1} \Big[W(c) + M \int_{t_0}^{\infty} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds \Big] < \infty, b_1 = \infty, \tag{3.18}$$

where $c = M|y_0| e^{\alpha t_0}$, then all solutions of (3.6) approch zero as $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.6), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is EAS. Using Lemma 2.3, (3.16), and (3.17), we have

$$
\begin{aligned}
|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| (\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|) ds \\
\le \quad & M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}[e^{-\alpha s}a(s)|y(s)| \\
& + e^{-\alpha s}b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + e^{-\alpha s}c(s)|y(s)|]ds. \\
\le \quad & M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}(a(s)+c(s))|y(s)|e^{\alpha s}ds \\
& + \int_{t_0}^{t} Me^{-\alpha t}b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau ds.
\end{aligned}
$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, it follows from Lemma 2.6 and (3.18) that

$$
\begin{aligned}
|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] \\
&\le e^{-\alpha t}M(t_0), \ t \ge t_0,
\end{aligned}
$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

REMARK 3.15. Letting $b(t) = c(t) = 0$ in Theorem 3.14, we obtain the same result as that of Corollary 3.8.

THEOREM 3.16. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbed term $g(t, y)$ satisfies*

$$
|g(t,y(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds\Big), \tag{3.19}
$$

and

$$
|h(t, y(t), Ty(t))| \le e^{-\alpha t}c(t)|y(t)|, \tag{3.20}
$$

where $\alpha > 0$*,* $a, b, c, k, w \in C(\mathbb{R}^+)$*,* $a, b, c, k, w \in L_1(\mathbb{R}^+)$ *and* $w(u)$ *is nondecreasing in u,* $u \leq w(u)$*, and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$*. If*
(3.21)

$$M(t_0) = W^{-1}\Big[W(c)+M\int_{t_0}^{\infty}(c(s)++\int_{t_0}^{s}a(\tau)d\tau+b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big] < \infty,$$

$b_1 = \infty$ *,where* $c = M|y_0|e^{\alpha t_0}$*, then all solutions of (3.6) approch zero as* $t \to \infty$*.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (3.6), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is EAS. Using Lemma 2.3, (3.19), and (3.20), we have

$$\begin{aligned}
|y(t)| \leq & |x(t)| + \int_{t_0}^{t}|\Phi(t, s, y(s))|(\int_{t_0}^{s}|g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\
\leq & M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t}Me^{-\alpha(t-s)}[\int_{t_0}^{s}(e^{-\alpha\tau}a(\tau)|y(\tau)| \\
& +e^{-\alpha\tau}b(\tau)\int_{t_0}^{\tau}k(r)w(|y(r)|)dr)d\tau + e^{-\alpha s}c(s)|y(s)|]ds. \\
\leq & M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t}Me^{-\alpha t}(c(s)|y(s)|e^{\alpha s}ds + \int_{t_0}^{s}a(\tau)|y(\tau)|e^{\alpha\tau}d\tau) \\
& + \int_{t_0}^{t}Me^{-\alpha t}b(s)\int_{t_0}^{s}k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, it follows from Corollary 2.8 and (3.21) that

$$\begin{aligned}
|y(t)| \leq & e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}a(\tau)d\tau + b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big] \\
\leq & e^{-\alpha t}M(t_0),\ t \geq t_0,
\end{aligned}$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Remark 3.17. Letting $b(t) = c(t) = 0$ in Theorem 3.16, we obtain the same result as that of Corollary 3.6.

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