LIFTABLE OPERATORS ON SOME BANACH SPACES

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ABSTRACT. In this paper we show that some operators defined on the Banach space with an unconditional basis and $L^1(\mu)$ into a Banach space with the RNP have liftable operators.

1. Introduction

The main purpose of this paper is to establish some conditions under which linear operators on the Banach spaces have some liftable operators. For this question, it is well known fact that every linear operator T on the space ℓ_1 is liftable. We begin our discussion with summary of known results concerning the corresponding problems for continuous linear operators between Banach spaces. Suppose that E, F and G are Banach spaces and q is a surjective linear map of G onto F which maps the closed unit ball in G onto the closed unit ball in F and that T is a bounded linear operator of E into F. When does have a norm preserving lifting, that is, when does there exist a continuous linear mapping $\tilde{T} : E \to G$ such that $\|\tilde{T}\| = \|T\|$ and such that the following diagram commutes with $q \circ \tilde{T} = T$?

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \tilde{T} \downarrow & \nearrow q & \\ G & & \end{array}$$

(1.1)

Grothendieck [3], Pelczyński [7] and Kőthe [5] have established the following results concerning this question.

THEOREM 1.1. *For a fixed Banach space E, such a lifting $\tilde{T}$ exists for arbitrary, F, G, q and T as above diagram if and only if $E = \ell_1(\Gamma)$ for some index set Γ. Moreover, if we add the restrictions that G, F are dual spaces and that q is an adjoint mapping, then $\ell_1(\Gamma)$ can be replaced by an AL-space.*

We now give the notion of the Radon-Nikodým property which is applied to one of our main results. The following definitions are useful to our study.

DEFINITION 1.2. A Banach space E has the Radon-Nikodým property with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $M : \Sigma \to E$ of the bounded variation, there exists $g \in L_1(\mu, E)$ such that $M(X) = \int_X g\, d\mu$ for all $X \in \Sigma$. Also a Banach space has the *Radon-Nikodým property(RNP)* if E has the Radon-Nikodým property with respect to every finite measure space.

For more about the Radon-Nikodým property, the reader can refer to the book [1]. Also we need the definition of the absolutely p -summing operator on a Banach space E.

DEFINITION 1.3. A bounded linear operator $T : E \to F$ is called *absolutely p-summing* if there is a constant $K > 0$ such that, for all finite subset (x_i) in E, we have

(1.2) $(\sum_{i=1}^{n} \|Tx_i\|^p)^{1/p} \le K \sup\{(\sum |x^*(x_i)|^p)^{1/p} : x^* \in B_{X^*}\}$

We will denote by $\pi_p(T)$ the smallest constant $K > 0$ satisfying (1.2). Moreover, we will denote by $\Pi_p(E, F)$ the set of all $p-$summing operators $T : E \to F$. It is known that $1 \le p < \infty$, $\pi_p(T)$ is a norm on $\Pi_p(E, F)$, with which this space is a Banach space. Also the space $\Pi_p(E, F)$ satisfies the following(so-called) "ideal property". If $T : E \to F$ is a $p-$summing, and if $S : G \to E$ and $U : F \to H$ are bounded operators

between Banach spaces, then the composition UTS is a $p-$summing, and we have $\pi_p(UTS) \leq \|U\|\pi_p(T)\|S\|$.

The following theorem gives some equivalent descriptions of absolutely 1-summing operators in [2].

THEOREM 1.4. *Any one of the following statements about a bounded linear operator implies all others.*
a) T is absolutely 1-summing.
b) T maps unconditionally convergent series in E into absolutely convergent series in F.
c) There exists a constant $K > 0$ such that for any finite set $x_1, x_2, \cdots, x_n \in E$ the following inequality obtains;

$$\sum_{i=1}^{n} \|Tx_i\| \leq K \sup\{\sum_{i=1}^{n} |<x_i, x^*>| : x^* \in B_{X^*}\}$$

DEFINITION 1.5. Let $(e_i)_{i \in \Gamma}$ be a family of nonzero elements in a Banach space E. We say that $(e_i)_{i \in \Gamma}$ is an *unconditional basis* for E if $(e_i)_{i \in \Gamma}$ is total in E and if there is a constant λ with the following property: For any finitely supported families of scalars $(\alpha_i)_{i \in \Gamma}$ and $(\beta_i)_{i \in \Gamma}$ such that $|\beta_i| \leq |\alpha_i|$ for all $i \in \Gamma$, we have $\|\sum \beta_i e_i\| \leq \lambda\|\sum \alpha_i e_i\|$.

The "unconditional basis constant" of $(e_i)_{i \in \Gamma}$ is defined as the smallest constant λ satisfying above inequality; it will be denote by $u((e_i)_{i \in \Gamma})$. The unconditional basis constant of E is defined as the infimum of $u((e_i)_{i \in \Gamma})$ over all possible unconditional bases $(e_i)_{i \in \Gamma}$ of E. We will denote it by $ub(E)$. Note that the spaces $\ell_p(\Gamma)$ $(1 \leq p < \infty)$ obviously possess an unconditional basis with unconditional constant 1.

DEFINITION 1.6. A bounded linear operator $T : E \to F$ is called *Pietsch integral* if there exists a $F-$valued countably additive vector measure M of bounded variation defined on the Borel(for the weak*-topology) sets of the closed units ball B_{E^*} of E^* such that for each $x \in E$

$$(1.3) \qquad T(x) = \int_{B_{E^*}} x^*(x) \, dM(x^*).$$

Note that the class of Pietsch integral operators from E into F becomes a Banach space under the norm

$$(1.4) \qquad \|T\|_{pint} = \inf\{|M|(B_{E^*})\}$$

where the infimum is taken over all measures M that satisfy the above definition 1.6. The Banach space of Pietsch integral operators from E into F will be denoted by $P(E, F)$. It is evident that $\|T\| \leq \|T\|_{pint}$.

2. Main results

In this paper, our main questions are based on Theorem 1.1, that is, which bounded linear operators on E into a Banach space F can have the lifting property? Due to Theorem 1.1, $E = \ell_1(\Gamma)$ for some index set Γ only have the lifting property. But if we impose more conditions, we can find a lifting on some Banach and on $L_1(\mu)$. Here any absolutely 1-summing operator on the Banach space E with a unconditional basis can have a liftable operator $\tilde{T}$ of T. The first main theorem is the following:

THEOREM 2.1. *Let E be a Banach space with an unconditional basis $(e_i)_{i \in \Gamma}$. For any Banach spaces F and G, let $T : E \to F$ be an absolutely 1-summing operator. Then for any surjective linear map $q : G \to F$, there exists a liftable operator $\tilde{T} : E \to G$ such that $q \circ \tilde{T} = T$, $\|\tilde{T}\| \leq \lambda \pi_1(T)$ and the following diagram commutes;*

$$
\begin{array}{ccc}
E & \stackrel{T}{\longrightarrow} & F \\
\tilde{T} \downarrow & \nearrow q & \\
G & &
\end{array}
$$

Proof. Let $(e_i)_{i \in \Gamma}$ be the unconditional basis constant and $\lambda = u((e_i)_{i \in \Gamma})$ be the unconditional basis constant. Since $T : E \to F$ is absolutely 1-summing, we have, for any finitely supported families of scalars $(\alpha_i)_{i \in \Gamma}$,

$$
\sum \|T(\alpha_i e_i)\| \leq \pi_1(T) \sup\{\| \sum \theta_i \alpha_i e_i\| : |\theta_i| = 1\}
$$

(2.1)
$$
\leq \lambda \pi_1(T) \| \sum \alpha_i e_i\|
$$

Now define $S : E \to \ell_1(\Gamma)$ by

$$
S(\sum \alpha_i e_i) = (\alpha_i \|T(e_i)\|)_{i \in \Gamma}
$$

and $U : \ell_1(\Gamma) \to F$ by

$$
U((\alpha_i)_{i \in \Gamma}) = \sum \alpha_i \frac{T(e_i)}{\|T(e_i)\|}
$$

with usual convention $\frac{0}{0} = 0$. Then clearly $U \circ S = T$ and $\|U\| \leq 1$.

On the other hand, the norm of S is given by

$$\|S(\sum \alpha_i e_i)\| = \sum |\alpha_i| \|T(e_i)\|$$

(2.2)
$$\leq \lambda \pi_1(T) \| \sum \alpha_i e_i \|, \qquad \text{by (2.1)}$$

Hence we have $\|S\| \leq \lambda \pi_1(T)$. Since $\ell_1(\Gamma)$ has the lifting property, for defined operator $U : \ell_1(\Gamma) \to F$ we define the lifting $\tilde{U} : \ell_1(\Gamma) \to G$ such that $q \circ \tilde{U} = U$ and $\|\tilde{U}\| = \|U\| \leq 1$. Hence to get a desired lifting $\tilde{T}$ of T, define $\tilde{T} : E \to F$ by $\tilde{T} = \tilde{U} \circ S$. Then we have

$$q \circ \tilde{T} = q \circ \tilde{U} \circ S$$
$$= U \circ S$$
$$= T.$$

Moreover the operator norm of $\tilde{T}$ is estimated by

$$\|\tilde{T}\| = \|\tilde{U} \circ S\|$$
$$\leq \|\tilde{U}\| \|S\|$$
$$\leq \lambda \pi_1(T), \qquad \text{by (2.2).}$$

This proves the theorem. □

The theorem 2.1 does not imply the Banach space E with unconditional basis has the lifting property. By the theorem 1.1, if every linear operator on E has the lifting property, then the Banach space E have to be isomorphic to $\ell_1(\Gamma)$ for some index Γ. But above theorem 2.1 is only applies to an absolutely 1-summing operator. In this direction of lifting, we can give another fact that if we take a target Banach space with the RNP, then every linear operator on $L_1(\mu)$ space has the liftable operator as the following theorem 2.4.

DEFINITION 2.2. We say that an operator $T : E \to F$, between Banach spaces, factors through $L_p(\mu)$ if there is a factorization

$$
\begin{array}{ccc}
 & L_p(\mu) & \\
A \nearrow & & \searrow B \\
E & \xrightarrow{\;\;T\;\;} & F
\end{array}
$$

of T through an L_p−space. We let $\tilde{\gamma}_p(T) = \inf\{\|A\|\|B\|\}$ over all such factorizations $T = BA$.

Let us denote by $i_F : F \longrightarrow F^{**}$ the canonical injection. We denote by $\Gamma_p(E, F)$ the space of all the operators $T : E \longrightarrow F$ such that $i_F T$ factors through L_p. Then we let $\gamma_p(T) = \tilde{\gamma}_p(i_F T)$. It is known that $\Gamma_p(E, F)$ equipped with $\gamma_p(T)$ is a Banach space.

LEMMA 2.3. *For any Banach spaces E and F, let $T : E \longrightarrow F$ be a bounded linear operator that factors through $\ell_1(\Gamma)$ for some index Γ . Then T has a lifting linear operator $\tilde{T}$ of T with $\|\tilde{T}\| \leq \gamma_1(T)$.*

Proof. Let G be a Banach space and q be a surjective linear map of G onto F which maps the closed unit ball in G onto the closed unit ball in F. Let T factor through $\ell_1(\Gamma)$ as $T = BA$ where $A : E \to \ell_1(\Gamma)$ and $B : \ell_1(\Gamma) \to F$ are bounded linear operators. Now consider the following factorization's diagram;

$$\ell_1(\Gamma)$$
$$A \nearrow \qquad \searrow B$$
$$E \qquad \overset{T}{\longrightarrow} \qquad F$$

Since $\ell_1(\Gamma)$ has the lifting property, for defined operator $B : \ell_1(\Gamma) \to F$, we can define the lifting $\tilde{B} : \ell_1(\Gamma) \to G$ such that $q \circ \tilde{B} = B$ and $\|\tilde{B}\| = \|B\|$. Hence to get a desired lifting $\tilde{T}$ of T, define $\tilde{T} : E \to F$ by $\tilde{T} = \tilde{B}A$. Then

$$q \circ \tilde{T} = q \circ \tilde{B} \circ A$$
$$= B \circ A$$
$$= T$$

Moreover, the operator norm of $\tilde{T}$ is estimated by $\|\tilde{T}\| \leq \|A\|\|B\|$. Hence taking the infimum over all such factorizations, we can say that $\|\tilde{T}\| \leq \gamma_1(T)$. This proves the lemma.   □

From above lemma, we can find a lifting operator on which was given in [4] as follows;

THEOREM 2.4. *Let (Ω, Σ, μ) be any finite measure space and F be a Banach space with the RNP. Then for any $\epsilon > 0$ and for any Banach space G if $T : L_1(\mu) \to F$ is a bounded linear operator and $q : G \to F$*

is any surjective map, then T has a lifting $\tilde{T} : L_1(\mu) \to G$ such that $\|\tilde{T}\| \le (1 + \epsilon)\|T\|$, and the following diagram commutes;

$$
\begin{array}{ccc}
L_1(\mu) & \xrightarrow{\ T\ } & F \\
\tilde{T} \downarrow & \nearrow q & \\
G & &
\end{array}
$$

Proof. Since F is a Banach space with the RNP, applying to Lewis-Stegall's theorem in [6], every bounded linear operator $T : L_1(\mu) \to F$ admits a factorization $T = BA$ where $A : L_1(\mu) \to \ell_1$ and $B : \ell_1 \to F$ are continuous linear operators, with $\|A\| \le (1 + \epsilon)\|T\|$ and $\|B\| \le 1$. This implies $\gamma_1(T) \le \|A\|\|B\| \le (1 + \epsilon)\|T\|$. Hence by above lemma 2.3, we can find a desired lifting $\tilde{T}$ of T with $\|\tilde{T}\| \le \gamma_1(T) \le (1 + \epsilon)\|T\|$. This proves the theorem. $\qquad\square$

COROLLARY 2.5. *For $1 \le p < \infty$, $F = \ell_p$ spaces and for $1 < p < \infty$, $F = L_p(\mu)$ spaces, if for any Banach space G, $T : L_1(\mu) \to F$ is a bounded linear operator and $q : G \to F$ is a surjective linear map, then T has a liftiable operator.*

Proof. For $1 \le p < \infty$, $F = \ell_p$ spaces have the Radon-Nikodým property and for $1 < p < \infty$, $F = L_p(\mu)$ spaces also have the Radon-Nikodým property. Then by above theorem 2.4, any bounded linear operator on $L_1(\mu)$ to a Banach space F with the RNP has a liftable linear operator. This proves the corollary. $\qquad\square$

Next, we can find a lifting property on $C(K)$ where K is a compact Hausdorff space. In this case, we impose more conditions on operators and target Banach spaces.

THEOREM 2.6. *Let F be a Banach space with the RNP and G be any Banach space. If $T : C(K) \to F$ is an absolutely 1-summing operator and $q : G \to F$ is any onto map, then there is $\tilde{T} : C(K) \to G$ such that $q \circ \tilde{T} = T$ and $\|\tilde{T}\| \le \lambda\pi_1(T)$, for some $\lambda > 0$.*

Proof. Let F be a Banach space with the RNP and $T : C(K) \to F$ be an absolutely 1-summing operator. Then by the Pietch factorization theorem [1, p.164], we can say

$$
\begin{array}{ccc}
C(K) & \xrightarrow{\ T\ } & F \\
J \searrow & & \nearrow S \\
& L^1(\mu) &
\end{array}
$$

where $J : C(K) \to L^1(\mu)$ is just natural inclusion with $\|J\| = \mu(K) = \pi_1(T)$ and $\|S\| \leq 1$.

Then by Theorem 2.4, S has a lifting $\tilde{S} : L^1(\mu) \to G$ such that $\|\tilde{S}\| \leq (1 + \epsilon)\|S\|$ and $q \circ \tilde{S} = S$.

Finally, if we take $\tilde{T} = \tilde{S} \circ J$ as a lifting operator, then we have

$$\begin{aligned}
q \circ \tilde{T} &= q \circ (\tilde{S} \circ J) \\
&= (q \circ \tilde{S}) \circ J \\
&= S \circ J \\
&= T
\end{aligned}$$

and $\|\tilde{T}\| \leq \|\tilde{S}\|\|J\| \leq (1 + \epsilon)\pi_1(T)$. $\qquad\qquad\qquad\square$

In this direction of a liftable operator, we can give another fact that every Pietsch integral operator from E into F with the RNP is liftable. For this proof, we need one of fundamental theorem about absolutely 1-summing operator on $C(K)$ where K is a compact Hausdorff space [1, p162, theorem 3].

THEOREM 2.7. *A bounded linear operator $T : C(K) \to F$ is absolutely 1-summing if and only if its representing measure M is of bounded variation. In this case $\pi_1(T) = |M|(K)$.*

Now one can prove the following theorem by using above theorem 2.7.

THEOREM 2.8. *Let E and F be Banach spaces and $T : E \to F$ be Pietsch integral operator. Then if F has the Radon-Nikodým property, then T has a lifting operator on E.*

Proof. Assume $T : E \to F$ is Pietsch integral and suppose $\epsilon > 0$. By definition of Pietsch integral operator, we can find a $F-$valued countably additive vector measure M on the weak* Borel sets B_{E^*} such that for each $x \in E$,

$$(2.3) \qquad\qquad T(x) = \int_{B_X^*} x^*(x) \, dM(x^*)$$

and

$$(2.4) \qquad\qquad \|T\|_{pint} \leq |M|(B_{E^*}) \leq (1 + \epsilon)\|T\|_{pint}.$$

Define $L : C(B_{E^*}) \to F$ by $L(f) = \int_{B_{E^*}} f \, dM$. Then by the above fundamental theorem 2.7, we can say that L is absolutely 1-summing

operator. Hence in the proof of theorem 2.6, L admits the factorization as following

$$(2.5) \qquad \begin{array}{ccc} C(B_{E^*}) & \xrightarrow{\;L\;} & F \\ {\scriptstyle J}\searrow & & \nearrow{\scriptstyle S} \\ & L^1(\mu) & \end{array}$$

with $\mu(K) = \pi_1(L) = |M|(B_{E^*})$, $\|S\| \leq 1$ and natural injection J. Now let R be the natural injection of E into $C(B_{E^*})$. Then for all $x \in E$

$$(2.6) \qquad \begin{aligned} LR(x) &= \int_{B_{E^*}} x^*(x)\, dM(x^*) \\ &= T(x). \end{aligned}$$

Therefore T admits the factorization

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & F \\ \downarrow R & & \uparrow S \\ C(B_{E^*}) & \xrightarrow{\;J\;} & L_1(\mu) \end{array}$$

with $\|R\| \leq 1$, $\|S\| \leq 1$ and $\|T\|_{pint} \leq \mu(B_{B_{E^*}}) \leq (1+\epsilon)\|T\|_{pint}$.

Since $S : L_1(\mu) \to F$ is a bounded linear operator and F has the RNP. Then by the theorem 2.4, we can find a lifting $\tilde{S}$ of S such that $q\tilde{S} = S$ and $\|\tilde{S}\| \leq (1+\epsilon)\|S\| \leq 1 + \epsilon$.

Finally, once we define a lifting operator $\tilde{T} = \tilde{S} \circ J \circ R$, then

$$\begin{aligned} q \circ \tilde{T} &= q \circ \tilde{S} \circ J \circ R \\ &= S \circ J \circ R \\ &= L \circ R \\ &= T \end{aligned}$$

and

$$\begin{aligned} \|\tilde{T}\| &\leq \|\tilde{S}\|\|J\|\|R\| \quad \text{since } \|T\|_{pint} \geq 1 \\ &\leq (1+\epsilon)\|T\|_{pint}. \end{aligned}$$

This proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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