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STRONG DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF NEW GENERALIZED DERIVATIVE OPERATOR

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Abstract. In this work, certain classes of admissible functions are considered. Some strong differential subordination and superordination properties of analytic functions associated with new generalized derivative operator $\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}$ are investigated. New strong differential sandwich-type results associated with the generalized derivative operator are also given.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots , a \in \mathbb{C} \tag{1}$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H}_1 \equiv \mathcal{H}[1, 1]$, and let $\mathcal{A}$ denote the class of all normalized analytic functions of the form

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$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}). \tag{2}$$

If f and F are members of $\mathcal{H}$ and there exists the Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))(z \in \mathbb{U})$, then we say that f is subordinate to F or F superordinate to f, and we write $f(z) \prec F(z)(z \in \mathbb{U})$. In particular, if F is univalent in $\mathbb{U}$, then $f(z) \prec F(z)$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [9]).

If $f(z)$ of the form (2) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ are two functions in $\mathcal{A}$, then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is denoted by $f(z) * g(z)$ and defined as

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, (z \in \mathbb{U}). \tag{3}$$

For parameters $\alpha_i \in \mathbb{C}(i = 1, ..., q)$, and $\beta_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}(j = 1, ..., s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined as:

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, ..., (\alpha_q)_k}{(\beta_1)_k, ..., (\beta_s)_k} \frac{z^k}{k!},$$

$(q \leq s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U})$,
where $(a)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & k = 0, a \in \mathbb{C}\backslash\{0\}; \\ a(a+1)(a+2)....(a+k-1), & k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

Dziok and Srivastava [6] defined the linear operator

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) = z + \sum_{k=2}^{\infty} \Upsilon_s^q a_k z^k, \tag{4}$$

where

$$\Upsilon_s^q = \frac{(\alpha_1)_{k-1}, ..., (\alpha_q)_{k-1}}{(\beta_1)_{k-1}, ..., (\beta_s)_{k-1}(k-1)!}. \tag{5}$$

Oshah and Darus [15] introduced a function $\mathcal{M}^{\mu}_{\lambda_1,\lambda_2,\ell,d}$ as follows

$$\mathcal{M}^{\mu}_{\lambda_1,\lambda_2,\ell,d}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\nabla^k_{\lambda_2,\ell,d} + \ell\lambda_1(k-1)}{\nabla^k_{\lambda_2,\ell,d}} \right]^{\mu} z^k, \tag{6}$$

where

$$\nabla^k_{\lambda_2,\ell,d} = \ell(1+\lambda_2(k-1)) + d, \tag{7}$$

$\mu, d \in \mathbb{N}_0 = \{0,1,2,...\}$, $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$, and $\ell + d > 0$.

By making use of Hadamard product, we define linear operator $\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{M}^{\mu}_{\lambda_1,\lambda_2,\ell,d}(z) * H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)f(z),$$

then, from (6) and (4), we have

$$\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{\nabla^k_{\lambda_2,\ell,d} + \ell\lambda_1(k-1)}{\nabla^k_{\lambda_2,\ell,d}} \right]^{\mu} \Upsilon^q_s a_k z^k, \tag{8}$$

where Υ^q_s and $\nabla^k_{\lambda_2,\ell,d}$ are defined in (5),(7), respectively.

One can easily verify from (8) that

$$\begin{aligned}\left[\nabla^k_{\lambda_2,\ell,d}\right] \mathfrak{B}^{\mu+1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) = \ell\lambda_1 z \big(\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z)\big)' + \\ \left[\nabla^k_{\lambda_2,\ell,d} - \ell\lambda_1\right] \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z).\end{aligned} \tag{9}$$

Note that, for $\mu = 0$ or $d = 1, \ell = 0$, we obtain

$$\mathfrak{B}^{0,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,0,1} f(z) = H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)f(z)$$

which was introduced and studied by Dziok and Srivastava (see [6]), which includes various other linear operators introduced and studied earlier in the literature. For example, when $q = 2, s = 1$, Hohlov in [7] studied this operator for α_1, α_2, and β_1, also for $\alpha_2 = 1$, this operator becomes the Carlson-Shaffer operator [4], and Ruscheweyh [16] studied this operator for $\alpha_1 = n+1, \alpha_2 = 1$, and $\beta_1 = 1$.

Further, if $q = 2, s = 1, \alpha_1 = n+1, \alpha_2 = 1, \beta_1 = 1$, we get $\mathfrak{B}^{\mu,2,1}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{D}^{n,\mu}_{\lambda_1,\lambda_2,\ell,d} f(z)$ which was introduced and studied by Oshah and Darus (see [15]).

Antonino and Romaguera in [1] have introduced the concept of strongly differential subordination which refered to the generalization of the notion of differential subordination developed by Oros and Oros [13], and

Oros [14] of strong differential subordination and superordination. In the present investigation, by making use of that notion of strong differential subordination, which is indeed an extension version of the theory of differential subordination introduced and developed by Miller and Mocanu [9, 10], we consider certain suitable classes of admissible functions. Here we investigate some strong differential subordination and strong differential superordination properties of analytic functions associated with the new generalized derivative operator, defined above in (8). New strong differential sandwich-type results associated with the generalized derivative operator are also obtained. Using various linear operators, strong differential subordinations were investigated by Jeyaraman et al. [8], Cho [5], and AL-Shaqsi [3] and of course many others.

To prove our results, we need the following definition and theorems considered by Antonino and Romaguera [1, 2], and Oros and Oros [13, 14].

Definition 1.1. ([1, 2, 13]) Let $\mathcal{H}(z,\zeta)$ be analytic in $\mathbb{U}\times\overline{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then, the function $\mathcal{H}(z,\zeta)$ is said to be strongly subordinate to $f(z)$, or $f(z)$ is said to be strongly superordinate to $\mathcal{H}(z,\zeta)$, written as $\mathcal{H}(z,\zeta)\prec\prec f(z)$, if, for $\zeta\in\overline{\mathbb{U}}, \mathcal{H}(z,\zeta)$ as the function of z is subordinate to $f(z)$. We note that $\mathcal{H}(z,\zeta)\prec\prec f(z)$ if and only if $\mathcal{H}(0,\zeta)=f(0)$ and $\mathcal{H}(\mathbb{U}\times\overline{\mathbb{U}})\subset f(\mathbb{U})$.

Definition 1.2. ([9, 13]) Let $\phi:\mathbb{C}^3\times\mathbb{U}\times\overline{\mathbb{U}}\to\mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the (second-order) differential subordination

$$\phi(p(z), zp'(z), zp''(z); z; \zeta)\prec\prec h(z), \tag{10}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solution of the strong differential subordination, or more simply a dominant, if $p(z)\prec q(z)$ for all $p(z)$ satisfying (10). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ for all dominants q(z) of (10) is said to be best dominant.

Recently, Oros [14] introduced the following strong differential superordinations as dual concept of strong differential subordination.

Definition 1.3. ([12,14]) Let $\varphi:\mathbb{C}^3\times\mathbb{U}\times\overline{\mathbb{U}}\to\mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If $p(z)$ and $\phi(p(z), zp'(z), z^2p''(z); z; \zeta)$ are univalent in $\mathbb{U}$ for $\zeta\in\mathbb{U}$ and satisfy the (second-order) strong differential superordination

$$h(z)\prec\prec\phi(p(z), zp'(z), z^2p''(z); z; \zeta), \tag{11}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the strong differential superordination, or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (11). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinantes $q(z)$ of (11) is said to be best subordinant.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{\eta \in \partial\mathbb{U} : \lim_{z \to \eta} q(z) = \infty\right\}$$

and are such that $q'(\eta) \neq 0, \eta \in \partial\mathbb{U} \setminus E(q)$.

Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a), \mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.4. ([13]) Let Ω be a set in $\mathbb{C}, q(z) \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition

$$\psi(r, s, t; z, \zeta) \notin \Omega,$$

whenever $r = q(\eta), s = k\eta q'(\eta)$ and

$$\mathfrak{Re}\left\{\frac{t}{s} + 1\right\} \geq k\mathfrak{Re}\left\{\frac{\eta q''(\eta)}{q'(\eta)} + 1\right\},$$

$(z \in \mathbb{U}, \eta \in \partial\mathbb{U} \setminus E(q), \zeta \in \overline{\mathbb{U}}, k \geq n)$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Definition 1.5. ([14]) Let Ω be a set in $\mathbb{C}$, and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition

$$\psi(r, s, t; \eta, \zeta) \in \Omega,$$

whenever $r = q(z), s = \frac{zq'(z)}{m}$ and

$$\mathfrak{Re}\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m}\mathfrak{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$(z \in \mathbb{U}, \eta \in \partial\mathbb{U}, \zeta \in \overline{\mathbb{U}}, m \geq n \geq 1)$. We write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

In order to prove the main results, we need the following theorem which was proved by Oros and Oros [13].

Theorem 1.1. ([13]) *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), z'p(z), z^2p''(z); z, \zeta) \in \Omega,$$

then $p(z) \prec q(z)$.

Furthermore, Oros [14] proved the following theorem.

Theorem 1.2. ([14]) *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) \in \mathfrak{Q}(a)$ *and*

$$\psi(p(z), z'p(z), z^2p''(z); z, \zeta),$$

is univalent in $\mathbb{U}$ *for* $\zeta \in \overline{\mathbb{U}}$, *then*

$$\Omega \subset \{\psi(p(z), z'p(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\}$$

implies that $q(z) \prec p(z)$.

2. The Main Subordination Result

First, we prove the subordination theorem by using the derivative operator $\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)$. For this purpose, we need the following class of admissible functions.

Definition 2.1. Let Ω be a set in $\mathbb{C}, \lambda_2 \geq \lambda_1 > 0, \ell > 0, \mu \geq 1, d \in \mathbb{N}_0$, and $q(z) \in \mathfrak{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{\mathfrak{B}}[\Omega, q]$ consists of those functions

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition

$$\phi(u, v, w; z, \zeta) \notin \Omega,$$

whenever

$$u = q(\eta), v = \frac{k\eta q'(\eta) + (\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1)q(\eta)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}},$$

and

$$\Re e\left\{\frac{(\nabla^k_{\lambda_2,\ell,d})^2 w-(\nabla^k_{\lambda_2,\ell,d}-\ell\lambda_1)^2 u}{\ell\lambda_1\left[\nabla^k_{\lambda_2,\ell,d})v-(\nabla^k_{\lambda_2,\ell,d}-\ell\lambda_1)u\right]}-2\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)\right\}\geq k\Re e\left\{\frac{\eta q''(\eta)}{q'(\eta)}+1\right\},$$

$(z\in\mathbb{U},\eta\in\partial\mathbb{U},\zeta\in\overline{\mathbb{U}},d\in\mathbb{N}_0;k\geq 1)$.

THEOREM 2.1. *Let $\phi\in\Phi_{\mathfrak{B}}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies*

(12)
$$\{\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z),\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z),\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z);z,\zeta\right):z\in\mathbb{U},\zeta\in\overline{\mathbb{U}}\}\subset\Omega,$$

then

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)\prec q(z),\quad(z\in\mathbb{U}).$$

Proof. From (9), we can see

$$\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)=\frac{z\left(\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}},\tag{13}$$

and hence

$$\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)=\frac{z\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}}.\tag{14}$$

Define the function p in $\mathbb{U}$ by

$$p(z)=\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z).\tag{15}$$

Making use of (14) and (15), we get

$$\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)=\frac{zp'(z)+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)p(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}}.\tag{16}$$

Also, making use of (13) and (15), and simple calculation we get

(17)
$$\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)=\frac{z^2p''(z)+\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)zp'(z)+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2p(z)}{\left(\frac{\nabla^k(\lambda_2,\ell,d)}{\ell\lambda_1}\right)^2}.$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

(18)
$$u=r,v=\frac{s+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)r}{\frac{\nabla^k(\lambda_2,\ell,d)}{\ell\lambda_1}},w=\frac{t+\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)s+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2r}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2}.$$

Let
$$\psi(r,s,t;z,\zeta)=\phi(u.v,w;z,\zeta)=$$

(19)
$$\phi\big(r,\frac{s+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)r}{\frac{\nabla^k(\lambda_2,\ell,d)}{\ell\lambda_1}},\frac{t+\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)s+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2r}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2};z,\zeta\big).$$

Using (15), (16) and (17), from (19) we obtain

(20)
$$\psi(p(z),zp'(z),z^2p''(z);z,\zeta)=\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z),\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z),\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z);z,\zeta\right).$$

Hence, (12) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z,\zeta)\in\Omega. \tag{21}$$

We note that

$$\frac{t}{s}+1=\frac{(\nabla^k_{\lambda_2,\ell,d})^2w-(\nabla^k_{\lambda_2,\ell,d}-\ell\lambda_1)^2u}{\ell\lambda_1\left[\nabla^k_{\lambda_2,\ell,d}v-(\nabla^k_{\lambda_2,\ell,d}-\ell\lambda_1)u\right]}, \tag{22}$$

and so the admissibility condition for $\phi\in\Phi_{\mathfrak{B}}[\Omega,q]$ in Definition 2.1 is equivalent to the admissibility condition for $\psi\in\Psi[\Omega,q]$. Therefore, by Theorem 1.1, we have $p(z)\prec q(z)$ or equivalently

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)\prec q(z), \tag{23}$$

which evidently completes the proof of Theorem 2.1. □

Next, if we consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathfrak{B}}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathfrak{B}}[h, q]$. The following result is an immediate consequence of Theorem 2.1.

THEOREM 2.2. *Let $\phi \in \Phi_{\mathfrak{B}}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta\right) \prec\prec h(z), \tag{24}$$

then

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \prec q(z). \tag{25}$$

Our next result is an extension of Theorem 2.1 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

COROLLARY 2.1. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 1$. Let $\phi \in \Phi_{\mathfrak{B}}[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies*

$$\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta) \in \Omega,$$

then

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \prec q(z). \tag{26}$$

Proof. From Theorem 2.1, we see $\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \prec q_\rho(z)$, and the result is deduced from $q_\rho(z) \prec q(z)$. □

THEOREM 2.3. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfy one of the following conditions:*

(*i*): *$\phi \in \Phi_{\mathfrak{B}}[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
(*ii*): *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathfrak{B}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (24), *then*

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \prec q(z). \tag{27}$$

Proof. The proof is similar to the one in [11, Theorem 2.3d, page 30] and therefore is omitted. □

Now, our next results give the best dominant of the strong differential subordination (24).

THEOREM 2.4. *Let h be univalent in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}}$. Suppose that the differential equation*

$$\phi\Bigg(q(z), \frac{zq'(z) + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right) q(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}}, \frac{z^2p''(z) + \left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right) zp'(z) + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)^2 p(z)}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2}; z, \zeta\Bigg) = h(z) \tag{28}$$

has a solution q with $q(0) = 0$ and satisfies one of the following conditions:

- ***(i):*** *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_{\mathfrak{B}}[h, q]$, or*
- ***(ii):*** *q is univalent in $\mathbb{U}$ and $\phi \in \Phi_{\mathfrak{B}}[h, q_\rho]$, for some $\rho \in (0, 1)$, or*
- ***(iii):*** *q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathfrak{B}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (24) *and*

$$\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z); z, \zeta)$$

is analytic in $\mathbb{U}$, then

$$\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) \prec q(z)$$

and q is the best dominant.

Proof. Using the same method given by [11, Theorem 2.3e, p.31], we deduce that from Theorems 2.2 and 2.3, q is a dominant of (24). Since q satisfies (28), q is also a solution of (24) and therefore q will be dominated by all dominants. Hence q should be the best dominant of (24). □

In the particular case, $q(z) = Mz, M > 0$, and in view of Definition 2.1, the class of admissible function $\Phi_{\mathfrak{B}}[h(\mathbb{U}), q]$, denoted by $\Phi_{\mathfrak{B}}[h(\mathbb{U}), M]$, is described below.

Definition 2.2. Let Ω be a set in $\mathbb{C}, \lambda_2 \geq \lambda_1 > 0, \ell > 0, \mu \geq 1, d \in \mathbb{N}_0$ and $M > 0$. The class of admissible functions $\Phi_{\mathfrak{B}}[\Omega, M]$ consists of those functions

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C},$$

such that

$$\phi\left(Me^{i\theta}, \frac{k + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}} Me^{i\theta}, \right.$$
$$\left. \frac{L + \left[\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right) k + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)^2\right] Me^{i\theta}}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2}; z, \zeta\right) \notin \Omega, \tag{29}$$

whenever $z \in \mathbb{U}, \eta \in \overline{\mathbb{U}}, \Re e\{Le^{-i\theta}\} \geq (k-1)kM, \theta$ is real number, and $k \geq 1$.

Corollary 2.2. *Let $\phi \in \Phi_{\mathfrak{B}}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta) \in \Omega,$$

then

$$|\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)| < M. \tag{30}$$

For the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathfrak{B}}[\Omega, M]$ is simply denoted by $\Phi_{\mathfrak{B}}[M]$.

Corollary 2.3. *Let $\phi \in \Phi_{\mathfrak{B}}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta)\right| < M,$$

then

$$|\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)| < M. \tag{31}$$

Corollary 2.4. *Let $\lambda_2 \geq \lambda_1 > 0, \ell > 0, M > 0$, and let $C(\eta)$ be an analytic function in $\overline{\mathbb{U}}$ with $\Re e\{\zeta C(\eta)\} \geq 0$ for $\zeta \in \partial\mathbb{U}$. If $f \in \mathcal{A}$*

satisfies

$$\Big|\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)-\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)-\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)+C(\eta)\Big|<\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)M,$$

then

$$|\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)|<M. \tag{32}$$

Proof. Let

$$\phi(u,v,w;z,\eta)=\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2w-\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}v-\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2u+C(\eta),$$

and $\Omega=h(\mathbb{U})$, where $h(z)=(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1)Mz$. Before using Corollary 2.2, we need to show that $\phi\in\Phi_{\mathfrak{B}}[\Omega,M]$, that means the admissible condition (29) is satisfied. This follows since

$$\Bigg|\phi\Bigg(Me^{i\theta},\frac{k+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}}Me^{i\theta},\frac{L+\left[\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)k+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2\right]Me^{i\theta}}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2};z,\zeta\Bigg)\Bigg|$$

$$=\Big|L+\left[\left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)k+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2\right]Me^{i\theta}-\left(k+\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)Me^{i\theta}-\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)^2Me^{i\theta}+C(\eta)\Big|$$

$$=\Big|L+\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}-1\right)(2k-1)Me^{i\theta}+C(\eta)\Big|$$

$$\geq \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)(2k-1)kM + \mathfrak{Re}\{Le^{-i\theta}\} + \mathfrak{Re}\{C(\eta)e^{-i\theta}\}$$

$$\geq k(k-1)M + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)(2k-1)M + \mathfrak{Re}\{C(\eta)e^{-i\theta}\}$$

$$\geq \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)M,$$

whenever $z \in \mathbb{U}, \eta \in \overline{\mathbb{U}}, \mathfrak{Re}\{Le^{-i\theta}\} \geq (k-1)kM, \theta$ is real number, and $k \geq 1$. Hence, the required result now follows from Corollary 2.2. □

3. Superordination and Sandwich Results

In this section, the dual problem of strong differential subordination (that is, strong differential superordination of the differential operator $\mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)$) is investigated. We will also give sandwich-type results, but first we will define the class of admissible functions as follows:

Definition 3.1. Let Ω be a set in $\mathbb{C}, q \in \mathcal{H}_0$ with $zq'(z) \neq 0, \lambda_2 \geq \lambda_1 > 0, \ell > 0, \mu \geq 1, d \in \mathbb{N}_0$. The class of admissible functions $\Phi'_{\mathfrak{B}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; \eta, \zeta) \in \Omega$ whenever

$$u = q(z), v = \frac{zq'(z)/m + (\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1)q(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}},$$

and

$$\mathfrak{Re}\left\{\frac{(\nabla^k_{\lambda_2,\ell,d})^2 w - (\nabla^k_{\lambda_2,\ell,d} - \ell\lambda_1)^2 u}{\ell\lambda_1\left[\nabla^k_{\lambda_2,\ell,d}v - (\nabla^k_{\lambda_2,\ell,d} - \ell\lambda_1)u\right]}\right\} \leq \frac{1}{m}\mathfrak{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in \mathbb{U}, \eta \in \partial\mathbb{U}, \zeta \in \overline{\mathbb{U}}$, *and* $m \geq 1$.

Theorem 3.1. *Let* $\phi \in \Phi'_{\mathfrak{B}}[\Omega, q]$. *If* $f \in \mathcal{A}, \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \in \mathfrak{Q}_0$, *and*

$$\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta)$$

is univalent in $\mathbb{U}$*, then*

(33)
$$\Omega \subset \{\phi(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta) : z \in \mathbb{U}, \eta \in \overline{\mathbb{U}}\},$$

which implies that

$$q(z) \prec \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z). \tag{34}$$

Proof. For p defined by (15) and ϕ by (19), the equations (20) and (33) yield

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\}.$$

From (18), the admissibility condition for $\phi \in \Phi'_{\mathfrak{B}}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence $\psi \in \Psi'_{\mathfrak{B}}[\Omega, q]$ and by Theorem 1.2, $q(z) \prec p(z)$ or equivalently $q(z) \prec \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z)$. □

Similar to the previous section, if we consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathfrak{B}}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathfrak{B}}[h, q]$. The following result is an immediate consequence of Theorem 3.1.

THEOREM 3.2. *Let* h *be analytic in the open unit disk* $\mathbb{U}$*,*$q \in \mathcal{H}_0$*, and we let* $\phi \in \Phi'_{\mathfrak{B}}[h, q]$*. If* $f \in \mathcal{A}$*,* $\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z) \in \mathcal{Q}_0$*, also*

$$\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta\right) \tag{35}$$

is univalent in $\mathbb{U}$*, then*

$$h(z) \prec\prec \phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z); z, \zeta\right), \tag{36}$$

which implies that

$$q(z) \prec \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d}f(z). \tag{37}$$

Theorems 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (33) or (36).

The following theorem proves the existence of the best subordinant of (36) for an appropriate ϕ.

THEOREM 3.3. *Let h be analytic in the open unit disk $\mathbb{U}$, and $\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\Bigg(q(z), \frac{zq'(z) + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right) q(z)}{\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}}, \frac{z^2p''(z) + \left(\frac{2\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right) zp'(z) + \left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1} - 1\right)^2 p(z)}{\left(\frac{\nabla^k_{\lambda_2,\ell,d}}{\ell\lambda_1}\right)^2}; z, \zeta\Bigg) = h(z)$$

has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_{\mathfrak{B}}[h, q]$, and

$$\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z); z, \zeta\right),$$

which implies that

$$q(z) \prec \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z)$$

and q is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4, and so it is being omitted here. □

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type result.

THEOREM 3.4. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, and let h_2 be analytic function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi'_{\mathfrak{B}}[h_1, q_1] \cap \Phi_{\mathfrak{B}}[h_2, q_2]$. If $f \in \mathcal{A}$, $\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) \in \mathcal{H}_0 \cap \mathcal{Q}_0$ and*

$$\phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu-1,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z), \mathfrak{B}^{\mu,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z); z, \zeta\right) \prec\prec h_2(z),$$

which implies that

$$q_1(z) \prec \mathfrak{B}^{\mu-2,q,s}_{\lambda_1,\lambda_2,\ell,d} f(z) \prec q_2(z).$$

References

[1] J. A. Antonino and S. Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equation **114** (1994), 101–105.

[2] J. A. Antonino, *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc. **43** (2006), 311–322.

[3] K. AL-Shaqsi, *Strong differential subordinations obtained with new integral operator defined by polylogarithm function*, Internat. J. Math. Math. Sci. **2014**, Article ID 260198 (2014).

[4] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM Journal on Mathematical Analysis **15** (1984), 737–745.

[5] N. E. Cho, *Strong differential subordination properties for analytic functions involving the Komatu integral operator*, Boundary Value Problems **2013**, Article ID 44 (2013).

[6] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric functiction*, Integral Transforms and Special Functions **14** (2003), 7–18.

[7] Y. E. Hohlov, *Operators and operations in the class of univalent functions*, Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika **10** (1978), 83–89 (Russian).

[8] M. P. Jeyaramana, T. K. Suresh and E. K. Reddy, *Strong differential subordination and superordination of analytic functions associated with Komatu operator*, Int. J. Nonlinear Anal. Appl. 4 (2013) (2), 26–44.

[9] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–172.

[10] S. S. Miller and P. T. Mocanu, *On some classes of first order differential subordinations*, Michigan Math. J. **32** (1981), 185–195.

[11] S. S. Miller and P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Pure and Applied Mathematics **225**, Marcel Dekker, New York, 2000.

[12] S. S. Miller and P. T. Mocanu, *Subordinates of differential superordinations*, Complex Var. Elliptic Equ. **48** (2003), 815–826.

[13] G. I. Oros and G. Oros, *Strong differential subordination*, Turkish J. Math. **33** (2009), 249–257.

[14] G. Oros, *Strong differential superordination*, Acta Universitatis Apulensis **19** (2009), 101–106.

[15] A. Oshah and M. Darus, *Differential sandwich theorems with a new generalized derivative operator*, Advances in Mathematics: Scientific J. **3** (2014), 117–124.

[16] S. Ruscheweyh, *A new criteria for univalent function*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

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