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EXISTENCE OF SOLUTIONS OF A CLASS OF IMPULSIVE PERIODIC TYPE BVPS FOR SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS

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Abstract. A class of periodic type boundary value problems of coupled impulsive fractional differential equations are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearities $p(t)f(t,x,y)$ and $q(t)g(t,x,y)$ in fractional differential equations to be singular at $t = 0, 1$ and be involved a sup-multiplicative-like function. So both f and g may be super-linear and sub-linear. The analysis relies on a well known fixed point theorem. An example is given to illustrate the efficiency of the theorems.

1. Introduction

Fractional calculus has many applications (see Chapter 10 in [36]). Boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during last decades. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Actually, fractional differential equations arise in many engineering and scientific

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disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media. There have been many results obtained on the existence of solutions of boundary value problems for nonlinear fractional differential equations (see [6, 7, 29, 31, 32, 43, 51, 54]).

In recent years, many authors [1,14,19,20,22,23,25,26,30,37,42,43,50, 55] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems (see [2–4, 39]), impulsive periodic boundary value problems (see [40]), impulsive initial value problems (see [9, 13, 28, 46]), two-point, three-point or multi-point impulsive boundary value problems (see [5, 41, 53]), impulsive boundary value problems on infinite intervals (see [52]).

In [40], the following periodic boundary value problem of impulse type fractional differential equation

$$\begin{cases} D^{\alpha}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (0,1], t \neq t_1, \\ x(1) - \lim\limits_{t\to 0} t^{1-\alpha}x(t) = 0, \\ \lim\limits_{t\to t_1^+} (t-t_1)^{1-\alpha}[x(t) - x(t_1)] = I(x(t_1)) \end{cases}$$

where $0 < \alpha < 1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\lambda \in R$ with $\lambda \neq 0$, $0 = t_0 < t_1 < t_2 = 1$, $I \in C(R, R)$, f is continuous at every point $(t, u) \in [0, 1] \times R$.

In [8], authors studied the following periodic boundary value problem of impulse type fractional differential equation

$$\begin{cases} D^{\alpha}_{t_k^+}x(t) - \lambda x(t) = f(t, x(t)), \quad t \in (t_k, t_{k+1}), k = 0, 1, \cdots, p, \\ x(1) - \lim\limits_{t\to 0} t^{1-\alpha}x(t) = 0, \\ \lim\limits_{t\to t_k^+} (t-t_k)^{1-\alpha}[x(t) - x(t_k)] = I(x(t_k)), k = 1, 2, \cdots, p, \end{cases}$$

where $0 < \alpha < 1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\lambda \in R$ with $\lambda \neq 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $I \in C(R, R)$, f is continuous at every point $(t, u) \in (t_k, t_{k+1}] \times R(k = 0, 1, 2, \cdots, p)$.

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others, see [10]. Diethehm [11] proposed the model of the type (which is called a multi-order fractional differential system):

$$^{c}D^{n_i}_{0^+} y_i(t) = f_i(t, y_1(t), \cdots, y_n(t)), i = 1, 2, \cdots, n$$

subjected to the initial conditions

$$y_j(0) = y_{j,0}(j = 1, 2, \cdots, n).$$

In [15, 33, 45], the fractional order nonlinear dynamical model of interpersonal relationships

$$\begin{cases} D^{\alpha}x(t) + \alpha_1 x(t) = A_1 + \beta_1 y(t)(1 - \epsilon y^2(t)), \\ \\ D^{\alpha}y(t) + \alpha_2 y(t) = A_2 + \beta_2 x(t)(1 - \epsilon x^2(t)), \end{cases}$$

was proposed, where $0 < \alpha \leq 1$, $\alpha_i, \beta_i, A_i, \epsilon$ are real constants. These systems contain many models as special cases, see Chen's fractional order system [47,48] with a double scroll attractor, Genesio-Tesi fractional-order system [18], Lu's fractional order system [12], Volta's fractional-order system [34, 35], Rossler's fractional-order system [24] and so on. To the authors knowledge, there has been no paper discussing the existence of solutions of impulsive periodic type boundary value problems of singular fractional differential systems.

Motivated by mentioned applications and reason, in this paper, we discuss the following impulsive periodic type boundary value problem of singular fractional differential system

(1)

$$\begin{cases} D^{\alpha}_{t_i^+}x(t) - \lambda x(t) = p(t)f(t, x(t), y(t)), t \in (t_i, t_{i+1}), i \in N[0, m], \\ D^{\beta}_{t_i^+}y(t) - \mu y(t) = q(t)g(t, x(t), y(t)), t \in (t_i, t_{i+1}), i \in N[0, m], \\ x(1) - a\lim\limits_{t\to 0} t^{1-\alpha}x(t) = \int_0^1 \phi(s)G(s, x(s), y(s))ds, \\ y(1) - b\lim\limits_{t\to 0} t^{1-\beta}y(t) = \int_0^1 \psi(s)H(s, x(s), y(s))ds, \\ \lim\limits_{t\to t_i^+}(t - t_i)^{1-\alpha}x(t) = I(t_i, x(t_i), y(t_i)), i \in N[1, m], \\ \lim\limits_{t\to t_i^+}(t - t_i)^{1-\beta}y(t) = J(t_i, x(t_i), y(t_i)), i \in N[1, m], \end{cases}$$

where

(a) $0 < \alpha, \beta < 1$, $D^{\alpha}_{t_i^+}$ (or $D^{\beta}_{t_i^+}$) is the Riemann-Liouville fractional derivative of order α (or β),

(b) $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$ with $m \geq 1$, $a, b \in R$ with $ab \neq 0$, $\lambda, \mu \in R$, $N[c, d] = \{c, c + 1, \cdots, d\}$ for integers c and d,

(c) $\phi, \psi : (0, 1) \to R$ satisfy $\phi, \psi \in L^1(0, 1)$,

(d) $p, q : \bigcup\limits_{i=0}^{m}(t_i, t_{i+1}) \to R$ satisfy the growth conditions: there exist constants $k_i, l_i (i = 1, 2)$ with $k_1 > -1, k_2 > -1$ and $\max\{-\alpha, -k_1 - 1\} \leq$

$l_1 \leq 0$ and $\max\{-\beta, -k_2-1\} \leq l_2 \leq 0$ such that $|p(t)| \leq (t-t_i)^{k_1}(t_{i+1}-t)^{l_1}$, $|q(t)| \leq (t-t_i)^{k_2}(t_{i+1}-t)^{l_2}$, $t \in (t_i, t_{i+1}), i = 0, 1, \cdots, m$,

(e) f, g, G, H defined on $(0,1) \times R \times R$ are *impulsive Caratheodory functions*(see Definition 2.3), I, J are *Caratheodory functions*(see Definition 2.4).

A pair of functions $x, y : (0,1] \to R$ is called a solution of BVP(1) if

$$x|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}],\ y|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}],\ \ k = 0, 1, 2, \cdots, m$$

and x, y satisfy all equations in (1). As in [40], for clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite minimal.

To the best of the authors knowledge, no one has studied the existence of solutions of BVP (1) in which the nonlinearities are singular functions. We fill this gap by establishing existence results on solutions of BVP(1). The assumptions (D) in Theorem 3.1 in this paper are more general that the assumptions (H1) and (H2) in Theorem 3.18 in [8,40]. Two examples are given to illustrate the efficiency of the main theorems.

The remainder of this paper is as follows: in Section 2, we present preliminary results. The main theorems and their proofs are given in Section 3. In Section 4, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [21, 36].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx,\ \ \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

$$E_{\delta,\delta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\delta k+\delta)}$$

respectively for $, \alpha > 0, p > 0, q > 0, \delta > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in R$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$ we have $E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x)$.

DEFINITION 2.1. ([21]) Let $c \in R$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \to R$ is given by

$$I^{\alpha}_{c^+} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s) ds,$$

provided that the right-hand side exists.

DEFINITION 2.2. ([21]) Let $c \in R$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (c, \infty) \to R$ is given by

$$D^{\alpha}_{c^+} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha < n \leq \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

For readers convenience, choose

$$\delta_{\alpha,\lambda}(t, t_i) = (t-t_i)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-t_i)^{\alpha}), t \in (t_i, t_{i+1}], i \in N[0, m],$$

$$\delta_{\beta,\mu}(t, t_i) = (t-t_i)^{\beta-1} E_{\beta,\beta}(\mu(t-t_i)^{\beta}), t \in (t_i, t_{i+1}], i \in N[0, m].$$

DEFINITION 2.3. We call $F : \bigcup_{i=0}^{m} (t_i, t_{i+1}) \times R^2 \to R$ an *impulsive Caratheodory function* if it satisfies

(i) $t \to F\left(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v\right)$ is measurable on $(t_i, t_{i+1})(i \in N[0, m])$ for any $(u, v) \in R^2$,

(ii) $(u, v) \to F\left(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v\right)$ is continuous on R^2 for almost all $t \in (t_i, t_{i+1})(i = 0, 1, 2, \cdots, m)$,

(iii) for each $r > 0$ there exists $M_r > 0$ such that

$$|F\left(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v\right)| \leq M_r, t \in (t_i, t_{i+1}), |u|, |v| \leq r, i \in N[0, m].$$

DEFINITION 2.4. We call $I : \{t_i : i \in N[1, m]\} \times R^2 \to R$ an *Caratheodory function* if it satisfies

(i) $(u, v) \to I\left(t_i, \delta_{\alpha,\lambda}(t_i, t_{i-1})u, \delta_{\beta,\mu}(t_i, t_{i-1})v\right)$ is continuous on R^2 for almost all $i = 1, 2, \cdots, m$,

(ii) for each $r > 0$ there exists $M_r > 0$ such that

$$|I\left(t_i, \delta_{\alpha,\lambda}(t_i, t_{i-1})u, \delta_{\beta,\mu}(t_i, t_{i-1})v\right)| \leq M_r, i \in N[1, m].$$

DEFINITION 2.5. ([19]) An odd homeomorphism Φ of the real line R onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0, +\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \geq 0$ it holds

$$\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2). \tag{2}$$

ω is called the supporting function of Φ.

REMARK 2.1. From [19], any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in R$$

is a sup-multiplicative-like function, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, \quad u\}$, $u \geq 0$.

REMARK 2.2. ([19]) It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{3}$$

for all $w_1, w_2 \geq 0$ and ν is called the supporting function of Φ^{-1}.

In this paper we suppose that $\Phi : R \to R$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1} : R \to R$ with supporting function ν.

Suppose that $\lambda > 0, \mu > 0$. We use the Banach spaces (similarly to [8], we can give the proofs)

$$X = \left\{ x : (0,1] \to R : \begin{array}{c} x|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in N[0,m], \\ \text{there exist the limits} \\ \lim\limits_{t \to t_i^+} \frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)}, \ i \in N[0,m] \end{array} \right\}$$

with the norm

$$||x|| = ||x||_X = \max\left\{ \sup_{t \in (t_i, t_{i+1}]} \frac{|x(t)|}{\delta_{\alpha,\lambda}(t,t_i)} : i \in N[0,m] \right\}$$

$$Y = \left\{ y : (0,1] \to R : \begin{array}{c} y|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in N[0,m], \\ \text{there exist the limits} \\ \lim\limits_{t \to t_i^+} \frac{y(t)}{\delta_{\beta,\mu}(t,t_i)}, \ i \in N[0,m] \end{array} \right\}$$

with the norm

$$||y|| = ||y||_Y = \max\left\{\sup_{t\in(t_i,t_{i+1}]}\frac{|y(t)|}{\delta_{\beta,\mu}(t,t_i)} : i \in N[0,m]\right\}.$$

Choose $E = X \times Y$ with the norm $||(x,y)|| = \max\{||x||_X, ||y||_Y\}$. Then E is a Banach space.

LEMMA 2.1. *Suppose that $\sigma : (0,1) \to R$ satisfies that there exist numbers $k > -1$ and $\max\{-\alpha, -k-1\} < l \le 0$ such that $|\sigma(t)| \le (t-t_i)^k(t_{i+1}-t)^l$ for all $t \in (t_i, t_{i+1})$, $i = 0, 1, \cdots, m$. The x is a solutions of*

$$(4)\qquad \begin{cases} D^{\alpha}_{t_i^+}x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}), i \in N[0,m], \\ x(1) - a\lim\limits_{t\to 0} t^{1-\alpha}x(t) = a_0, \\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t) = I_i, i \in N[1,m] \end{cases}$$

if and only if $x \in X$ and

(5)

$$x(t) = \begin{cases} \Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{I_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m)+\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds - a_0}{a} \\ \quad + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I_i + \int_{t_i}^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_i,t_{i+1}], i \in N[1,m]. \end{cases}$$

Proof. Let x be a solution of (4). One sees from $l \le 0$, for $t \in (t_i, t_{i+1}]$, that

$$\begin{aligned} &(t-t_i)^{1-\alpha}\left|\int_{t_i}^t \delta_{\alpha,\alpha}(t,s)\sigma(s)ds\right| \\ &\le (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})(s-t_i)^k(t_{i+1}-s)^l ds \\ &= (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha-1}\sum_{i=0}^{\infty}\frac{\lambda^i(t-s)^{\alpha i}}{\Gamma(\alpha i+\alpha)}(s-t_i)^k(t_{i+1}-s)^l ds \\ &\le (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha+l-1}\sum_{i=0}^{\infty}\frac{\lambda^i(t-s)^{\alpha i}}{\Gamma(\alpha i+\alpha)}(s-t_i)^k ds \end{aligned}$$

$$= (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}\int_{t_i}^{t}(t-s)^{\alpha+\alpha i+l-1}(s-t_i)^k ds$$

$$= (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}(t-t_i)^{\alpha+\alpha i+l+k}\int_0^1(1-w)^{\alpha+\alpha i+l-1}w^k dw$$

$$\leq (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}(t-t_i)^{\alpha+\alpha i+l+k}\int_0^1(1-w)^{\alpha+l-1}w^k dw$$

$$= (t-t_i)^{1+l+k}\mathbf{B}(\alpha+l,k+1)\sum_{i=0}^{\infty}\frac{\lambda^i(t-t_i)^{\alpha i}}{\Gamma(\alpha i+\alpha)}$$

$$= (t-t_i)^{1+l+k}\mathbf{B}(\alpha+l,k+1)E_{\alpha,\alpha}(\lambda(t-t_i)^{\alpha}).$$

From $k+l+1>0$, we get

$$\lim_{t\to t_i^+}(t-t_i)^{1-\alpha}\left|\int_{t_i}^{t}\delta_{\alpha,\lambda}(t,s)\sigma(s)ds\right| = 0.$$

By (3.26) in [7], we know that there exist numbers A_i such that
(6)
$$x(t) = A_i\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i) + \int_{t_i}^{t}\delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t\in(t_i,t_{i+1}], i\in N[0,m].$$

Note $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$. It follows from the boundary conditions and the impulse assumption in (4) that

$$A_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m) + \int_{t_m}^{1}\delta_{\alpha,\lambda}(1,s)\sigma(s)ds - aA_0 = a_0,$$

$$A_i = I_i,\ i\in N[1,m].$$

Then

$$A_0 = \frac{I_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m)+\int_{t_m}^{1}\delta_{\alpha,\lambda}(1,s)\sigma(s)ds-a_0}{a}.$$

Substituting $A_i(i=0,1,2,\cdots,m)$ into (6), we get (5) obviously.

It is easy to see that both $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim_{t\to 0}t^{1-\alpha}x(t)$ and $\lim_{t\to t_1}x(t)$. So $x\in X$.

On the other hand, if x satisfies (5), we can prove that $x\in X$ and x satisfies (4). The proof is completed. □

LEMMA 2.2. *Suppose that* $\sigma : (0,1) \to R$ *satisfies that there exist numbers* $k > -1$ *and* $\max\{-\beta, -k-1\} < l \leq 0$ *such that* $|\sigma(t)| \leq (t-t_i)^k(t_{i+1}-t)^l$ *for all* $t \in (t_i, t_{i+1}), i \in N[0,m]$. *The* y *is a solutions of*

$$(7) \qquad \begin{cases} D^{\beta}_{t_i^+} y(t) - \mu y(t) = \sigma(t), t \in (t_i, t_{i+1}), i \in N[0,m], \\ y(1) - b\lim_{t\to 0} t^{1-\beta} y(t) = b_0, \\ \lim\limits_{t\to t_i^+} (t-t_i)^{1-\beta} y(t) = J_i, i \in N[1,m] \end{cases}$$

if and only if $y \in Y$ *and*

(8)

$$y(t) = \begin{cases} \Gamma(\beta)\delta_{\beta,\mu}(t,0)\frac{J_m\Gamma(\beta)\delta_{\beta,\mu}(1,t_m)+\int_{t_m}^1 \delta_{\beta,\mu}(1,s)\sigma(s)ds - b_0}{b} \\ \quad + \int_0^t \delta_{\beta,\mu}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J_i + \int_{t_i}^t \delta_{\beta,\mu}(t,s)\sigma(s)ds, t \in (t_i, t_{i+1}], i \in N[1,m]. \end{cases}$$

Proof. The proof is similar to that of the proof of Lemma 2.1 and is omitted.

□

Define the nonlinear operator T on E by

$$T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t)) \text{ with}$$

$$(T_1(x,y))(t) =$$

$$\begin{cases} \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{a} I(t_m, x(t_m), y(t_m)) \\ + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a} \int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)p(s)f(s,x(s),y(s))ds \\ - \Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{\int_0^1 \phi(s)G(s,x(s),y(s))ds}{a} \\ + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f(s,x(s),y(s))ds, t \in (0,t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I(t_i,x(t_i),y(t_i)) \\ + \int_{t_i}^t \delta_{\alpha,\lambda}(t,s)p(s)f(s,x(s),y(s))ds, t \in (t_i,t_{i+1}], i \in N[1,m]. \end{cases}$$

$$(T_2(x,y))(t) =$$
$$\begin{cases} \frac{\Gamma(\beta)^2\delta_{\beta,\mu}(t,0)\delta_{\beta,\mu}(1,t_m)}{b}J(t_m,x(t_m),y(t_m)) \\ \frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_{t_m}^1 \delta_{\beta,\mu}(1,s)q(s)g(s,x(s),y(s))ds \\ -\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_0^1 \psi(s)H(s,x(s),y(s))ds \\ +\int_0^t \delta_{\beta,\mu}(t,s)q(s)g(s,x(s),y(s))ds, t\in(0,t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J(t_i,x(t_i),y(t_i)) \\ +\int_{t_i}^t \delta_{\beta,\mu}(t,s)q(s)g(s,x(s),y(s))ds, t\in(t_i,t_{i+1}], i\in N[1,m] \end{cases}$$

for $(x,y)\in E$.

LEMMA 2.3. *Suppose that (a)-(e) hold and $\lambda>0,\mu>0$. Then $T:E\to E$ is well defined and is completely continuous.*

Proof. **Step (i)** We prove that $T:E\to E$ is well defined. It comes from that $T_j(x,y)|_{(t_i,t_{i+1}]}(i=0,1,\cdots,m,j=1,2)$ are continuous and the limits

$$\lim_{t\to t_i^+}\delta_{\alpha,\lambda}(t,t_i)(T_1(x,y))(t)(i=0,1,\cdots,m),$$

$$\lim_{t\to t_i}\delta_{\beta,\mu}(t,t_i)(T_2(x,y))(t)(i=0,1,\cdots,m) \text{ exist.}$$

Step (ii) We prove that T is continuous.

Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We can show that $T(x_n,y_n)\to T(x_0,y_0)$ as $n\to\infty$ by using the dominant convergence theorem. We refer the readers to the papers [38, 44, 49].

Step (iii) Prove that T is compact, i.e., prove that $T(\overline{\Omega})$ is relatively compact for every bounded closed subset $\overline{\Omega}\subset E$.

Let $\overline{\Omega}$ be a bounded closed nonempty subset of E. We have $||(x,y)||\le r<+\infty$ for all $(x,y)\in\overline{\Omega}$. Since f,g,G,H are *impulsive Caratheodory functions*, I,J are *Caratheodory functions*, then there exists a constant

$M_I, M_J, M_f, M_g, M_G, M_H \geq 0$ such that
(9)
$|f(t,x(t),y(t))| = \left|f\left(t, \delta_{\alpha,\lambda}(t,t_i)\frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)}, \delta_{\beta,\mu}(t,t_i)\frac{y(t)}{\delta_{\beta,\mu}(t,t_i)}\right)\right|$
$\leq M_f, t \in (t_i, t_{i+1}], i \in N[0,m],$

$|g(t,x(t),y(t))| \leq M_g, t \in (t_i, t_{i+1}], i \in N[0,m],$

$|G(t,x(t),y(t))| \leq M_G, t \in (t_i, t_{i+1}], i \in N[0,m],$

$|H(t,x(t),y(t))| \leq M_H, t \in (t_i, t_{i+1}], i \in N[0,m],$

$|I(t_i,x(t_i),y(t_i))| = \left|I\left(t_i, \delta_{\alpha,\lambda}(t_i,t_{i-1})\frac{x(t_i)}{\delta_{\alpha,\lambda}(t_i,t_{i-1})}, \delta_{\beta,\mu}(t_i,t_{i-1})\frac{y(t_i)}{\delta_{\beta,\mu}(t_i,t_{i-1})}\right)\right|$
$\leq M_I, i \in N[1,m],$

$|J(t_i,x(t_i),y(t_i))| \leq M_J, i \in N[1,m].$

This step is done by the following two sub-steps:

Sub-step (iii1) Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using (d), (10), $\lambda > 0, \mu > 0$ and the definition of T_1, we have for $t \in (0, t_1]$ that

$$\begin{aligned}
\frac{|(T_1(x,y))(t)|}{\delta_{\alpha,\lambda}(t,0)} \leq\ & \frac{1}{\delta_{\alpha,\lambda}(t,0)}\frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I \\
& + \frac{1}{\delta_{\alpha,\lambda}(t,0)}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{|a|}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)(s-t_m)^{k_1}(1-s)^{l_1}M_f ds \\
& + \frac{1}{\delta_{\alpha,\lambda}(t,0)}\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{||\phi||_1 M_G}{|a|} \\
& + \frac{1}{\delta_{\alpha,\lambda}(t,0)}\int_0^t \delta_{\alpha,\lambda}(t,s)s^{k_1}(t_1-s)^{l_1}M_f ds \\
\leq\ & \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I + \frac{\Gamma(\alpha)}{|a|}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)(s-t_m)^{k_1}(1-s)^{l_1}M_f ds \\
& + \Gamma(\alpha)\frac{||\phi||_1 M_G}{|a|} + \frac{1}{\delta_{\alpha,\lambda}(t,0)}\int_0^t \delta_{\alpha,\lambda}(t,s)s^{k_1}(t_1-s)^{l_1}M_f ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I + \frac{\Gamma(\alpha)||\phi||_1}{|a|}M_G \\
&\quad + \frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}\int_{t_m}^1 (1-s)^{\alpha-1}(s-t_m)^{k_1}(1-s)^{l_1}dsM_f \\
&\quad + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{k_1}(t-s)^{l_1}dsM_f \\
&= \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I + \frac{\Gamma(\alpha)||\phi||_1}{|a|}M_G \\
&\quad + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}(1-t_m)^{1+k_1+l_1} + t^{1+k_1+l_1}\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f \\
&\leq \frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}M_I + \frac{\Gamma(\alpha)||\phi||_1}{|a|}M_G \\
&\quad + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned}$$

For $t \in (t_i, t_{i+1}](i \in N[1,m])$, similarly we have

$$\begin{aligned}
&\frac{|(T_1(x,y))(t)|}{\delta_{\alpha,\lambda}(t,t_i)} \leq \Gamma(\alpha)M_I + \frac{1}{\delta_{\alpha,\lambda}(t,t_i)}\int_{t_i}^t \delta_{\alpha,\lambda}(t,s)(s-t_i)^{k_1}(t_{i+1}-s)^{l_1}dsM_f \\
&\leq \Gamma(\alpha)M_I + (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha-1}(s-t_i)^{k_1}(t-s)^{l_1}dsM_f \\
&\leq \Gamma(\alpha)M_I + \mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned}$$

It follows that

$$\begin{aligned}
||T_1(x,y)|| &\leq \left(\frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)M_I + \frac{\Gamma(\alpha)||\phi||_1}{|a|}M_G \\
&\quad + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned} \tag{10}$$

Similarly we have

$$\begin{aligned}
||T_2(x,y)|| &\leq \left(\frac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)M_J + \frac{\Gamma(\beta)||\psi||_1}{|b|}M_H \\
&\quad + \left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)M_g.
\end{aligned} \tag{11}$$

Then $T(\overline{\Omega})$ is uniformly bounded.

From above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2) Prove that both $\left\{t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ and $\left\{t \to \frac{(T_2(x,y))(t)}{\delta_{\beta,\mu}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ are equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$, respectively.

Let

$$\frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} = \begin{cases} \lim\limits_{t \to t_i^+} \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}, t = t_i, \\ \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}, t \in (t_i, t_{i+1}] \end{cases}$$

Since $t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}$ is continuous on $[t_i, t_{i+1}]$, $\left\{t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ is equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$. We can prove similarly that $\left\{t \to \frac{(T_2(x,y))(t)}{\delta_{\beta,\mu}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ is equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$.

So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proofs are completed. □

3. Main results

Now, we prove that main theorem in this paper by using the Schauder's fixed point theorem [27]. We need the following assumptions:

(C) Φ is a sup-multiplicative-like function with its supporting function w, the inverse function of Φ is Φ^{-1} with supporting function ν.

(D) f, g, H, G are *impulsive caratheodory functions*, I, J are continuous functions and satisfy that there exist nonnegative constants I_0, J_0, $b_i, a_i (i = 1,2)$, $B_i, A_i (i = 1,2)$ and $\overline{B}_i, \overline{A}_i (i = 1,2)$, bounded measurable functions $\phi_i, \psi_i : (0,1) \to R (i = 1,2)$ such that

$$\left| f\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \phi_1(t) \right| \leq b_1|x| + a_1\Phi^{-1}(|y|), t \in (t_i, t_{i+1}],$$

$$\left| g\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \phi_2(t) \right| \leq b_2\Phi(|x|) + a_2|y|, t \in (t_i, t_{i+1}],$$

$$\left| G\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \psi_1(t) \right| \leq B_1|x| + A_1\Phi^{-1}(|y|), t \in (t_i, t_{i+1}],$$

$$\left| H\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \psi_2(t) \right| \leq B_2\Phi(|x|) + A_2|y|, t \in (t_i, t_{i+1}]$$

hold for $x, y \in R, i \in N[0, m]$ and

$$\left| I\left(t_i, \frac{x}{\delta_{\alpha,\lambda}(t_i, t_{i-1})}, \frac{y}{\delta_{\beta,\mu}(t_i - t_{i-1})}\right) - I_0 \right| \leq \overline{B}_1|x| + \overline{A}_1\Phi^{-1}(|y|),$$

$$\left| J\left(t_i, \frac{x}{\delta_{\alpha,\lambda}(t_i, t_{i-1})}, \frac{y}{\delta_{\beta,\mu}(t_i - t_{i-1})}\right) - J_0 \right| \leq \overline{B}_2\Phi(|x|) + \overline{A}_2|y|$$

hold for $i \in N[1, m], x, y \in R$.

Denote

$$\Phi_1(t) = \begin{cases} \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{a} I_0 + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a} \int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)p(s)\phi_1(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{\int_0^1 \phi(s)\psi_1(s)}{a} + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t \in (0, t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I_0 + \int_{t_i}^t \delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t \in (t_i, t_{i+1}], i \in N[1, m], \end{cases}$$

$$\Phi_2(t) = \begin{cases} \frac{\Gamma(\beta)^2\delta_{\beta,\mu}(t,0)\delta_{\beta,\mu}(1,t_m)}{b} J_0 + \frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b} \int_{t_m}^1 \delta_{\beta,\mu}(1,s)q(s)\phi_2(s)ds \\ -\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b} \int_0^1 \psi(s)\psi_2(s)ds + \int_0^t \delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t \in (0, t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J_0 + \int_{t_i}^t \delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t \in (t_i, t_{i+1}], i \in N[1, m] \end{cases}$$

and

$$\begin{aligned} M_2 = & \left(\frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\overline{B}_1 + \frac{\Gamma(\alpha)||\phi||_1}{|a|}B_1 \\ & + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\mathbf{B}(\alpha + l_1, k_1 + 1)b_1, \\ M_3 = & \left(\frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\overline{A}_1 \\ & + \frac{\Gamma(\alpha)||\phi||_1}{|a|}A_1 + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\mathbf{B}(\alpha + l_1, k_1 + 1)a_1, \end{aligned}$$

$$
\begin{aligned}
N_2 = & \left(\frac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\overline{B}_2+\frac{\Gamma(\beta)||\psi||_1}{|b|}B_2 \\
& +\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)b_2, \\
N_3 = & \left(\frac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\overline{A}_2+\frac{\Gamma(\beta)||\psi||_1}{|b|}A_2 \\
& +\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)a_2.
\end{aligned}
$$

THEOREM 3.1. *Suppose that $\lambda>0,\mu>0$ and (a)-(e), (C), (D) hold. Then BVP(1) has at least one solution if*

$$
M_2<1,\ \ N_3<1,\ \ \lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r}<\frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1} \tag{12}
$$

or

$$
M_2<1,\ \ N_3<1,\ \ \lim_{r\to+\infty}\omega(1/\Phi^{-1}(r))r>\frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right). \tag{13}
$$

Proof. To apply the Schauder's fixed point theorem, we should define an closed convex bounded subset Ω of E such that $T(\Omega)\subseteq\Omega$.

For $r_1>0,r_2>0$, denote $\Omega=\{(x,y)\in E:||x-\Phi_1||\le r_1,||y-\Phi_2||\le r_2\}$. For $(x,y)\in\Omega$, we get

$$
\begin{aligned}
&||x||\le||x-\Phi_1||+||\Phi_1||\le r_1+||\Phi_1||, \\
&||y||\le||y-\Phi_2||+||\Phi_2||\le r_2+||\Phi_2||.
\end{aligned} \tag{14}
$$

Then

$$
\begin{aligned}
&|f(t,x(t),y(t))-\phi_1(t)| \\
&=\left|f\left(t,\delta_{\alpha,\lambda}(t,t_i)\frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)},\delta_{\beta,\mu}(t,t_i)\frac{y(t)}{\delta_{\beta,\mu}(t,t_i)}\right)-\phi_1(t)\right| \\
&\le b_1\delta_{\alpha,\lambda}(t,t_i)|x(t)|+a_1\Phi^{-1}(\delta_{\beta,\mu}(t,t_i)|y(t)|) \\
&\le b_1||x||+a_1\Phi^{-1}(||y||)\le b_1[r_1+||\Phi_1||]+a_1\Phi^{-1}(r_2+||\Phi_2||),
\end{aligned}
$$

$$|g(t,x(t),y(t))-\phi_2(t)|\le b_2\Phi(r_1+||\Phi_1||)+a_2[r_2+||\Phi_2||],$$

$$|G(t,x(t),y(t))-\psi_1(t)|\le B_1[r_1+||\Phi_1||]+A_1\Phi^{-1}(r_2+||\Phi_2||),$$

$$|H(t,x(t),y(t))-\psi_2(t)|\le B_2\Phi(r_1+||\Phi_1||)+A_2[r_2+||\Phi_2||]$$

hold for $t\in(t_i,t_{i+1}]$, $i\in N[0,m]$ and

$$|I(t_i,x(t_i),y(t_i))-I_0|\le \overline{B}_1[r_1+||\Phi_1||]+\overline{A}_1\Phi^{-1}(r_2+||\Phi_2||),$$

$$|J(t_i,x(t_i),y(t_i))-J_0|\le \overline{B}_2\Phi(r_1+||\Phi_1||)+\overline{A}_2[r_2+||\Phi_2||]$$

hold for $i\in N[1,m]$.

By the definition of T, using the methods proving (10) and (11), in Step (iii1) of the proof of Lemma 2.3, we have that

$$\begin{aligned}
&||T_1(x,y)-\Phi_1||\\
&\le\left(\tfrac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)[\overline{B}_1[r_1+||\Phi_1||]+\overline{A}_1\Phi^{-1}(r_2+||\Phi_2||)]\\
&+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}[B_1[r_1+||\Phi_1||]+A_1\Phi^{-1}(r_2+||\Phi_2||)]\\
&+\left(\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)[b_1[r_1+||\Phi_1||]+a_1\Phi^{-1}(r_2+||\Phi_2||)],
\end{aligned}$$

and

$$\begin{aligned}
&||T_2(x,y)-\Phi_2||\\
&\le\left(\tfrac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)[\overline{B}_2\Phi(r_1+||\Phi_1||)+\overline{A}_2[r_2+||\Phi_2||]]\\
&+\tfrac{\Gamma(\beta)||\psi||_1}{|b|}[B_2\Phi(r_1+||\Phi_1||)+A_2[r_2+||\Phi_2||]]\\
&+\left(\tfrac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)[b_2\Phi(r_1+||\Phi_1||)+a_2[r_2+||\Phi_2||]].
\end{aligned}$$

It follows that

$$\begin{aligned}
&||T_1(x,y)-\Phi_1||\le M_2(r_1+||\Phi_1||)+M_3\Phi^{-1}(r_2+||\Phi_2||),\\
&||T_2(x,y)-\Phi_2||\le N_2\Phi(r_1+||\Phi_1||)+N_3(r_2+||\Phi_2||).
\end{aligned}\tag{15}$$

We claim that there exists $r_1, r_2 > 0$ such that

$$\begin{aligned} &M_2(r_1 + ||\Phi_1||) + M_3\Phi^{-1}(r_2 + ||\Phi_2||) \leq r_1, \\ &N_2\Phi(r_1 + ||\Phi_1||) + N_3(r_2 + ||\Phi_2||) \leq r_2. \end{aligned} \tag{16}$$

We consider two cases:

Case (i) $M_2 < 1,\ \ N_3 < 1,\ \lim\limits_{r\to+\infty} \frac{\nu(\Phi(r))}{r} < \frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1}$.

First we prove that that exists $r_1 > 0$ such that

$$r_1 \geq \frac{M_2||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r_1 + ||\Phi_1||) + \frac{||\Phi_2||}{1-N_3}\right). \tag{17}$$

In fact, if

$$r < \frac{M_2||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r + ||\Phi_1||) + \frac{||\Phi_2||}{1-N_3}\right)$$

for every $r > 0$, using (3), we get

$$\begin{aligned} 1 &< \frac{M_2||\Phi_1||}{1-M_2}\frac{1}{r} + \frac{M_3}{1-M_2}\frac{1}{r}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r + ||\Phi_1||) + \frac{||\Phi_2||}{1-N_3}\right) \\ &\leq \frac{M_2||\Phi_1||}{1-M_2}\frac{1}{r} + \frac{M_3}{1-M_2}\frac{\nu(\Phi(r))}{r}\Phi^{-1}\left(\frac{\frac{N_2}{1-N_3}\Phi(r+||\Phi_1||)+\frac{||\Phi_2||}{1-N_3}}{\Phi(r)}\right). \end{aligned}$$

Let $r \to +\infty$, we get

$$1 \leq \frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r},$$

which contradicts

$$\lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r} < \frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1}.$$

Then there exists $r_1 > 0$ such that (17) holds. Choose $r_2 > 0$ satisfying $r_2 \geq \frac{N_2}{1-N_3}\Phi(r_1+||\Phi_1||)+\frac{N_3||\Phi_2||}{1-N_3}$. Then $r_1 > 0$ and $r_2 > 0$ satisfy (16).

Case (ii) $M_2 < 1,\ \ N_3 < 1,\ \lim\limits_{r\to+\infty} \omega(1/\Phi^{-1}(r))r > \frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right)$.

First we prove that that exists $r_2 > 0$ such that

$$r_2 \geq \frac{N_2}{1-N_3}\Phi\left(\frac{||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}(r_2 + ||\Phi_2||)\right) + \frac{N_3||\Phi_2||}{1-N_3}. \tag{18}$$

In fact, if

$$r < \frac{N_2}{1-N_3}\Phi\left(\frac{||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}(r + ||\Phi_2||)\right) + \frac{N_3||\Phi_2||}{1-N_3}$$

holds for all $r > 0$. using (2), we get $\Phi(xy) \leq \frac{1}{\omega(1/x)}\Phi(y)$. Then

$$1 < \frac{\frac{N_2}{1-N_3}\Phi\Big(\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)\Big)}{r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r}$$

$$= \frac{N_2}{1-N_3}\Phi\left(\frac{\Phi^{-1}(r)\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)}{\Phi^{-1}(r)}\right)\frac{1}{r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r}$$

$$\leq \frac{N_2}{1-N_3}\Phi\left(\frac{\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)}{\Phi^{-1}(r)}\right)\frac{1}{\omega(1/\Phi^{-1}(r))r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r}.$$

Let $r \to \infty$. We get

$$1 \leq \frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right)\frac{1}{\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r}.$$

Hence there is $r_2 > 0$ such that (18) holds. Now choose $r_1 > 0$ such that

$$r_1 \geq \frac{M_2||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}(r_2 + ||\Phi_2||).$$

Then $r_1 > 0$ and $r_2 > 0$ satisfy (16).

We choose $\Omega = \{(x, y) \in E : ||x - \Phi_1|| \leq r_1, ||y - \Phi_2|| \leq r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x, y) \in \Omega$. So (x, y) is a solution of BVP(1). The proof of Theorem 3.1 is complete. □

REMARK 3.1. When the limits $\lim\limits_{r\to+\infty}\frac{\nu(\Phi(r))}{r}$ and $\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r$ exist, we note, from Theorem 3.1, that (12) and (13) hold for sufficiently small nonnegative constants $I_0, J_0,\ b_i, a_i(i = 1, 2),\ B_i, A_i(i = 1, 2)$ and $\overline{B}_i, \overline{A}_i(i = 1, 2)$. So it is easy to see that BVP(1) has at least one solution if the nonnegative constants $I_0, J_0,\ b_i, a_i(i = 1, 2),\ B_i, A_i(i = 1, 2)$ and $\overline{B}_i, \overline{A}_i(i = 1, 2)$ are very small.

REMARK 3.2. In BVP(1) when $\lambda < 0, \mu < 0$, or $\lambda < 0, \mu > 0$, or $\lambda > 0, \mu < 0$, similar result to Theorem 3.1 can be obtained. The details are omitted.

REMARK 3.3. Consider the following periodic boundary value problem

$$(19)\quad \begin{cases} D^{\alpha}_{t_i^+}x(t)-\lambda x(t)=p(t)f(t,x(t),y(t)), \quad t\in(t_i,t_{i+1}], i=0,1,\\ D^{\beta}_{t_i^+}y(t)-\mu y(t)=q(t)g(t,x(t),y(t)), \quad t\in(t_i,t_{i+1}], i=0,1,\\ x(1)-\lim\limits_{t\to 0}t^{1-\alpha}x(t)=0,\ \ y(1)-\lim\limits_{t\to 0}t^{1-\beta}y(t)=0,\\ \lim\limits_{t\to t_1^+}(t-t_t)^{1-\alpha}x(t)-x(t_1)=\lim\limits_{t\to t_1^+}(t-t_1)^{1-\beta}y(t)-y(t_1)=0, \end{cases}$$

where

(i) $0<\alpha,\beta<1$, $\lambda,\mu\in R$ with $\lambda\neq 0,\mu\neq 0$, $D^{\alpha}_{t_i^+}$ (or $D^{\beta}_{t_i^+}$) is the Riemann-Liouville fractional derivative of order α (or β),

(ii) $0=t_0<t_1<t_2=1$,

(iii) $p,q:(0,1)\to R$ satisfy the growth conditions: there exist constants $k_i,l_i(i=1,2)$ with $k_1>-1,k_2>-1$ and $\max\{-\alpha,-k_1-1\}\le l_1\le 0$ and $\max\{-\beta,-k_2-1\}\le l_2\le 0$ such that

$$|p(t)|\le(t-t_i)^{k_1}(t_{i+1}-t)^{l_1},\ |q(t)|\le(t-t_i)^{k_2}(t_{i+1}-t)^{l_2},\ t\in(t_i,t_{i+1}), i=0,1,$$

(iv) f,g defined on $(0,1]\times R\times R$ are *impulsive Caratheodory functions.*

THEOREM 3.2. *Suppose that $\lambda>0,\mu>0$ and (i)-(iv), (C) hold and (D1) f,g are impulsive caratheodory functions, and satisfy that there exist nonnegative constants $b_i,a_i(i=1,2)$ and bounded measurable functions $\phi_i:(0,1)\to R(i=1,2)$ such that*

$$|f\left(t,\delta_{\alpha,\lambda}(t,t_i)x,\delta_{\beta,\mu}(t,t_i)y\right)-\phi_1(t)|\le b_1|x|+a_1\Phi^{-1}(|y|),t\in(t_i,t_{i+1}],$$

$$|g\left(t,\delta_{\alpha,\lambda}(t,t_i)x,\delta_{\beta,\mu}(t,t_i)y\right)-\phi_2(t)|\le b_2\Phi(|x|)+a_2|y|,t\in(t_i,t_{i+1}]$$

hold for $x,y\in R,i\in N[0,m]$.

Then BVP(19) has at least one solution if

$$(20)\qquad M_2<1,\ \ N_3<1,\ \ \lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r}<\frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1}$$

or

$$(21)\qquad M_2<1,\ \ N_3<1,\ \ \lim_{r\to+\infty}\omega(1/\Phi^{-1}(r))r>\frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right),$$

where

$$\Phi_1(t)=\begin{cases}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a}\int_{t_m}^1\delta_{\alpha,\lambda}(1,s)p(s)\phi_1(s)ds\\ +\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t\in(0,t_1],\\ \int_{t_i}^t\delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t\in(t_i,t_{i+1}], i\in N[1,m],\end{cases}$$

$$\Phi_2(t)=\begin{cases}\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_{t_m}^1\delta_{\beta,\mu}(1,s)q(s)\phi_2(s)ds\\ +\int_0^t\delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t\in(0,t_1],\\ \int_{t_i}^t\delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t\in(t_i,t_{i+1}], i\in N[1,m],\end{cases}$$

and

$$M_2=\left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)b_1,$$

$$M_3=\left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)a_1,$$

$$N_2=\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)b_2,$$

$$N_3=\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)a_2.$$

Proof. In Theorem 3.1, choose $G(t,x,y)\equiv H(t,x,y)\equiv 0$, $I(t,x,y)\equiv J(t,x,y)\equiv 0$. The theorem follows Theorem 3.1. The details of proof is omitted. □

REMARK 3.4. Similar results can be obtained for BVP(19) when $\lambda<0,\mu<0$, $\lambda<0,\mu>0$ and $\lambda>0,\mu<0$ respectively. The details are omitted. When the limits $\lim\limits_{r\to+\infty}\frac{\nu(\Phi(r))}{r}$ and $\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r$ exist, we note, from Theorem 3.2, that (18) and (19) hold for sufficiently small nonnegative constants $b_i,a_i(i=1,2)$. So it is easy to see that BVP(19) has at least one solution if the nonnegative constants $b_i,a_i(i=1,2)$ are very small.

4. Applications

Now, we present an example, which can not be covered by known results, to illustrate Theorem 3.1.

EXAMPLE 4.1. *Consider the following periodic type boundary value problem for fractional differential equation*

(22)
$$\begin{cases} D^{\frac{2}{3}}_{t_i^+} x(t) - x(t) = (t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}} f(t,x(t),y(t)), t \in (t_i, t_{i+1}], \\ D^{\frac{1}{2}}_{t_i^+} y(t) - y(t) = (t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}} g(t,x(t),y(t)), t \in (t_i, t_{i+1}], \\ x(1) - \lim\limits_{t\to 0} t^{\frac{1}{3}} x(t) = \frac{1}{2}\int_0^1 s^{-\frac{1}{2}} G(s,x(s),y(s))ds, \\ y(1) - \lim\limits_{t\to 0} t^{\frac{1}{2}} y(t) = \frac{1}{2}\int_0^1 s^{-\frac{1}{2}} H(s,x(s),y(s))ds, \\ \lim\limits_{t\to \frac{1}{2}^+} \left(t-\frac{1}{2}\right)^{\frac{1}{3}} x(t) - x(1/2) = 1, \lim\limits_{t\to \frac{1}{2}^+} \left(t-\frac{1}{2}\right)^{\frac{1}{2}} y(t) - y(1/2) = 1. \end{cases}$$

where $0 = t_0 < t_1 = \frac{1}{2} < t_2 = 1$ *and*

$$f(t,x,y) = c_1 + b_1\delta_{2/3,1}(t,t_i)x + a_1[\delta_{1/2,1}(t,t_i)]^{\frac{1}{3}} y^{\frac{1}{3}}, t \in (t_i, t_{i+1}],$$

$$g(t,x,y) = c_2 + b_2[\delta_{2/3,1}(t,t_i)]^3 x^3 + a_2\delta_{1/2,1}(t,t_i)y, t \in (t_i, t_{i+1}],$$

$$G(t,x,y) = C_1 + B_1\delta_{2/3,1}(t,t_i)x + A_1[\delta_{1/2,1}(t,t_i)]^{\frac{1}{3}} y^{\frac{1}{3}}, t \in (t_i, t_{i+1}],$$

$$H(t,x,y) = C_2 + B_2[\delta_{2/3,1}(t,t_i)]^3 x^3 + A_2\delta_{1/2,1}(t,t_i)y, t \in (t_i, t_{i+1}],$$

with $c_i, b_i, a_i, C_i, B_i, A_i (i = 1,2)$ *being nonnegative numbers. Then, BVP(22) has at least one solution for sufficiently small* $b_i, a_i, B_i, A_i (i = 1,2)$..

Proof. Corresponding to BVP(1), $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$, $\lambda = \mu = 1$, $a = b = 1$, $t_1 = \frac{1}{2}$, $p(t) = q(t) = (t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}}$ for $t \in (t_i, t_{i+1})(i = 0,1)$, $\phi(t) = \psi(t) = \frac{1}{2}t^{-\frac{1}{2}}$, $\Phi(x) = x^3$ with $\Phi^{-1}(x) = x^{\frac{1}{3}}$, the supporting function of Φ is $\omega(x) = x^3$ and the supporting function of Φ^{-1} is $\nu(x) = x^{\frac{1}{3}}$, $I(t,x,y) = J(t,x,y) = 1$.

It is easy to see that $k_1 = l_1 = k_2 = l_2 = -\frac{1}{4}$, $||\phi||_1 = ||\psi||_1 = 1$ and

$$f\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = c_1 + b_1 x + a_1 \Phi^{-1}(y), t \in (t_i, t_{i+1}], i = 0, 1,$$

$$g\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = c_2 + b_2 \Phi(x) + a_1 y, t \in (t_i, t_{i+1}], i = 0, 1,$$

$$G\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = C_1 + B_1 x + A_1 \Phi^{-1}(y), t \in (t_i, t_{i+1}], i = 0, 1,$$

$$H\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = C_2 + B_2 \Phi(x) + A_2 y, t \in (t_i, t_{i+1}], i = 0, 1.$$

It is easy to see that $I_0 = J_0 = 1$ and $\overline{B}_1 = \overline{B}_2 = \overline{A}_1 = \overline{A}_2 = 0$.

One sees that (C) and (D) hold. By computation, we get by direct computation that

$$M_2 = \Gamma(2/3)B_1 + \left(\Gamma(2/3)\mathbf{E}_{2/3,2/3}(1) + 1\right)\mathbf{B}(5/12, 3/4)b_1,$$

$$M_3 = \Gamma(2/3)A_1 + \left(\Gamma(2/3)\mathbf{E}_{2/3,2/3}(1) + 1\right)\mathbf{B}(5/12, 3/4)a_1,$$

$$N_2 = \Gamma(1/2)B_1 + \left(\Gamma(1/2)\mathbf{E}_{1/2,1/2}(1) + 1\right)\mathbf{B}(1/4, 3/4)b_2,$$

$$N_3 = \Gamma(1/2)A_2 + \left(\Gamma(1/2)\mathbf{E}_{1/2,1/2}(1) + 1\right)\mathbf{B}(1/4, 3/4)a_2.$$

From Theorem 3.1, we know that BVP(22) has at least one solution if

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r \to +\infty} \frac{\nu(\Phi(r))}{r} = 1 < \frac{1-M_2}{M_3} \sqrt[3]{\frac{1-N_3}{N_2}}$$

or

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r \to +\infty} \omega(1/\Phi^{-1}(r))r = 1 > \frac{N_2}{1-N_3} \sqrt[3]{\frac{M_3}{1-M_2}}.$$

So BVP(22) has at least one solution for sufficiently small $b_i, a_i, B_i, A_i (i = 1, 2)$. □

References

[1] A. Arara, M. Benchohra, N. Hamidi and J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Analysis **72** (2010), 580–586.

[2] R. P. Agarwal, M. Benchohra and B. A. Slimani, *Existence results for differential equations with fractional order and impulses*, Mem. Differential Equations Math. Phys. **44** (2008), 1–21.

[3] B. Ahmad and J. J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topological Methods in Nonlinear Analysis **35** (2010), 295–304.

[4] B. Ahmad and J. J. Nieto, *Existence of solutions for impulsive anti-periodic boundary value problems of fractional order*, Taiwanese Journal of Mathematics **15** (3) (2011), 981–993.

[5] B. Ahmad and S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Analysis: Hybrid Systems **3** (2009), 251–258.

[6] M. Benchohra, J. Graef and S. Hamani, *Existence results for boundary value problems with nonlinear frational differential equations*, Applicable Analysis **87** (2008), 851–863.

[7] M. Belmekki, Juan J. Nieto and Rosana Rodriguez-Lopez, *Existence of periodic solution for a nonlinear fractional differential equation*, Boundary Value Problems **2009** (2009), Article ID 324561, doi:10.1155/2009/324561.

[8] M. Belmekki, Juan J. Nieto and Rosana Rodriguez-Lopez, *Existence of solution to a periodic boundary value problem for a nonlinear impulsive fractional differential equation*, Electronic Journal of Qualitative Theory of Differential Equations **16** (2014), 1–27.

[9] M. Benchohra and B. A. Slimani, *Impulsive fractional differential equations*, Electron. J. Differential Equations **10** (2009), 1–11.

[10] R. Caponetto, G. Dongola and L. Fortuna, *Frational order systems Modeling and control applications*, World Scientific Series on nonlinear science, Ser. A, Vol. 72, World Scientific, Publishing Co. Pte. Ltd. Singapore, 2010.

[11] K. Diethelm, *Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution*, J. Eur. Syst. Autom. **42** (2008), 665–676.

[12] W. H. Deng and C. P. Li, *Chaos synchronization of the fractional Lu system*, Physica A **353** (2005), 61–72.

[13] J. Dabas, A. Chauhan, and M. Kumar, Existence of the Mild Solutions for Impulsive Fractional Equations with Infinite Delay, International Journal of Differential Equations 2011 (2011), Article ID 793023, 20 pages.

[14] R. Dehghant and K. Ghanbari, *Triple positive solutions for boundar*, Bulletin of the Iranian Mathematical Society **33** (2007), 1–14.

[15] S. Das and P.K.Gupta, *A mathematical model on fractional Lotka-Volterra equations*, Journal of Theoretical Biology **277** (2011), 1–6.

[16] H. Ergoren and A. Kilicman, *Some Existence Results for Impulsive Nonlinear Fractional Differential Equations with Closed Boundary Conditions*, Abstract and Applied Analysis **2012** (2012), Article ID 387629, 15 pages.

[17] M. Feckan, Y. Zhou and J. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun Nonlinear Sci Numer Simulat **17** (2012), 3050–3060.

[18] L. J. Guo, *Chaotic dynamics and synchronization of fractional-order Genesio-Tesi systems*, Chinese Physics **14** (2005), 1517–1521.

[19] G. L. Karakostas, *Positive solutions for the $\Phi-$Laplacian when Φ is a sup-multiplicative-like function*, Electron. J. Diff. Eqns. **68** (2004), 1–12.
[20] E. Kaufmann, E. Mboumi, *Positive solutions of a boundary value problem for a nonlinear fractional differential equation*, Electronic Journal of Qualitative Theory of Differential Equations, **3** (2008), 1–11.
[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Frational Differential Equations*, Elsevier Science B. V. Amsterdam, 2006.
[22] Y. Liu, *Positive solutions for singular FDES*, U.P.B. Sci. Series A, **73** (2011), 89–100.
[23] Y. Liu, *Solvability of multi-point boundary value problems for multiple term Riemann-Liouville fractional differential equations*, Comput. Math. Appl. **64** (4) (2012), 413–431.
[24] C. Li and G. Chen, *Chaos and hyperchaos in the fractional-order Rossler equations*, Physica A **341** (2004), 55–61.
[25] Z. Liu and X. Li, *Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations, Communications in Nonlinear Science and Numerical Simulation*, **18** (6) (2013), 1362–1373.
[26] Z. Liu, L. Lu and I. Szanto, *Existence of solutions for fractional impulsive differential equations with p-Laplacian operator*, Acta Mathematica Hungarica **141** (3) (2013), 203–219.
[27] J. Mawhin, *Topological degree methods in nonlinear boundary value problems, in: NSFCBMS Regional Conference Series in Math.*, American Math. Soc. Providence, RI, 1979.
[28] G. M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal. **72** (2010), 1604–1615.
[29] K. S. Miller and S. G. Samko, *Completely monotonic functions*, Integr. Transf. Spec. Funct. **12** (2001), 389–402.
[30] A. M. Nakhushev, *The Sturm-Liouville Problem for a Second Order Ordinary Differential equations with fractional derivatives in the lower terms*, Dokl. Akad. Nauk SSSR **234** (1977), 308–311.
[31] J. J. Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions*, Applied Mathematics Letters **23** (2010), 1248–1251.
[32] J. J. Nieto, *Comparison results for periodic boundary value problems of fractional differential equations*, Fractional Differential Equations **1** (2011), 99–104.
[33] N Ozalp and I Koca, *A fractional order nonlinear dynamical model of interpersonal relationships*, Advances in Difference Equations, (2012) 2012, 189.
[34] I. Petras, *Chaos in the fractional-order Volta's system: modeling and simulation*, Nonlinear Dyn. **57** (2009), 157–170.
[35] I. Petras, *Fractional-Order Feedback Control of a DC Motor*, J. of Electrical Engineering **60** (2009), 117–128.
[36] I. Podlubny, Frational Differential Equations, Mathematics in Science and Engineering, Academic Press, San Diego, USA, 1999.
[37] S. Z. Rida, H.M. El-Sherbiny and A. Arafa, *On the solution of the fractional nonlinear Schrodinger equation*, Physics Letters A **372** (2008), 553–558.

[38] M. Rehman and R. Khan, *A note on boundaryvalueproblems for a coupled system of fractional differential equations*, Computers and Mathematics with Applications **61** (2011), 2630–2637.

[39] G. Wang, B. Ahmad and L. Zhang, *Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order*, Nonlinear Analysis **74** (2011), 792–804.

[40] X. Wang, C. Bai, Periodic boundary value problems for nonlinear impulsive fractional differential equations, Electronic Journal of Qualitative Theory and Differential Equations, 3 (2011), 1-15.

[41] X. Wang and H. Chen, *Nonlocal Boundary Value Problem for Impulsive Differential Equations of Fractional Order*, Advances in Difference Equations, (2011) 2011, 404917.

[42] Z. Wei, W. Dong and J. Che, *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative, Nonlinear Analysis: Theory*, Methods and Applications **73** (2010), 3232–3238.

[43] Z. Wei and W. Dong, *Periodic boundary value problems for Riemann-Liouville fractional differential equations, Electronic Journal of Qualitative Theory of Differential Equations*, **87** (2011), 1–13.

[44] J. Wang, H. Xiang and Z. Liu, *Positive Solution to Nonzero Boundary Values Problem for a Coupled System of Nonlinear Fractional Differential Equations*, International Journal of Differential Equations **2010** (2010), Article ID 186928, 12 pages, doi:10.1155/2010/186928.

[45] P. K. Singh and T Som, *Fractional Ecosystem Model and Its Solution by Homotopy Perturbation Method*, International Journal of Ecosystem **2** (5) (2012), 140–149.

[46] Y. Tian and Z. Bai, *Existence results for the three-point impulsive boundary value problem involving fractional differential equations*, Computers and Mathematics with Applications **59** (2010), 2601–2609.

[47] M. S. Tavazoei and M. Haeri, *Chaotic attractors in incommensurate fractional order systems*, Physica D **327** (2008), 2628–2637.

[48] M. S. Tavazoei and M. Haeri, *Limitations of frequency domain approximation for detecting chaos in fractional order systems*, Nonlinear Analysis **69** (2008), 1299–1320.

[49] A. Yang and W. Ge, *Positive solutions for boundary value problems of N-dimension nonlinear fractional differential systems*, Boundary Value Problems, 2008, article ID 437453, doi: 10.1155/2008/437453.

[50] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **252** (2000), 804–812.

[51] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional differential equation*, Electron. J. Diff. Eqns. **36** (2006), 1–12.

[52] X. Zhao and W. Ge, *Some results for fractional impulsive boundary value problems on infinite intervals*, Applications of Mathematics **56** (4) (2011), 371–387.

[53] X. Zhang, X. Huang and Z. Liu, *The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay*, Nonlinear Analysis: Hybrid Systems **4** (2010), 775–781.

[54] Y. Zhao, S. Sun, Z. Han, M. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Applied Mathematics and Computation, 217 (2011), 6950–6958.
[55] Z. Liu, L. Lu and I. Szanto, *Existence of solutions for fractional impulsive differential equations with p-Laplacian operator*, Acta Math. Hungar. **141** (2013), 203–219.

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