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EXISTENCE OF SOLUTION FOR A FRACTIONAL DIFFERENTIAL INCLUSION VIA NONSMOOTH CRITICAL POINT THEORY

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Abstract. This paper is concerned with the existence of solutions to the following fractional differential inclusion

$$\begin{cases} -\frac{d}{dx}\left(p\ {}_0D_x^{-\beta}(u'(x)) + q\ {}_xD_1^{-\beta}(u'(x))\right) \in \partial F_u(x,u), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where ${}_0D_x^{-\beta}$ and ${}_xD_1^{-\beta}$ are left and right Riemann-Liouville fractional integrals of order $\beta \in (0,1)$ respectively, $0 < p = 1 - q < 1$ and $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with respect to the second variable. Due to the general assumption on the constants p and q, the problem does not have a variational structure. Despite that, here we study it combining with an iterative technique and nonsmooth critical point theory, we obtain an existence result for the above problem under suitable assumptions. The result extends some corresponding results in the literatures.

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1. Introduction

Fractional differential equations and inclusions have been proved that they are very useful tools in modeling of many phenomena in various fields of science and engineering, such as, viscoelasticity, electrochemistry, electromagnetism, economics, optimal control and so forth. For details and examples, see [1,3,4,6,7,11,16,19] and the references therein. In consequence, more and more attention has been paid to fractional differential equations and inclusions.

The study of fractional differential inclusions was initiated by Sayed and Ibrahim, see [21]. Very recently several qualitative results for fractional differential inclusions were obtained in [2, 5, 11, 14, 23] and the references therein. Especially, in [23], Teng et al. considered the fractional differential inclusion

(1.1)
$$\begin{cases} -\frac{d}{dx}\left(\frac{1}{2}\ {}_0D_x^{-\beta}(u'(x)) + \frac{1}{2}\ {}_xD_1^{-\beta}(u'(x))\right) \in \partial F_u(x,u), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

by using nonsmooth mountain pass theorem and nonsmooth symmetric mountain pass theorem, they derived the existence and multiplicity of solutions, where ${}_0D_x^{-\beta}$ and ${}_xD_1^{-\beta}$ are left and right Riemann-Liouville fractional integrals of order $\beta \in (0,1)$ respectively, defined by

$${}_0D_x^{-\beta}u = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}u(s)ds, \quad {}_xD_1^{-\beta}u = \frac{1}{\Gamma(\beta)}\int_x^1 (s-x)^{\beta-1}u(s)ds.$$

Obviously, in (1.1), the coefficient $\frac{1}{2}$ is very special. So a natural question is what will happen for the existence with coefficient p and q, which only satisfy $p+q=1$?

We will give a positive answer in present paper, so we attempt to use nonsmooth mountain pass theorem and iterative technique to study the existence of nontrivial solutions of fractional differential inclusion

(1.2)
$$\begin{cases} -\frac{d}{dx}\left(p\ {}_0D_x^{-\beta}(u'(x)) + q\ {}_xD_1^{-\beta}(u'(x))\right) \in \partial F_u(x,u), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < p = 1-q < 1$, for $s \in \mathbb{R}, F(\cdot, s)$ is measurable, and for a.e. $x \in [0,1], F(x,\cdot)$ is locally Lipschitz, $\partial F_s(x,s)$ denotes the generalized subdifferential in the sense of Clarke [9].

In order to use variational method, we consider a family fractional differential inclusions with variational structure, that is, for given $w \in H_0^\alpha(0,1)$, we discuss the following problem

$$(1.3)\quad \begin{cases} -\frac{d}{dx}\Big(q\ {}_0D_x^{-\beta}(u'(x)) + q\ {}_xD_1^{-\beta}(u'(x)) \\ \qquad\qquad +(p-q)\ {}_0D_x^{-\beta}(w'(x))\Big) \in \partial F_u(x,u),\ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

which can be solved by variational method. Then, for each $w \in H_0^\alpha(0,1)$, we find a solution $u_w \in H_0^\alpha(0,1)$ with some bounds. Next, by iterative technique, one gets the existence of solutions of (1.2) under suitable assumptions.

This paper is organized as follows. In Section 2, we recall some basic knowledge of nonsmooth analysis and abstract results which we are going to apply. Section 3 is devoted to present the preliminaries about fractional calculus to derive our result, and list the assumptions on the problem and state our main result. In the final Section, we give the proof of the main result. The result extends that in [22, 23].

2. Nonsmooth analysis

We collect some basic notions and results of nonsmooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [9], Motreanu and Panagiotopoulos [18], Chang [10], Gasinki and Papageorgiou [13].

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \to \mathbb{R}$ be a functional. We recall that φ is locally Lipschitz (l.L.) if for $u \in X$, there exist a neighborhood U of u and a real number $K_U > 0$ such that

$$|\varphi(v) - \varphi(\omega)| \le K_U \|v - \omega\|_X \quad \text{for} \quad v, \omega \in U.$$

If φ is l.L. and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^0(u; v) = \overline{\lim_{\substack{\omega \to u \\ t \downarrow 0}}} \frac{\varphi(\omega + tv) - \varphi(\omega)}{t},$$

and the generalized gradient of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \le \varphi^0(u; v) \ \text{ for } \ v \in X\}.$$

Then for $u \in X, \partial\varphi(u) \in 2^{X^*}\backslash\{\emptyset\}$ is a convex and weakly*-compact subset [9, Proposition 1].

LEMMA 2.1. ([18, Proposition 1.1]) *If $\varphi \in C^1(X, \mathbb{R})$, then φ is l.L. and*

$$\varphi^0(u; v) = \langle \varphi'(u), v \rangle, \quad \partial\varphi(u) = \{\varphi'(u)\}, \quad u, v \in X.$$

LEMMA 2.2. ([18, Proposition 1.3]]) *Let $\varphi : X \to \mathbb{R}$ be a l.L. functional. Then for $u \in X$, $\varphi^0(u; \cdot)$ is subadditive and positively homogeneous and*

$$\varphi^0(u; v) \leq K_U \|v\|, \quad v \in X$$

with $K_U > 0$ being a Lipschitz constant for φ around u.

Assume φ is a l.L. functional defined on Banach space X, set

$$\lambda(u) = \min\{\|u^*\|_{X^*} : \ u^* \in \partial\varphi(u)\}, \qquad u \in X,$$

then $\lambda(u)$ exists and is lower semi-continuous [10, 13]. $u \in X$ is said to be a critical point of φ if $0 \in \partial\varphi(u)$.

A l.L. functional $\varphi : X \to \mathbb{R}$ is said to satisfy the non-smooth (PS) condition at level $c \in \mathbb{R}$, if any sequence $\{u_n\} \subset X$ with $\varphi(u_n) \to c$ and $\lambda(u_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence [10, 13].

3. Fractional calculus and main result

For convenience, hereafter, we denote $\alpha = 1 - \frac{\beta}{2}$. In view of $\beta \in (0, 1)$, we have $\alpha \in (\frac{1}{2}, 1)$. The fractional Sobolev space $H_0^\alpha(0, 1)$ is defined as the completion of $C_0^\infty(0, 1)$ under the norm

$$\|u\|_\alpha = \|{}_0D_x^\alpha u\|_{L^2}.$$

From [12, Theorem 2.13], we know $H_0^\alpha(0, 1)$ is a reflexive Banach space, and $H_0^\alpha(0, 1) \hookrightarrow C[0, 1]$ is compact, moreover, if $u \in H_0^\alpha(0, 1)$, then $u(0) = u(1) = 0$.

For the space $H_0^\alpha(0, 1)$, we have the following results.

LEMMA 3.1. *([22, Lemma 2.2]) If $u \in H_0^\alpha(0, 1)$, then*

$$\|u\|_\infty \leq \frac{1}{\Gamma(\alpha)(2\alpha - 1)^{\frac{1}{2}}} \|u\|_\alpha, \quad \|u\|_{L^2} \leq \frac{1}{\Gamma(\alpha + 1)} \|u\|_\alpha,$$

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq - \int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2,$$

$$\int_0^1 |{}_xD_1^\alpha u|^2 dx \leq \frac{1}{|\cos(\pi\alpha)|^2}\|u\|_\alpha^2.$$

DEFINITION 3.2. By a weak solution of problem (1.2), it is understood an element $u \in H_0^\alpha(0,1)$ for which there corresponds to a mapping $[0,1] \ni x \mapsto u^*(x)$ with $u^*(x) \in \partial F_u(x, u(x))$ for a.e. $x \in [0,1]$, and having the property that for every $v \in H_0^\alpha(0,1)$, $u^*v \in L^1[0,1]$ and

$$-p\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha v dx - q\int_0^1 {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u dx = \int_0^1 u^*(x)v(x)dx.$$

Similarly, a function $\tilde{u} \in H_0^\alpha(0,1)$ is called a weak solution of problem (1.3) if there exists a corresponding mapping $[0,1] \ni x \mapsto \tilde{u}^*(x)$ with $\tilde{u}^*(x) \in \partial F_{\tilde{u}}(x, \tilde{u}(x))$ for a.e. $x \in [0,1]$, and having the property that for every $v \in H_0^\alpha(0,1)$, $\tilde{u}^*v \in L^1[0,1]$ and

$$-q\int_0^1 \big({}_0D_x^\alpha \tilde{u} \cdot {}_xD_1^\alpha v + {}_0D_x^\alpha v \cdot {}_xD_1^\alpha \tilde{u}\big)dx \\ - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha v dx = \int_0^1 \tilde{u}^*(x)v(x)dx.$$

DEFINITION 3.3. A function $u \in H_0^\alpha(0,1)$ is called a solution of problem (1.2) if $p\,{}_0D_x^{2\alpha-1}u - q\,{}_xD_1^{2\alpha-1}u$ is derivable with respect to $x \in (0,1)$ and

$$-\frac{d}{dx}(p\,{}_0D_x^{2\alpha-1}u - q\,{}_xD_1^{2\alpha-1}u) = u^*, \quad a.e.\ x \in (0,1),$$

where $u^*(x) \in \partial F_u(x, u(x))$ for a.e. $x \in [0,1]$.

LEMMA 3.4. *If $u \in H_0^\alpha(0,1)$ is a weak solution of problem* (1.2), *then u is a solution of problem* (1.2).

Proof. Suppose $u \in H_0^\alpha(0,1)$ is a weak solution of problem (1.2), then there exists $u^* \in \partial F_u(x,u)$ satisfying

$$(3.1)\quad -p\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha v dx - q\int_0^1 {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u dx = \int_0^1 u^*(x)v(x)dx$$

for all $v \in H_0^\alpha(0,1)$. Similar to the argument of [15, Theorem 4.2], we can get

$$\int_0^1 \Big(p\,{}_0D_x^{2\alpha-1}u - q\,{}_xD_1^{2\alpha-1}u + \int_0^x u^*(s)ds\Big)v'(x)dx = 0$$

for all $v \in C_0^\infty(0,1)$. So there exists constant C, such that

$$p\ {}_0D_x^{2\alpha-1}u - q\ {}_xD_1^{2\alpha-1}u + \int_0^x u^*(s)ds = C,$$

and then

$$-\frac{d}{dx}(p\ {}_0D_x^{2\alpha-1}u - q\ {}_xD_1^{2\alpha-1}u) = u^* \quad a.e.\ x \in (0,1).$$

According to Definition 3.3, we know that u is a solution of problem (1.2). □

We impose F the following conditions.

(F1) For $s \in \mathbb{R}$, the function $x \mapsto F(x,s)$ is measurable, for a.e. $x \in [0,1]$, $s \mapsto F(x,s)$ is l.L. and $F(x,0) = 0$;

(F2) there exist $a, b \in L^1([0,1], \mathbb{R}_+)$ and $r \in [1,\infty)$, such that

$$|s^*| \leq a(x) + b(x)|s|^{r-1}, \quad \text{a.e. } x \in [0,1], \quad s \in \mathbb{R} \text{ and } s^* \in \partial F_s(x,s);$$

(F3) there exist $\mu \in (0, \frac{1}{2}), c_0 > 0$ and $M > 0$, such that

$$c_0 < F(x,s) \leq -\mu F^0(x,s;-s) \text{ for a.e. } x \in [0,1], \text{and } s \in \mathbb{R} \text{ with } |s| \geq M;$$

(F4) for $s^* \in \partial F_s(x,s)$, $\lim\limits_{s\to 0} \frac{s^*}{s} = 0$ in a.e. $x \in [0,1]$.

Remark 3.5. Noting that from conditions (F1), (F2) and (F4), using the Lebourg's mean value theorem, we obtain that for given $\epsilon > 0$, there exist $\tilde{b} \in L^1([0,1], \mathbb{R}_+), \eta > 2$, such that

$$|F(x,s)| \leq \epsilon|s|^2 + \tilde{b}(x)|s|^\eta, \quad x \in [0,1], \ s \in \mathbb{R}.$$

Remark 3.6. From conditions (F1), (F2) and the Lebourg's mean value theorem, one has

$$|F(x,s)| \leq a(x)|s| + b(x)|s|^r, \ |F^0(x,s;-s)| \leq a(x)|s| + b(x)|s|^r, \ x \in [0,1].$$

Lemma 3.7. *Assume conditions (F1)(F3) hold, then*

$$F(x,s) \geq \tilde{c}(x)\left(\frac{|s|}{M}\right)^{\frac{1}{\mu}}, \quad x \in [0,1] \setminus \mathcal{N}, |s| \geq M,$$

where $\mathcal{N}$ is the Lebesgue-null set outside which the hypothesis (F3) uniformly holds, and

$$\tilde{c}(x) = \min\{F(x,M),\ F(x,-M)\}, \quad x \in [0,1] \setminus \mathcal{N}, \tag{3.2}$$

clearly, $\tilde{c}(x) \geq c_0$ for a.e. $x \in [0,1]$.

Proof. For given $s \in \mathbb{R}$ with $|s| \geq M$, set

$$\mathcal{F}(x,\lambda) = F(x,\lambda s), \quad x \in [0,1] \setminus \mathcal{N}, \quad \lambda \in \mathbb{R},$$

then $\mathcal{F}(x,\cdot)$ is l.L.. Via Rademarchers theorem, we see that $\lambda \mapsto \mathcal{F}(x,\lambda)$ is differentiable a.e. on $\mathbb{R}$, and at a point of differentiability $\lambda \in \mathbb{R}$, it gets $\frac{d}{d\lambda}\mathcal{F}(x,\lambda) = \partial\mathcal{F}_\lambda(x,\lambda)$. Moreover, it follows from Chain rule that $\partial\mathcal{F}_\lambda(x,\lambda) = s\partial F_\xi(x,\xi)|_{\lambda s}$, hence $\lambda\partial\mathcal{F}_\lambda(x,\lambda) = \lambda s\partial F_\xi(x,\xi)|_{\lambda s}$. At a point of differentiability, condition (F3) reduces to

$$\mu F_s(x,s)s \geq F(x,s), \quad x \in [0,1] \setminus \mathcal{N}, \quad |s| \geq M,$$

so one presents

$$\frac{\lambda d\mathcal{F}(x,\lambda)}{d\lambda} \geq \frac{1}{\mu}\mathcal{F}(x,\lambda),$$

i.e.

$$\frac{\frac{d}{d\lambda}\mathcal{F}(x,\lambda)}{\mathcal{F}(x,\lambda)} \geq \frac{1}{\lambda\mu}.$$

Integrating from 1 to $\lambda_0 (\lambda_0 \geq 1)$, it gives $\ln \frac{\mathcal{F}(x,\lambda_0)}{\mathcal{F}(x,1)} \geq \ln \lambda_0^{\frac{1}{\mu}}$, hence $\mathcal{F}(x,\lambda_0) \geq \lambda_0^{\frac{1}{\mu}}\mathcal{F}(x,1)$, that is $F(x,\lambda_0 s) \geq \lambda_0^{\frac{1}{\mu}} F(x,s)$. Thus, for $x \in [0,1] \setminus \mathcal{N}$, $|s| \geq M$, we have

$$F(x,s) = F\left(x, \frac{|s|}{M}\frac{Ms}{|s|}\right) \geq \left(\frac{|s|}{M}\right)^{\frac{1}{\mu}} F(x, \frac{Ms}{|s|}) \geq \tilde{c}(x)\left(\frac{|s|}{M}\right)^{\frac{1}{\mu}}.$$

□

For problem (1.2), since the symmetric position of the constants p and q lying in, without loss of generality, one can assume that $p \geq q$.

The functional $I_w : H_0^\alpha(0,1) \mapsto \mathbb{R}$ corresponding to the problem (1.3) is defined by

$$I_w(u) = -q\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u dx - \int_0^1 F(x,u)dx.$$

Proposition 3.8. *Assume that F satisfies the hypotheses (F1),(F2), then the functional $I_w : H_0^\alpha(0,1) \mapsto \mathbb{R}$ is l.L., and every critical point $u \in H_0^\alpha(0,1)$ of I_w is a solution of the problem* (1.3).

Proof. Let $I_w(u) = I_1(u) + I_2(u)$, where

$$I_1(u) = -q\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u dx,$$

$$I_2(u) = -\int_0^1 F(x,u)dx.$$

Clearly $I_1 \in C^1(H_0^\alpha(0,1),\mathbb{R})$. By Lemma 2.1, I_1 is l.L. on $H_0^\alpha(0,1)$. From condition (F2), one knows that I_2 is l.L. on $L^r[0,1]$. Moreover $H_0^\alpha(0,1)$ is compactly embedded into $L^r[0,1]$. So I_2 is also l.L. on $H_0^\alpha(0,1)$, see [10, Proposition 2.3 and Theorem 2.2] and [17], furthermore,

$$\partial I_2(u) \subset -\int_0^1 \partial F_u(x,u)dx. \tag{3.3}$$

The interpretation of (3.3) is as follows: for every $u^* \in \partial I_2(u)$, we have $u^*(x) \in -\partial F_u(x,u(x))$ for a.e. $x \in [0,1]$, and for every $v \in H_0^\alpha(0,1)$, the function $u^*v \in L^1[0,1]$ and $\langle u^*, v\rangle = \int_0^1 u^*(x)v(x)dx$. Therefore I_w is l.L. on $H_0^\alpha(0,1)$.

Now we shall show that each critical point u of I_w is a weak solution of problem (1.3). Let $u \in H_0^\alpha(0,1)$ be a critical point of I_w, then

$$0 \in \partial I_w(u) = \{u^* \in (H_0^\alpha(0,1))^* : \langle u^*, v\rangle \le I_w^0(u;v) \ \text{ for } \ v \in H_0^\alpha(0,1)\}. \tag{3.4}$$

Set

$$\begin{aligned} &\langle A_w(u), v\rangle \\ &= -q\int_0^1 \left({}_0D_x^\alpha u \cdot {}_xD_1^\alpha v + {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u\right)dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha v dx, \end{aligned} \tag{3.5}$$

$u, v \in H_0^\alpha(0,1)$. It follows from Lemma 2.1, (3.4) that

$$A_w(u) + u^* = 0 \ \text{ with } u^* \in \partial I_2(u),$$

hence $u^*(x) \in -\partial F_u(x,u(x))$ a.e. on $[0,1]$ and for every $v \in H_0^\alpha(0,1)$, one obtains

$$\begin{aligned} -q\int_0^1 &\Big({}_0D_x^\alpha u \cdot {}_xD_1^\alpha v + {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u\Big)dx \\ &- (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha v dx + \int_0^1 u^* v dx = 0. \end{aligned}$$

By Definition 3.2, u is a weak solution of problem (1.3), similar to the proof of Lemma 3.4, it gets u is a solution of problem (1.3). □

For the μ, M given in condition (F3), denote

$$a = q(1-2\mu)|\cos(\pi\alpha)|, \qquad b = \frac{(p-q)(1-\mu)}{|\cos(\pi\alpha)|}. \tag{3.6}$$

Assume that $a^2 > b^2 + b$, we take

$$\epsilon_1 = \frac{a - \sqrt{a^2 - b(b+1)}}{2(1+b)}, \ \epsilon_2 = \frac{a + \sqrt{a^2 - b(b+1)}}{2(1+b)}. \tag{3.7}$$

For $\epsilon \in (\epsilon_1, \epsilon_2)$, define

$$\bar{t} = \bar{t}(\epsilon) = \left[\frac{2\mu \left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p-q)^2}{4\varepsilon|\cos(\pi\alpha)|^2} \right)}{(\frac{1}{M})^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x)|\varphi(x)|^{\frac{1}{\mu}} dx} \right]^{\frac{\mu}{1-2\mu}}, \tag{3.8}$$

$$\begin{aligned} C(\epsilon) = \frac{q\bar{t}^2}{|\cos(\pi\alpha)|} &- \left(\frac{\bar{t}}{M}\right)^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x)|\varphi(x)|^{\frac{1}{\mu}} dx + \frac{\bar{t}^2(p-q)^2}{4\epsilon|\cos(\pi\alpha)|^2} \\ &+ (1+\mu) \int_0^1 (Ma(x) + M^r b(x)) dx, \end{aligned} \tag{3.9}$$

(3.10)

$$R_1 = R_1(\epsilon) = \left(\frac{4\epsilon C(\epsilon)}{4a\epsilon - 4(b+1)\epsilon^2 - b} \right)^{\frac{1}{2}}, \quad R_2 = R_2(\epsilon) = \frac{R_1}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}},$$

where $\tilde{c}$ is defined in (3.2), $\varphi \in H_0^\alpha(0,1)$ is a fixed function with $\|\varphi\|_\alpha = 1$.

The main result of this paper is the following.

THEOREM 3.9. *Assume that F satisfies the hypotheses (F1)-(F4). If there exists $\epsilon \in (\epsilon_1, \epsilon_2)$, such that*

$$L_{R_2} := \sup\left\{ \frac{|s_1^* - s_2^*|}{|s_1 - s_2|}, \quad |s_1|, |s_2| \le R_2, \ s_1 \ne s_2 \right\} \text{ exists,} \tag{3.11}$$

(3.12)

$$\left(\frac{q|\cos(\pi\alpha)|}{2} - \frac{\epsilon}{(\Gamma(\alpha+1))^2} \right) \frac{\left(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}\right)^\eta}{\int_0^1 \tilde{b}(x)dx} > \left(\frac{2(p-q)R_1}{q|\cos(\pi\alpha)|^2} \right)^{\eta-2}$$

and

$$L_{R_2} < (\Gamma(\alpha+1))^2 \left(2q|\cos(\pi\alpha)| - \frac{p-q}{|\cos(\pi\alpha)|} \right) \tag{3.13}$$

hold. Then the problem (1.2) *has at least one nonzero solution, where* $s_i^* \in \partial F_{s_i}(x, s_i), x \in [0,1], i = 1, 2,$ ϵ_1, ϵ_2 *are defined in* (3.7), $\tilde{b}, \eta$ *are defined in Remark 3.5.*

4. Proof of main result

In this section, we give the proof of Theorem 3.9 by the nonsmooth mountain pass theorem and iterative technique, the proof idea inspired from [22].

Proof of Theorem 3.9. We proceed by three steps to prove the main result.

Step 1: For given $w \in H_0^\alpha(0,1)$ with $\|w\|_\alpha \le R_1$, one shows that I_w has a nontrivial critical point in $H_0^\alpha(0,1)$ by the nonsmooth mountain pass theorem.

Firstly, we check that I_w satisfies the nonsmooth (PS) condition. Suppose $\{u_n\} \subset H_0^\alpha(0,1)$ satisfies

$$I_w(u_n) \to C \quad \text{and} \quad \lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

For every $n \ge 1$, since $\partial I_w(u_n) \subset (H_0^\alpha(0,1))^*$ is a weakly* compact set and the norm function is weakly lower semi-continuous in Banach space, we can find $u_n^* \in \partial I_w(u_n)$, such that

$$\lambda(u_n) = \|u_n^*\|_{(H_0^\alpha(0,1))^*} \quad \text{and} \quad u_n^* = A_w u_n - v_n \tag{4.2}$$

with $v_n(x) \in \partial F_{u_n}(x, u_n(x))$ for a.e. $x \in [0,1]$. Hence, by (4.1), (4.2), (3.5), Lemma 3.1, condition (F3) and Remark 3.6, it shows

$$\begin{aligned}
&C + 1 + \mu\|u_n\|_\alpha \\
&\ge I_w(u_n) - \mu\langle u_n^*, u_n\rangle \\
&= -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx - \int_0^1 F(x, u_n)dx \\
&\quad + \mu q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx + \mu q\int_0^1 {}_xD_1^\alpha u_n \cdot {}_0D_x^\alpha u_n dx \\
&\quad + \mu(p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx - \mu\langle v_n, -u_n\rangle
\end{aligned}$$

$$= q(2\mu-1)\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx + (p-q)(\mu-1)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx$$
$$- \int_0^1 F(x,u_n)dx - \mu\langle v_n, -u_n\rangle$$
$$\geq q(1-2\mu)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \frac{(p-q)(1-\mu)}{|\cos(\pi\alpha)|}\|w\|_\alpha\|u_n\|_\alpha$$
$$- \int_{\{|u_n|\leq M\}} \big(F(x,u_n) + \mu F^0(x,u_n;-u_n)\big)\,dx$$
$$- \int_{\{|u_n|> M\}} \big(F(x,u_n) + \mu F^0(x,u_n;-u_n)\big)\,dx$$
$$\geq a\|u_n\|_\alpha^2 + b\|w\|_\alpha\|u_n\|_\alpha - (1+\mu)\int_0^1 (Ma(x) + M^r b(x))dx,$$

where a, b are defined in (3.6). So the sequence $\{u_n\}$ is bounded. Thus, by passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ in $H_0^\alpha(0,1)$. Via Rellich-Kondrachov compactness theorem, one gets

$$u_n \to u \text{ in } L^2[0,1], \quad \text{and} \quad u_n \to u \text{ in } C[0,1]. \tag{4.3}$$

By Lemma 3.1 and (3.5), it gives

$$\langle A_w u_n - A_w u, u_n - u\rangle = -2q\int_0^1 \left({}_0D_x^\alpha(u_n(x)-u(x)), {}_xD_1^\alpha(u_n(x)-u(x))\right)dx$$
$$\geq 2q|\cos(\pi\alpha)|\|u_n - u\|_\alpha^2.$$

Consequently, in order to prove $u_n \to u$, it suffices to prove the following fact

$$\overline{\lim_n}\langle A_w u_n - A_w u, u_n - u\rangle \leq 0. \tag{4.4}$$

Indeed, from (4.1) and (4.2), there holds

$$\epsilon_n\|u_n - u\|_\alpha \geq \langle u_n^*, u_n - u\rangle = \langle A_w u_n, u_n - u\rangle - \int_0^1 v_n\cdot(u_n-u)dx$$

with $\epsilon_n \to 0$. In view of (4.3) and Hölder's inequality, one has $\int_0^1 v_n \cdot (u_n - u)dx \to 0$ as $n \to \infty$. So $\overline{\lim_n}\langle A_w u_n, u_n - u\rangle \leq 0$. Via $u_n \rightharpoonup u$ in $H_0^\alpha(0,1)$, it is easy to get $\lim_{n\to\infty}\langle A_w u, u_n - u\rangle = 0$. Hence

$$\overline{\lim_{n\to\infty}}\langle A_w u_n - A_w u, u_n - u\rangle \leq \overline{\lim_{n\to\infty}}\langle A_w u_n, u_n - u\rangle - \underline{\lim_{n\to\infty}}\langle A_w u, u_n - u\rangle \leq 0.$$

that is, (4.4) holds, we obtain $u_n \to u$ in $H_0^\alpha(0,1)$.

On the other hand, via Lemma 3.1 and Remark 3.5, one derives

$$\begin{aligned}
&I_w(u)\\
&= -q\int_0^1 {}_0D_x^\alpha u\cdot {}_xD_1^\alpha u dx - (p-q)\int_0^1 {}_0D_x^\alpha w\cdot {}_xD_1^\alpha u dx - \int_0^1 F(x,u)dx\\
&\geq q|\cos(\pi\alpha)|\|u\|_\alpha^2 - (p-q)\|w\|_\alpha\|_xD_1^\alpha u\|_{L^2} - \epsilon\|u\|_{L^2}^2 - \int_0^1 \tilde{b}(x)|u(x)|^\eta dx\\
&\geq q|\cos(\pi\alpha)|\|u\|_\alpha^2 - \frac{(p-q)R_1}{|\cos(\pi\alpha)|}\|u\|_\alpha - \frac{\epsilon}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2 - \|u\|_\infty^\eta\int_0^1 \tilde{b}(x)dx\\
&\geq q|\cos(\pi\alpha)|\|u\|_\alpha^2 - \frac{(p-q)R_1}{|\cos(\pi\alpha)|}\|u\|_\alpha - \frac{\epsilon}{(\Gamma(\alpha+1))^2}\|u\|_\alpha^2\\
&\quad - \frac{\int_0^1\tilde{b}(x)dx}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^\eta}\|u\|_\alpha^\eta\\
&= \left(\frac{q|\cos(\pi\alpha)|}{2} - \frac{\epsilon}{(\Gamma(\alpha+1))^2} - \frac{\int_0^1\tilde{b}(x)dx}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^\eta}\|u\|_\alpha^{\eta-2}\right)\|u\|_\alpha^2\\
&\quad + \left(\frac{q|\cos(\pi\alpha)|}{2}\|u\|_\alpha - \frac{(p-q)R_1}{|\cos(\pi\alpha)|}\right)\|u\|_\alpha.
\end{aligned}$$

by the assumption (3.12), one can choose $\rho > \frac{2(p-q)R_1}{q|\cos(\pi\alpha)|^2}$, such that

$$\frac{q|\cos(\pi\alpha)|}{2} - \frac{\epsilon}{(\Gamma(\alpha+1))^2} > \frac{\int_0^1\tilde{b}(x)dx}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^\eta}\rho^{\eta-2}.$$

Now, let $u \in H_0^\alpha(0,1)$ with $\|u\|_\alpha = \rho$, then there exists $\beta_1 > 0$, such that

$$I_w(u) \geq \beta_1 \quad \text{uniformly for} \quad w \in H_0^\alpha(0,1) \text{ with } \|w\|_\alpha \leq R_1. \tag{4.5}$$

For $\varphi \in H_0^\alpha(0,1)$ with $\|\varphi\|_\alpha = 1$, we will prove

$$I_w(t\varphi) \to -\infty \quad \text{as} \quad t \to \infty.$$

In fact, by Lemma 3.1 and Remark 3.6, as $t \to \infty$, it gives

$$
\begin{aligned}
& I_w(t\varphi) \\
& = -qt^2 \int_0^1 {}_0D_x^\alpha \varphi \cdot {}_xD_1^\alpha \varphi dx - (p-q)t \int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha \varphi dx - \int_0^1 F(x, t\varphi) dx \\
& \leq \frac{qt^2}{|\cos(\pi\alpha)|} \|\varphi\|_\alpha^2 + t(p-q)\|w\|_\alpha \|{}_xD_1^\alpha \varphi\|_{L^2} - \int_0^1 \tilde{c}(x) \left(\frac{|t\varphi(x)|}{M} \right)^{\frac{1}{\mu}} dx \\
& \leq \frac{qt^2}{|\cos(\pi\alpha)|} + \frac{t(p-q)\|w\|_\alpha}{|\cos(\pi\alpha)|} \|{}_0D_x^\alpha \varphi\|_{L^2} - \left(\frac{t}{M} \right)^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x) |\varphi(x)|^{\frac{1}{\mu}} dx \\
& \leq \frac{qt^2}{|\cos(\pi\alpha)|} + \frac{t(p-q)R_1}{|\cos(\pi\alpha)|} - \left(\frac{t}{M} \right)^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x) |\varphi(x)|^{\frac{1}{\mu}} dx \\
& \to -\infty.
\end{aligned}
$$

Thus, there exists $t_0 > 0$ such that $\|t_0\varphi\|_\alpha > \rho$ and $I_w(t_0\varphi) < 0$.

Then noting that $I_w(0) = 0$, combining with (4.5) and the nonsmooth mountain pass theorem [11, 13], we obtain that there is $u_w \in H_0^\alpha(0,1) \backslash \{\theta\}$ with $0 \in \partial I_w(u_w)$ and

$$
I_w(u_w) = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I_w(u) \geq \beta_1 > 0,
$$

where $\Gamma = \{g \in C([0,1], H_0^\alpha(0,1)) | g(0) = 0,\ g(1) = t_0\varphi\}$. By Proposition 3.8, we get u_w is a solution of problem (1.3).

Step 2: We construct an iterative sequence $\{u_n\}$ and estimate its norm in $H_0^\alpha(0,1)$.

For $u_1 \equiv 0$, by Step 1, we know I_{u_1} has a nontrivial critical point u_2. If we can prove $\|u_2\|_\alpha \leq R_1$, then by Step 1, one gets I_{u_2} has a critical point u_3. So in order to obtain iterative sequence $\{u_n\}$, we need prove that if we assume $\|u_{n-1}\|_\alpha \leq R_1$, then u_n, the nontrivial critical point of $I_{u_{n-1}}$ obtained by Step 1, satisfies $\|u_n\|_\alpha \leq R_1$.

Indeed, by Lemma 3.1, and Cauchy's inequality with the positive constant ϵ, for $\varphi \in H_0^\alpha(0,1)$ satisfying $\|\varphi\|_\alpha = 1$, it gives

$$
\begin{aligned}
(4.6)\quad & \max_{t\in[0,\infty)} I_{u_{n-1}}(t\varphi) \\
\leq\ & \frac{qt^2}{|\cos(\pi\alpha)|} - \int_0^1 \tilde{c}(x)\left(\frac{|t\varphi(x)|}{M}\right)^{\frac{1}{\mu}} dx + \frac{t(p-q)\|u_{n-1}\|_\alpha}{|\cos(\pi\alpha)|} \\
\leq\ & \frac{qt^2}{|\cos(\pi\alpha)|} - \left(\frac{t}{M}\right)^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x)|\varphi(x)|^{\frac{1}{\mu}} dx + \epsilon\|u_{n-1}\|_\alpha^2 + \frac{t^2(p-q)^2}{4\epsilon|\cos(\pi\alpha)|^2} \\
\leq\ & \frac{q\bar{t}^2}{|\cos(\pi\alpha)|} - \left(\frac{\bar{t}}{M}\right)^{\frac{1}{\mu}} \int_0^1 \tilde{c}(x)|\varphi(x)|^{\frac{1}{\mu}} dx + \epsilon\|u_{n-1}\|_\alpha^2 + \frac{\bar{t}^2(p-q)^2}{4\epsilon|\cos(\pi\alpha)|^2} \\
=\ & C(\epsilon) + \epsilon\|u_{n-1}\|_\alpha^2 - (1+\mu)\int_0^1 (Ma(x) + M^r b(x))dx,
\end{aligned}
$$

where $\epsilon \in (\epsilon_1, \epsilon_2), \epsilon_1, \epsilon_2$ are defined in (3.7), $\bar{t}, C(\epsilon)$ are defined in (3.8), (3.9) respectively.

On the other hand, since $0 \in \partial I_{u_{n-1}}(u_n)$, one has

$$
(4.7)\quad 2q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx + (p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx + \int_0^1 u_n^* u_n dx = 0,
$$

where $u_n^*(x) \in \partial F_{u_n}(x, u_n(x))$ for a.e. $x \in [0,1]$. Take (4.7), Lemma 3.1, (F3), Remark 3.6 into account, it derives

$$
\begin{aligned}
& I_{u_{n-1}}(u_n) \\
=\ & -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \int_0^1 F(x,u_n)dx \\
& -(p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx \\
=\ & -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \int_0^1 F(x,u_n)dx \\
& -(p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx + 2\mu q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx \\
& +\mu(p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx + \mu\int_0^1 u_n^* u_n dx
\end{aligned}
$$

$$
\begin{aligned}
&= (2\mu-1)q\int_0^1 {}_0D_x^\alpha u_n\cdot {}_xD_1^\alpha u_n dx\\
&\quad +(p-q)(\mu-1)\int_0^1 {}_0D_x^\alpha u_{n-1}\cdot {}_xD_1^\alpha u_n dx\\
&\quad -\int_0^1 (F(x,u_n)+\mu u_n^*(-u_n))\,dx\\
&\geq q(1-2\mu)|\cos(\pi\alpha)|\|u_n\|_\alpha^2+(\mu-1)(p-q)\|u_{n-1}\|_\alpha\|{}_xD_1^\alpha u_n\|_{L^2}\\
&\quad -\int_{|u_n|\leq M}\big(F(x,u_n)+\mu F^0(x,u_n;-u_n)\big)\,dx\\
&\quad -\int_{|u_n|>M}\big(F(x,u_n)+\mu F^0(x,u_n;-u_n)\big)\,dx\\
&\geq a\|u_n\|_\alpha^2-b\|u_{n-1}\|_\alpha\|u_n\|_\alpha-(1+\mu)\int_0^1(Ma(x)+M^rb(x))dx\\
&\geq a\|u_n\|_\alpha^2-b\left(\epsilon\|u_n\|_\alpha^2+\frac{1}{4\epsilon}\|u_{n-1}\|_\alpha^2\right)\\
&\quad -(1+\mu)\int_0^1(Ma(x)+M^rb(x))dx, \qquad (4.8)
\end{aligned}
$$

where $\epsilon\in(\epsilon_1,\epsilon_2),\epsilon_1,\epsilon_2$ are defined in (3.7). According to the nonsmooth mountain pass characterization of the critical level, we have

$$
\max_{t\in[0,\infty)} I_{u_{n-1}}(t\varphi)\geq I_{u_{n-1}}(u_n). \qquad (4.9)
$$

So combining with (4.6), (4.8) and (4.9), it concludes

$$
a\|u_n\|_\alpha^2\leq \epsilon\|u_{n-1}\|_\alpha^2+b\left(\epsilon\|u_n\|_\alpha^2+\frac{1}{4\epsilon}\|u_{n-1}\|_\alpha^2\right)+C(\epsilon),
$$

i.e.

$$
(a-b\epsilon)\|u_n\|_\alpha^2\leq(\epsilon+\frac{b}{4\epsilon})\|u_{n-1}\|_\alpha^2+C(\epsilon).
$$

Since $\epsilon \in (\epsilon_1, \epsilon_2)$, it holds $a - b\epsilon > \epsilon + \frac{b}{4\epsilon}$. Then

$$\begin{aligned}\|u_n\|_\alpha^2 &\leq \frac{\epsilon + \frac{b}{4\epsilon}}{a - b\epsilon}\|u_{n-1}\|_\alpha^2 + \frac{C(\epsilon)}{a - b\epsilon} \\ &\leq \left(\frac{\epsilon + \frac{b}{4\epsilon}}{a - b\epsilon}\right)^{n-1} \|u_1\|_\alpha^2 + \frac{C(\epsilon)}{a - b\epsilon}\sum_{k=0}^{n-2}\left(\frac{\epsilon + \frac{b}{4\epsilon}}{a - b\epsilon}\right)^k \\ &\leq \|u_1\|_\alpha^2 + \frac{4\epsilon C(\epsilon)}{4a\epsilon - 4(b+1)\epsilon^2 - b}.\end{aligned}$$

Consequently, if we take $u_1 \equiv 0$ and let u_n be a critical point of $I_{u_{n-1}}$ for $n = 2, 3, \ldots$, then from the above argument, one knows $\|u_n\|_\alpha \leq R_1$ and $I_{u_{n-1}}(u_n) \geq \beta_1 > 0$ for $n = 2, 3, \ldots$.

Step 3: We show the iterative sequence $\{u_n\}$ constructed in Step 2 is convergent to a nontrivial solution of the problem (1.2).

We intend to prove $\{u_n\}$ is a Cauchy sequence in $H_0^\alpha(0,1)$. Indeed, since $\|u_n\|_\alpha \leq R_1$, in view of Lemma 3.1 and the definition of R_2, one derives $\|u_n\|_\infty \leq R_2$. By $0 \in \partial I_{u_{n-1}}(u_n)(u_{n+1} - u_n), 0 \in \partial I_{u_n}(u_{n+1})(u_{n+1} - u_n)$, we get

$$\begin{aligned}(4.10)\quad -q\int_0^1 &\left({}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha(u_{n+1} - u_n) + {}_xD_1^\alpha u_n \cdot {}_0D_x^\alpha(u_{n+1} - u_n)\right) dx \\ &-(p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha(u_{n+1} - u_n)dx = \int_0^1 u_n^*(u_{n+1} - u_n)dx\end{aligned}$$

and

$$\begin{aligned}(4.11)\quad & \\ &- q\int_0^1 \left({}_0D_x^\alpha u_{n+1} \cdot {}_xD_1^\alpha(u_{n+1} - u_n) + {}_xD_1^\alpha u_{n+1} \cdot {}_0D_x^\alpha(u_{n+1} - u_n)\right) dx \\ &- (p-q)\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha(u_{n+1} - u_n)dx = \int_0^1 u_{n+1}^*(u_{n+1} - u_n)dx,\end{aligned}$$

where $u_n^*(x) \in \partial F_{u_n}(x, u_n(x)), u_{n+1}^*(x) \in \partial F_{u_{n+1}}(x, u_{n+1}(x))$ for a.e. $x \in [0,1]$. (4.11) subtracting (4.10), combining lemma 3.1 and (3.11), one concludes

$$
\begin{aligned}
&2q|\cos(\pi\alpha)|\|u_{n+1}-u_n\|_\alpha^2 \\
&\le -2q\int_0^1 {}_0D_x^\alpha(u_{n+1}-u_n)\cdot {}_xD_1^\alpha(u_{n+1}-u_n)dx \\
&= (p-q)\int_0^1 {}_0D_x^\alpha(u_n-u_{n-1})\cdot {}_xD_1^\alpha(u_{n+1}-u_n)dx \\
&\quad + \int_0^1 (u_{n+1}^*-u_n^*)(u_{n+1}-u_n)dx \\
&\le \frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha + \int_0^1 (u_{n+1}^*-u_n^*)(u_{n+1}-u_n)dx \\
&\le \frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha + L_{R_2}\|u_{n+1}-u_n\|_{L^2}^2 \\
&\le \frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha + \frac{L_{R_2}}{(\Gamma(\alpha+1))^2}\|u_{n+1}-u_n\|_\alpha^2 \\
&= \left(\frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha + \frac{L_{R_2}}{(\Gamma(\alpha+1))^2}\|u_{n+1}-u_n\|_\alpha\right)\|u_{n+1}-u_n\|_\alpha.
\end{aligned}
$$

Hence,

(4.12)

$$
\left(2q|\cos(\pi\alpha)| - \frac{L_{R_2}}{(\Gamma(\alpha+1))^2}\right)\|u_{n+1}-u_n\|_\alpha \le \frac{(p-q)}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha.
$$

By the assumptions (3.13) and (4.12), we know $\{u_n\}$ is a Cauchy sequence in $H_0^\alpha(0,1)$. So we suppose that $u_n \to u$ in $H_0^\alpha(0,1)$. In view of the definition of $\{u_n\}$, we know u is a weak solution and then by Lemma 3.4, it is a solution of problem (1.2). Note that $I_{u_{n-1}}(u_n) \ge \beta_1$ and $\beta_1 > 0$ does not depend on n, we derive that u is a nontrivial solution of problem (1.2). □

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References

[1] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973–1033.

[2] B. Ahmad and V. Otero-Espinar, *Existence of solutions for fractional differential inclusions with anti-periodic boundary conditions*, Bound. Value Probl. **2009** (2009). Article ID 625347.

[3] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resour. Res. **36** (2000), 1403–1412.

[4] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, *The fractional-order governing equation of Lévy motion*, Water Resour. Res. **36** (2000), 1413–1423.

[5] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential inclusions with infinite delay and applications to control theory*, Fract. Calc. Appl. Anal. **11** (2008), 35–56.

[6] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag Wien GmbH, 1997.

[7] B.A. Carreras, V.E. Lynch and G.M. Zaslavsky, *Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model*, Phys. Plasmas, **8** (2001), 5096–5103.

[8] Y.K. Chang and J.J. Nieto, *Some new existence results for fractional differential inclusions with boundary conditions*, Math. Comput. Model. **49** (2009), 605–609.

[9] F. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[10] K.C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[11] Y.K. Chang and J. J. Nieto, *Some new existence results for fractional differential inclusions with boundary conditions*, Math. Comput. Model. **49** (2009), 605–609.

[12] V.J. Ervin and J.P. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Partial Differential Equations, **22** (2006), 558–576.

[13] L. Gasinki and N. Papageorgiou, *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, Chapman Hall/CRC, Boca Raton, 2005.

[14] J. Henderson and A. Ouahab, *Fractional functional differential inclusions with finite delay*, Nonlinear Anal. **70** (2009), 2091–2105.

[15] F. Jiao and Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl. **62** (2011), 1181–1199.

[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[17] A. Kristály, *Infinitely many solutions for a differential inclusion problem in* $\mathbb{R}^N$, J. Differential Equations, **220** (2006), 511–530.

[18] D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Kluwer Academic Publishers, Dordrecht, 1999.
[19] I. Podlubny, *Fractional Differential Equations*, Academic press, New York, 1999.
[20] M.F. Shlesinger, B.J. West and J. Klafter, *Lévy dynamics of enhanced diffusion: application to turbulence*, Phys. Rev. Lett. **58** (1987), 1100–1103.
[21] E. Sayed, *A.G. Ibrahim, Multivalued fractional differential equations of arbitrary orders*, Appl. Math. Comput. **68** (1995), 15–25.
[22] H.R. Sun and Q.G. Zhang, *Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique*, Comput. Math. Appl. **64** (2012), 3436–3443.
[23] K.M. Teng, H.G. Jia and H.F. Zhang, *Existence and multiplicity results for fractional differential inclusions with Dirichlet boundary conditions*, Appl. Math. Comput. **220** (2013), 792–801.

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