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WEAKLY SUBNORMAL WEIGHTED SHIFTS NEED NOT BE 2-HYPONORMAL

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Abstract. In this paper we give an example which is a weakly subnormal weighted shift but not 2-hyponormal. Also, we show that every partially normal extension of an isometry T needs not be 2-hyponormal even though p.n.e.(T) is weakly subnormal.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([1], [5, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$(1) \qquad \begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).$$

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Condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([7], [6]).

On the other hand, note that an operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{3}$$

The operator $\widehat{T}$ is called a *normal extension* of T. We also say that $\widehat{T}$ in $\mathcal{L}(\mathcal{K})$ is a *minimal normal extension* (briefly, m.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a normal extension of T. It is known that

$$\widehat{T} = \text{ m.n.e.}(T) \iff \mathcal{K} = \bigvee \{\widehat{T}^{*n} h : \ h \in \mathcal{H}, \ n \geq 0\},$$

and the m.n.e.(T) is unique.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (3) hold:

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*, \tag{4}$$

or equivalently, there is an extension $\widehat{T}$ of T such that

$$\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad \text{ for all } f \in \mathcal{H}.$$

The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $\mathcal{L}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([4, Lemma 2.5 and Corollary 2.7]) that

$$\widehat{T} = \text{ m.p.n.e.}(T) \iff \mathcal{K} = \bigvee\{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\},$$

and the m.p.n.e.(T) is unique. For convenience, if $\widehat{T} =$ m.p.n.e. (T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally,

$$\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}},$$

which will be called the *n-th minimal partially normal extension of* T. It was ([4], [3]) shown that

(5) 2-hyponormal $\Longrightarrow$ weakly subnormal $\Longrightarrow$ hyponormal

and the converses of both implications in 5 are not true in general. It was ([4]) known that

(6) T is weakly subnormal $\Longrightarrow T(\ker{[T^*, T]}) \subseteq \ker{[T^*, T]}$

and it was ([3]) known that if $\widehat{T} :=$ m.p.n.e.(T) then for any $k \geq 1$,

T is $(k+1)$-hyponormal $\iff T$ is weakly subnormal and $\widehat{T}$ is k-hyponormal.

So, in particular, one can see that

(7) if T is subnormal then $\widehat{T}$ is subnormal.

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [5, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t) \ \ (k \geq 1)$.

In a talk at Kyoto University entitled 'On 2-hyponormal operators', W.Y. Lee posed the following question.

QUESTION 1. *Is every weakly subnormal weighted shift 2-hyponomal ?*

In this paper we negatively answer to the Question 1. To do so, we need the next Lemma.

LEMMA 2. ([2, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*

EXAMPLE 3. If W_α is the weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, where

$$\alpha_0 = a, \quad \alpha_1 = b, \quad \alpha_n = 1 \ \ (n \geq 2, \quad a < b < 1)$$

then W_α is weakly subnormal, but W_α is not 2-hyponormal.

Proof. For the weak subnormality, let

$$A := \begin{pmatrix} a & 0 & 0 \\ 0 & \sqrt{b^2-a^2} & 0 \\ 0 & 0 & \sqrt{1-b^2} \end{pmatrix} \bigoplus 0 \ \text{ and}$$

$$B := \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{b^2-a^2} & 0 & 0 \\ 0 & b\sqrt{\frac{1-b^2}{b^2-a^2}} & 0 \end{pmatrix} \bigoplus 0.$$

Observe that $[W_\alpha^*, W_\alpha] = A^2 = AA^*$ and $A^*W_\alpha = BA^*$. Thus, $\widehat{W_\alpha} := \begin{pmatrix} W_\alpha & A \\ 0 & B \end{pmatrix}$ is a partially normal extension of W_α(cf. [4, Theorem 5.4]). Since α has two equal weights, by Lemma 2 W_α cannot be 2-hyponormal without being flat. Thus, W_α is not 2-hyponormal.

□

REMARK 4. In particular, the weighted shift W_α in Example 3 is a partially normal extension of the unilateral shift U: indeed, observe that

$$W_\alpha \cong \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & U & & 0 & 0 & 0 \\ & & & \vdots & \vdots & \vdots \\ \hline & & & 0 & a & 0 \\ & 0 & & 0 & 0 & b \\ & & & 0 & 0 & 0 \end{array}\right) = \ \text{p.n.e.}(U).$$

So, we need not expect that every partially normal extension of an isometry T is 2-hyponormal even though p.n.e.(T) is weakly subnormal.

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