Korean J. Math. **23** (2015), No. 4, pp. 557–569
http://dx.doi.org/10.11568/kjm.2015.23.4.557

FUZZY LATTICES AS FUZZY RELATIONS

Inheung Chon

Abstract. We define a fuzzy lattice as a fuzzy relation, develop some basic properties of the fuzzy lattice, show that the operations of join and meet in fuzzy lattices are isotone and associative, characterize a fuzzy lattice by its level set, and show that the direct product of two fuzzy lattices is a fuzzy lattice.

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh ([5]) and this concept was applied by Goguen ([3]) and Sanchez ([4]) to define and study fuzzy relations. Ajmal and Thomas ([1]) defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy lattices. Chon ([2]) defined a fuzzy lattice as a fuzzy relation, which is a fuzzification of a crisp lattice, and developed some properties of the fuzzy lattice. However the fuzzy lattice defined by Chon ([2]) turns out to be somewhat inadequate. We redefine a fuzzy lattice as a fuzzy relation and study fuzzy lattices in this note.

In Section 2, we give some definitions and develop some basic properties of fuzzy lattices which will be used in Section 3. In Section 3, we show that the operations of join and meet in fuzzy lattices are isotone

Received August 28, 2015. Revised September 14, 2015. Accepted October 15, 2015.

2010 Mathematics Subject Classification: 06B99.

Key words and phrases: fuzzy partially ordered set, fuzzy lattice.

This work was supported by a research grant from Seoul Women's University (2014).

and associative, characterize a fuzzy lattice by its level set, and show that the direct product of two fuzzy lattices is a fuzzy lattice.

2. Preliminaries

In this section, we give some definitions and develop some basic properties of fuzzy lattices which will be used in the next section.

Definition 2.1. Let X be a set. A function $A : X \times X \to [0, 1]$ is called a *fuzzy relation* in X. The fuzzy relation A in X is *reflexive* if $A(x, x) = 1$ for all $x \in X$ and A is *transitive* if $A(x, z) \geq \underset{y\in X}{\to}$ sup $\min(A(x, y), A(y, z))$.

The following definition of an anti-symmetric fuzzy relation is due to Zadeh ([6]).

Definition 2.2. Let X be a set. A fuzzy relation A in X is *antisymmetric* if $A(x, y) > 0$ and $A(y, x) > 0$ implies $x = y$. A fuzzy relation A in X is a *fuzzy partial order relation* if A is reflexive, antisymmetric, and transitive. If A is a fuzzy partial order relation in X, then (X, A) is called a *fuzzy partially ordered set.*

Chon defined a fuzzy lattice as a fuzzy partial order relation (see Definition 3.2 in [2]). However the definition turns out to be somewhat inadequate. We redefine a fuzzy lattice in Proposition 2.4.

Definition 2.3. Let (X, A) be a fuzzy partially ordered set and let $S \subseteq X$. An element $u \in X$ is said to be an *upper bound* for a set S if $A(u, b) \leq A(b, u)$ and $A(b, u) > 0$ for all $b \in S$. An upper bound u_0 for S is the *least upper bound* of S if $A(u, u_0) \leq A(u_0, u)$ and $A(u_0, u) > 0$ for every upper bound u for S. An element $v \in X$ is said to be a *lower bound* for S if $A(b, v) \leq A(v, b)$ and $A(v, b) > 0$ for all $b \in S$. A lower bound v_0 for S is the *greatest lower bound* of S iff $A(v_0, v) \leq A(v, v_0)$ and $A(v, v_0) > 0$ for every lower bound v for S.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$. $\vee$ is called the join and $\wedge$ is called the meet.

Definition 2.4. Let (X, A) be a fuzzy partially ordered set. Then (X, A) is a *fuzzy lattice* if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

Example of a fuzzy lattice. Let $X = \{x, y, z\}$ and let $A : X \times X \to [0, 1]$ be a fuzzy relation such that $A(x, x) = A(y, y) = A(z, z) = 1$, $A(x, y) = 0.2$, $A(x, z) = 0.1$, $A(y, z) = 0.1$, $A(y, x) = 0.5$, $A(z, x) = 0.3$, and $A(z, y) = 0.2$. Then it is easily checked that A is a fuzzy partial order relation. Also $x \vee y = y \vee x = x$, $x \vee z = z \vee x = x$, $y \vee z = z \vee y = y$, $x \wedge y = y \wedge x = y$, $x \wedge z = z \wedge x = z$, and $y \wedge z = z \wedge y = z$. Thus (X, A) is a fuzzy lattice.

Proposition 2.5. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

(1) *$A(x, x \vee y) > 0$ and $A(x \vee y, x) \le A(x, x \vee y)$.*
(2) *$A(y, x \vee y) > 0$ and $A(x \vee y, y) \le A(y, x \vee y)$.*
(3) *$A(x \wedge y, x) > 0$ and $A(x, x \wedge y) \le A(x \wedge y, x)$.*
(4) *$A(x \wedge y, y) > 0$ and $A(y, x \wedge y) \le A(x \wedge y, y)$.*
(5) *If $A(x, z) > 0$, $A(z, x) \le A(x, z)$, $A(y, z) > 0$, and $A(z, y) \le A(y, z)$, then $A(x \vee y, z) > 0$ and $A(z, x \vee y) \le A(x \vee y, z)$.*
(6) *If $A(z, x) > 0$, $A(x, z) \le A(z, x)$, $A(z, y) > 0$, and $A(y, z) \le A(z, y)$, then $A(z, x \wedge y) > 0$ and $A(x \wedge y, z) \le A(z, x \wedge y)$.*
(7) *$A(x, y) > 0$ and $A(y, x) \le A(x, y)$ if and only if $x \vee y = y$.*
(8) *$A(x, y) > 0$ and $A(y, x) \le A(x, y)$ if and only if $x \wedge y = x$.*

Proof. (1), (2), (3), and (4) are straightforward.

(5) Since $A(x, z) > 0$, $A(z, x) \le A(x, z)$, $A(y, z) > 0$, and $A(z, y) \le A(y, z)$, z is an upper bound of $\{x, y\}$. Since $x \vee y$ is the least upper bound of $\{x, y\}$, $A(x \vee y, z) > 0$ and $A(z, x \vee y) \le A(x \vee y, z)$.

(6) Since $A(z, x) > 0$, $A(x, z) \le A(z, x)$, $A(z, y) > 0$, and $A(y, z) \le A(z, y)$, z is a lower bound of $\{x, y\}$. Since $x \wedge y$ is the greatest lower bound of $\{x, y\}$, $A(z, x \wedge y) > 0$ and $A(x \wedge y, z) \le A(z, x \wedge y)$.

(7) Suppose $A(y, x) \le A(x, y)$ and $A(x, y) > 0$. Since $A(y, y) = 1 > 0$ and $A(y, y) \le A(y, y)$, $A(x \vee y, y) > 0$ and $A(y, x \vee y) \le A(x \vee y, y)$ by (5). Since $A(x \vee y, y) \le A(y, x \vee y)$ by (2), $A(y, x \vee y) = A(x \vee y, y) > 0$. Since A is antisymmetric, $x \vee y = y$. Conversely, suppose $x \vee y = y$. Then $A(y, x) = A(x \vee y, x) \le A(x, x \vee y) = A(x, y)$ by (1). Also $A(x, y) = A(x, x \vee y) > 0$ by (1).

(8) Suppose $A(y, x) \le A(x, y)$ and $A(x, y) > 0$. Since $A(x, x) = 1 > 0$ and $A(x, x) \le A(x, x)$, $A(x, x \wedge y) > 0$ and $A(x \wedge y, x) \le A(x, x \wedge y)$ by (6). Since $A(x, x \wedge y) \le A(x \wedge y, x)$ by (3), $A(x \wedge y, x) = A(x, x \wedge y) > 0$. Since A is antisymmetric, $x \wedge y = x$. Conversely, suppose $x \wedge y = x$.

Then $A(y,x) = A(y, x \wedge y) \leq A(x \wedge y, y) = A(x,y)$ by (4). Also $A(x,y) = A(x \wedge y, y) > 0$ by (4). $\square$

PROPOSITION 2.6. *Let (X, A) be a fuzzy lattice and let $x, y \in X$. Then*

(1) $x \vee x = x$, $x \wedge x = x$.
(2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$.
(3) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$.

Proof. (1) and (2) are straightforward.

(3) Let $B = \{x \vee y, x\}$. By Proposition 2.5 (1), $A(x, x \vee y) > 0$ and $A(x \vee y, x) \leq A(x, x \vee y)$. Since $A(x,x) = 1 > 0$, x is a lower bound of B. If z is a lower bound of B, then $A(z,x) > 0$ and $A(x,z) \leq A(z,x)$. Thus x is the greatest lower bound of B. Hence $(x \vee y) \wedge x = x$. Similarly we may show $(x \wedge y) \vee x = x$. $\square$

3. Some properties of fuzzy lattices

In this section, we show that the operations of join and meet in fuzzy lattices are isotone and associative, characterize the fuzzy lattice by its level set, and show that the direct product of two fuzzy lattices is a fuzzy lattice.

THEOREM 3.1. *Let (X, A) be a fuzzy lattice and $x \in X$. Suppose that $A(z,y) \leq A(y,z)$ and $A(y,z) > 0$. Then*

(1) $A(x \wedge y,\ x \wedge z) > 0$ *and* $A(x \wedge z,\ x \wedge y) \leq A(x \wedge y,\ x \wedge z)$.
(2) $A(x \vee y,\ x \vee z) > 0$ *and* $A(x \vee z,\ x \vee y) \leq A(x \vee y,\ x \vee z)$.

Proof. (1) (i) We consider the case of $A(z,y) < A(y,z)$ and $A(z,y) < A(x \wedge y, y)$. Since $A(x \wedge y,\ z) \geq \min\,[A(x \wedge y,\ y), A(y,z)]$, $A(x \wedge y, z) \geq \min\,[A(x \wedge y, y), A(z,y)]$. Since $A(z,y) < A(x \wedge y, y)$, $A(x \wedge y, z) \geq A(z,y)$. Also $A(z,y) \geq \min\,[A(z, x \wedge y), A(x \wedge y, y)]$. Since $A(z,y) < A(x \wedge y, y)$, $A(z,y) \geq A(z, x \wedge y)$. Thus $A(x \wedge y, z) \geq A(z, x \wedge y)$.

(ii) We consider the case of $A(z,y) < A(y,z)$ and $A(x \wedge y, y) \leq A(z,y)$. Since $A(y,z) > A(x \wedge y, y)$, $A(x \wedge y, z) \geq A(x \wedge y, y)$. Since $A(x \wedge y, y) \geq A(y, x \wedge y)$ by Proposition 2.5 (4), $A(x \wedge y, y) \geq A(y, x \wedge y) \geq \min\,[A(y,z), A(z, x \wedge y)]$. Since $A(y,z) > A(z,y) \geq A(x \wedge y, y)$, $A(x \wedge y, y) \geq A(z, x \wedge y)$. Thus $A(x \wedge y, z) \geq A(x \wedge y, y) \geq A(z, x \wedge y)$.

(iii) We consider the case of $A(z, y) = A(y, z) > 0$. Since A is antisymmetric, $z = y$. Thus $A(x \wedge y, z) = A(x \wedge z, z) \geq A(z, x \wedge z) = A(z, x \wedge y)$.

From (i), (ii), and (iii), $A(x \wedge y, z) \geq A(z, x \wedge y)$. Also $A(x \wedge y,\ z) \geq \min\,[A(x \wedge y,\ y), A(y, z)] > 0$. Since $A(x \wedge y,\ x) > 0$ and $A(x \wedge y,\ x) \geq A(x, x \wedge y)$ by Proposition 2.5 (3), $x \wedge y$ is a lower bound of $\{x, z\}$. Since $x \wedge z$ is the greatest lower bound of $\{x, z\}$, $A(x \wedge y,\ x \wedge z) > 0$ and $A(x \wedge z,\ x \wedge y) \leq A(x \wedge y,\ x \wedge z)$.

(2) (i) We consider the case of $A(z, y) < A(y, z)$ and $A(z, y) < A(z, x \vee z)$. Since $A(y, x \vee z) \geq \min\,[A(y, z),\ A(z, x \vee z)]$, $A(y, x \vee z) \geq \min\,[A(z, y), A(z, x \vee z)] = A(z, y)$. Also $A(z, y) \geq \min\,[A(z, x \vee z), A(x \vee z,\ y)]$. Since $A(z, y) < A(z, x \vee z)$, $A(z, y) \geq A(x \vee z, y)$. Thus $A(y, x \vee z) \geq A(x \vee z, y)$.

(ii) We consider the case of $A(z, y) < A(y, z)$ and $A(z, x \vee z) \leq A(z, y)$. Since $A(y, z) > A(z, x \vee z)$, $A(y, x \vee z) \geq A(z, x \vee z)$. Since $A(z, x \vee z) \geq A(x \vee z, z) \geq \min\,[A(x \vee z, y), A(y, z)]$ and $A(y, z) > A(z, y) \geq A(z, x \vee z)$, $A(z, x \vee z) \geq A(x \vee z, y)$. Thus $A(y, x \vee z) \geq A(z, x \vee z) \geq A(x \vee z, y)$.

(iii) We consider the case of $A(z, y) = A(y, z) > 0$. Since A is antisymmetric, $z = y$. Thus $A(y, x \vee z) = A(z, x \vee z) \geq A(x \vee z, z) = A(x \vee z, y)$.

From (i), (ii), and (iii), $A(y, x \vee z) \geq A(x \vee z, y)$. Also $A(y,\ x \vee z) \geq \min\,[A(y,\ z),\ A(z, x \vee z)] > 0$. Since $A(x, x \vee z) > 0$ and $A(x \vee z,\ x) \leq A(x, x \vee z)$ by Proposition 2.5 (1), $x \vee z$ is an upper bound of $\{x, y\}$. Since $x \vee y$ is the least upper bound of $\{x, y\}$, $A(x \vee y,\ x \vee z) > 0$ and $A(x \vee z,\ x \vee y) \leq A(x \vee y,\ x \vee z)$. □

Theorem 3.2. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

$$(x \vee y) \vee z = x \vee (y \vee z).$$

Proof. Since $A(y, y \vee z) \geq A(y \vee z, y)$,

$$\begin{aligned} A(y, x \vee (y \vee z)) &\geq \min\,[A(y, y \vee z),\ A(y \vee z, x \vee (y \vee z))] \\ &\geq \min\,[A(y \vee z, y),\ A(y \vee z, x \vee (y \vee z))]. \end{aligned}$$

(i) We consider the case of $A(y \vee z, y) \geq A(y \vee z, x \vee (y \vee z))$.
Clearly

$$A(y, x \vee (y \vee z)) \geq A(y \vee z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), y \vee z).$$

Since $A(y \vee z, y) \geq A(y \vee z, x \vee (y \vee z))$,

$$\begin{aligned} A(x \vee (y \vee z), y \vee z) &\geq \min\,[A(x \vee (y \vee z), y), A(y, y \vee z)] \\ &\geq \min\,[A(x \vee (y \vee z), y), A(y \vee z, y)] \\ &\geq \min\,[A(x \vee (y \vee z), y), A(y \vee z, x \vee (y \vee z))]. \end{aligned}$$

If $A(y \vee z, x \vee (y \vee z)) > A(x \vee (y \vee z), y \vee z)$, then $A(x \vee (y \vee z), y \vee z) \geq A(x \vee (y \vee z), y)$, and hence $A(y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), y)$. If $A(y \vee z, x \vee (y \vee z)) = A(x \vee (y \vee z), y \vee z) > 0$, then $x \vee (y \vee z) = y \vee z$, and hence $A(y, x \vee (y \vee z)) = A(y, y \vee z) \geq A(y \vee z, y) = A(x \vee (y \vee z), y)$. That is, $A(y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), y)$.
(ii) We consider the case of $A(y \vee z, x \vee (y \vee z)) > A(y \vee z, y)$.
Then $A(y, x \vee (y \vee z)) \geq A(y \vee z, y)$. Clearly

$$A(y \vee z, y) \geq \min\,[A(y \vee z, x \vee (y \vee z)),\ A(x \vee (y \vee z), y)].$$

Since $A(y \vee z, x \vee (y \vee z)) > A(y \vee z, y)$, $A(y \vee z, y) \geq A(x \vee (y \vee z), y)$. Thus $A(y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), y)$.

From (i) and (ii), $A(y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), y)$. Also $A(y, x \vee (y \vee z)) \geq \min\,[A(y, y \vee z), A(y \vee z, x \vee (y \vee z))] > 0$. Clearly $A(x, x \vee (y \vee z)) \geq A(x \vee (y \vee z), x)$ and $A(x, x \vee (y \vee z)) > 0$. By Proposition 2.5 (5),

$$A(x \vee y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), x \vee y) \text{ and } A(x \vee y, x \vee (y \vee z)) > 0.$$

Clearly

$$\begin{aligned} A(z, x \vee (y \vee z)) &\geq \min\,[A(z, y \vee z),\ A(y \vee z, x \vee (y \vee z))] \\ &\geq \min\,[A(y \vee z, z),\ A(y \vee z, x \vee (y \vee z))]. \end{aligned}$$

$(i)'$ We consider the case of $A(y \vee z, z) \geq A(y \vee z, x \vee (y \vee z))$.
Then

$$\begin{aligned} A(z, x \vee (y \vee z)) &\geq A(y \vee z, x \vee (y \vee z)) \\ &\geq A(x \vee (y \vee z), y \vee z). \end{aligned}$$

Since $A(y \vee z, z) \geq A(y \vee z, x \vee (y \vee z))$,

$$\begin{aligned} A(x \vee (y \vee z), y \vee z) &\geq \min\,[A(x \vee (y \vee z), z), A(z, y \vee z)] \\ &\geq \min\,[A(x \vee (y \vee z), z), A(y \vee z, z)] \\ &\geq \min\,[A(x \vee (y \vee z), z), A(y \vee z, x \vee (y \vee z))]. \end{aligned}$$

If $A(y \vee z, x \vee (y \vee z)) > A(x \vee (y \vee z), y \vee z)$, $A(x \vee (y \vee z), y \vee z) \geq A(x \vee (y \vee z), z)$, and hence $A(z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), z)$. If $A(y \vee z, x \vee (y \vee z)) = A(x \vee (y \vee z), y \vee z) > 0$, then $y \vee z = x \vee (y \vee z)$,

and hence $A(z, x \vee (y \vee z)) = A(z, y \vee z) \geq A(y \vee z, z) = A(x \vee (y \vee z), z)$. Thus $A(z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), z)$.
$(ii)'$ We consider the case of $A(y \vee z, x \vee (y \vee z)) > A(y \vee z, z)$. Then $A(z, x \vee (y \vee z)) \geq A(y \vee z, z)$. Also $A(y \vee z, z) \geq \min\,[A(y \vee z, x \vee (y \vee z)),\ A(x \vee (y \vee z), z)]$. Since $A(y \vee z, x \vee (y \vee z)) > A(y \vee z, z)$, $A(y \vee z, z) \geq A(x \vee (y \vee z), z)$. Thus $A(z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), z)$.

From $(i)'$ and $(ii)'$, $A(z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), z)$. Since $A(z, x \vee (y \vee z)) \geq \min\,[A(z, y \vee z), A(y \vee z, x \vee (y \vee z))]$, $A(z, x \vee (y \vee z)) > 0$. From the above,

$$A(x \vee y, x \vee (y \vee z)) \geq A(x \vee (y \vee z), x \vee y) \text{ and } A(x \vee y, x \vee (y \vee z)) > 0.$$

By Proposition 2.5 (5),

$$A((x \vee y) \vee z, x \vee (y \vee z)) \geq A(x \vee (y \vee z), (x \vee y) \vee z) \text{ and } A((x \vee y) \vee z, x \vee (y \vee z)) > 0.$$

By the same way as shown in the above, we may show that

$$A((x \vee y) \vee z, x \vee (y \vee z)) \leq A(x \vee (y \vee z), (x \vee y) \vee z) \text{ and } A(x \vee (y \vee z), (x \vee y) \vee z) > 0.$$

Thus

$$A((x \vee y) \vee z, x \vee (y \vee z)) = A(x \vee (y \vee z), (x \vee y) \vee z) > 0.$$

Since A is antisymmetric,

$$(x \vee y) \vee z = x \vee (y \vee z).$$

□

Theorem 3.3. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

$$(x \wedge y) \wedge z = x \wedge (y \wedge z).$$

Proof. Clearly

$$\begin{aligned} A((x \wedge y) \wedge z, x) &\geq \min\,[A((x \wedge y) \wedge z, x \wedge y),\ A(x \wedge y, x)] \\ &\geq \min\,[A((x \wedge y) \wedge z, x \wedge y),\ A(x, x \wedge y)]. \end{aligned}$$

(i) We consider the case of $A(x, x \wedge y) \geq A((x \wedge y) \wedge z, x \wedge y)$. Then $A((x \wedge y) \wedge z, x) \geq A((x \wedge y) \wedge z, x \wedge y) \geq A(x \wedge y, (x \wedge y) \wedge z)$. Also

$$\begin{aligned} A(x \wedge y, (x \wedge y) \wedge z) &\geq \min\,[A(x \wedge y, x), A(x, (x \wedge y) \wedge z)] \\ &\geq \min\,[A(x, x \wedge y), A(x, (x \wedge y) \wedge z)] \\ &\geq \min\,[A((x \wedge y) \wedge z, x \wedge y), A(x, (x \wedge y) \wedge z)]. \end{aligned}$$

If $A((x\wedge y)\wedge z, x\wedge y) > A(x\wedge y, (x\wedge y)\wedge z)$, then $A(x\wedge y, (x\wedge y)\wedge z) \geq A(x, (x\wedge y)\wedge z)$, and hence $A((x\wedge y)\wedge z, x) \geq A(x, (x\wedge y)\wedge z)$. If $A((x\wedge y)\wedge z, x\wedge y) = A(x\wedge y, (x\wedge y)\wedge z)$, then $(x\wedge y)\wedge z = x\wedge y$, and hence $A((x\wedge y)\wedge z, x) = A(x\wedge y, x) \geq A(x, x\wedge y) = A(x, (x\wedge y)\wedge z)$.
(ii) We consider the case of $A((x\wedge y)\wedge z, x\wedge y) > A(x, x\wedge y)$.
Then $A((x\wedge y)\wedge z, x) \geq A(x, x\wedge y)$. Also $A(x, x\wedge y) \geq \min\,[A(x, (x\wedge y)\wedge z), A((x\wedge y)\wedge z, x\wedge y)]$. Since $A((x\wedge y)\wedge z, x\wedge y) > A(x, x\wedge y)$, $A(x, x\wedge y) \geq A(x, (x\wedge y)\wedge z)$. Thus $A((x\wedge y)\wedge z, x) \geq A(x, (x\wedge y)\wedge z)$.

From (i) and (ii), $A((x\wedge y)\wedge z, x) \geq A(x, (x\wedge y)\wedge z)$. Also $A((x\wedge y)\wedge z, x) \geq \min\,[A((x\wedge y)\wedge z, x\wedge y), A(x\wedge y, x)] > 0$.
Clearly

$$\begin{aligned} A((x\wedge y)\wedge z, y) &\geq \min\,[A((x\wedge y)\wedge z, x\wedge y), A(x\wedge y, y)] \\ &\geq \min\,[A((x\wedge y)\wedge z, x\wedge y),\ A(y, x\wedge y)]. \end{aligned}$$

$(i)'$ We consider the case of $A(y, x\wedge y) \geq A((x\wedge y)\wedge z, x\wedge y)$.
Then $A((x\wedge y)\wedge z, y) \geq A((x\wedge y)\wedge z, x\wedge y) \geq A(x\wedge y, (x\wedge y)\wedge z)$. Also

$$\begin{aligned} A(x\wedge y, (x\wedge y)\wedge z) &\geq \min\,[A(x\wedge y, y), A(y, (x\wedge y)\wedge z)] \\ &\geq \min\,[A(y, x\wedge y), A(y, (x\wedge y)\wedge z)] \\ &\geq \min\,[A((x\wedge y)\wedge z, x\wedge y), A(y, (x\wedge y)\wedge z)]. \end{aligned}$$

If $A((x\wedge y)\wedge z, x\wedge y) > A(x\wedge y, (x\wedge y)\wedge z)$, then $A(x\wedge y, (x\wedge y)\wedge z) \geq A(y, (x\wedge y)\wedge z)$, and hence $A((x\wedge y)\wedge z, y) \geq A(y, (x\wedge y)\wedge z)$. If $A((x\wedge y)\wedge z, x\wedge y) = A(x\wedge y, (x\wedge y)\wedge z) > 0$, then $(x\wedge y)\wedge z = x\wedge y$, and hence $A((x\wedge y)\wedge z, y) = A(x\wedge y, y) \geq A(y, x\wedge y) = A(y, (x\wedge y)\wedge z)$.
$(ii)'$ We consider the case of $A((x\wedge y)\wedge z, x\wedge y) > A(y, x\wedge y)$.
Then $A((x\wedge y)\wedge z, y) \geq A(y, x\wedge y)$. $A(y, x\wedge y) \geq \min\,[A(y, (x\wedge y)\wedge z),\ A((x\wedge y)\wedge z, x\wedge y)]$. Since $A((x\wedge y)\wedge z, x\wedge y) > A(y, x\wedge y)$, $A(y, x\wedge y) \geq A(y, (x\wedge y)\wedge z)$. Thus $A((x\wedge y)\wedge z, y) \geq A(y, (x\wedge y)\wedge z)$.

From $(i)'$ and $(ii)'$, $A((x\wedge y)\wedge z, y) \geq A(y, (x\wedge y)\wedge z)$. Since $A((x\wedge y)\wedge z, y) \geq \min\,[A((x\wedge y)\wedge z, x\wedge y), A(x\wedge y, y)]$, $A((x\wedge y)\wedge z, y) > 0$.
Clearly

$$A((x\wedge y)\wedge z, z) \geq A(z, (x\wedge y)\wedge z) \text{ and } A((x\wedge y)\wedge z, z) > 0.$$

By Proposition 2.5 (6),

$$A((x\wedge y)\wedge z, y\wedge z) \geq A(y\wedge z, (x\wedge y)\wedge z) \text{ and } A((x\wedge y)\wedge z, y\wedge z) > 0.$$

From the above,

$$A((x \wedge y) \wedge z, x) \geq A(x, (x \wedge y) \wedge z) \text{ and } A((x \wedge y) \wedge z, x) > 0.$$

By Proposition 2.5 (6),

$$A((x\wedge y)\wedge z, x\wedge(y\wedge z)) \geq A(x\wedge(y\wedge z), (x\wedge y)\wedge z) \text{ and } A((x\wedge y)\wedge z, x\wedge(y\wedge z)) > 0.$$

As shown in the above, we may show that $A(x \wedge (y \wedge z), (x \wedge y) \wedge z) \geq A((x \wedge y) \wedge z, x \wedge (y \wedge z))$ and $A(x \wedge (y \wedge z), (x \wedge y) \wedge z) > 0$. Thus

$$A((x \wedge y) \wedge z, x \wedge (y \wedge z)) = A(x \wedge (y \wedge z), (x \wedge y) \wedge z) > 0.$$

Since A is antisymmetric, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. □

We define the level set $B_p = \{(x, y) \in X \times X : B(x, y) \geq p\}$ of a fuzzy relation B in a set X and characterize the relationships between a fuzzy lattice and its level set.

PROPOSITION 3.4. *Let B be a fuzzy relation in a set X and let $B_p = \{(x, y) \in X \times X : B(x, y) \geq p\}$. Then B is a fuzzy partial order relation if and only if the level set B_p is a partial order relation in $X \times X$ for all p such that $0 < p \leq 1$.*

Proof. See Proposition 2.4 of ([2]). □

PROPOSITION 3.5. *Let B be a fuzzy relation in a set X. Suppose that (X, B_p) is a lattice for all p such that $0 < p \leq 1$ and that if $B(p, q) > 0$,then $B(p, q) = B(q, p)$ or $B(q, p) = 0$. Then (X, B) is a fuzzy lattice.*

Proof. Let (X, B_p) be a lattice for all p such that $0 < p \leq 1$. Then (X, B) is a fuzzy partial order relation by Proposition 3.4. Let $x, y \in X$. Then there exists $r \in X$ such that $(x, r) \in B_p$, $(y, r) \in B_p$, and $(r, u) \in B_p$ for every upper bound u of $\{x, y\}$. Thus there exists $r \in X$ such that $B(x, r) \geq p > 0$, $B(y, r) \geq p > 0$, and $B(r, u) \geq p > 0$ for every upper bound u of $\{x, y\}$. By our hypothesis, $B(x, r) \geq B(r, x)$, $B(y, r) \geq B(r, y)$, and $B(r, u) \geq B(u, r)$. Thus there exists a least upper bound $r \in X$ of $\{x, y\}$. Similarly we may show that there exists a greatest lower bound $c \in X$ of $\{x, y\}$. Hence (X, B) is a fuzzy lattice. □

PROPOSITION 3.6. *Let B be a fuzzy relation in a set X. If (X, B) is a fuzzy lattice, then (X, B_p) is a lattice for some $p > 0$.*

Proof. The proof is similar to that of Proposition 3.6 in ([2]). □

We now turn to the direct product of fuzzy lattices.

Definition 3.7. Let (P, A) and (Q, B) be fuzzy partially ordered sets. The *direct product* PQ of P and Q is defined by $(PQ, A \times B)$, where $A \times B : PQ \to [0,1]$ is a fuzzy relation defined by $(A \times B)((p_1, q_1), (p_2, q_2)) = \min\,[A(p_1, p_2), B(q_1, q_2)]$.

Theorem 3.8. *Let (P, A) and (Q, B) be fuzzy lattices. Then the direct product $(PQ, A \times B)$ of (P, A) and (Q, B) is a fuzzy lattice.*

Proof. Let $(p_1, q_1), (p_2, q_2) \in PQ$. Clearly

$$(A \times B)((p_1, q_1), (p_1, q_1)) = \min\,[A(p_1, p_1), B(q_1, q_1)] = 1.$$

Suppose that $(A \times B)((p_1, q_1), (p_2, q_2)) = (A \times B)((p_2, q_2), (p_1, q_1)) > 0$. Then $\min\,[A(p_1, p_2), B(q_1, q_2)] > 0$ and $\min\,[A(p_2, p_1), B(q_2, q_1)] > 0$. Thus $A(p_1, p_2) > 0$, $A(p_2, p_1) > 0$, $B(q_1, q_2) > 0$, and $B(q_2, q_1) > 0$. Thus $p_1 = p_2$ and $q_1 = q_2$, and hence $(p_1, q_1) = (p_2, q_2)$. $(A \times B)((p_1, q_1), (p_2, q_2)) = \min\,[A(p_1, p_2), B(q_1, q_2)]$
$\geq \min\,[\underset{p \in P}{\rightarrow} \sup \min\,(A(p_1, p), A(p, p_2)),\ \underset{q \in Q}{\rightarrow} \sup \min\,(B(q_1, q), B(q, q_2))]$.
Since

$$\begin{aligned}
&\min\,[\underset{p \in P}{\rightarrow} \sup \min\,(A(p_1, p), A(p, p_2)),\ \underset{q \in Q}{\rightarrow} \sup \min\,(B(q_1, q), B(q, q_2))] \\
&= \underset{(p,q) \in PQ}{\rightarrow} \sup \min\,[\min(A(p_1, p), A(p, p_2)),\ \min\,(B(q_1, q), B(q, q_2))] \\
&= \underset{(p,q) \in PQ}{\rightarrow} \sup \min[A(p_1, p), B(q_1, q), A(p, p_2), B(q, q_2)] \\
&= \underset{(p,q) \in PQ}{\rightarrow} \sup \min\,[\min\,(A(p_1, p), B(q_1, q)),\ \min\,(A(p, p_2), B(q, q_2))] \\
&= \underset{(p,q) \in PQ}{\rightarrow} \sup\min\,[(A \times B)((p_1, q_1), (p, q)),\ (A \times B)((p, q), (p_2, q_2))],
\end{aligned}$$

$(A \times B)((p_1, q_1), (p_2, q_2)) \geq \underset{(p,q) \in PQ}{\rightarrow} \sup \min\,[(A \times B)((p_1, q_1), (p, q)),\ (A \times B)((p, q), (p_2, q_2))]$. Thus $(PQ, A \times B)$ is a fuzzy partially ordered set.

Let $(p_1, q_1), (p_2, q_2) \in PQ$. Then

$$(A \times B)((p_1, q_1), (p_1 \vee p_2, q_1 \vee q_2)) = \min[A(p_1, p_1 \vee p_2),\ B(q_1, q_1 \vee q_2)] > 0$$

and

$$\begin{aligned}
(A \times B)((p_1, q_1), (p_1 \vee p_2, q_1 \vee q_2)) &\geq \min\,[A(p_1 \vee p_2, p_1),\ B(q_1 \vee q_2, q_1)] \\
&= (A \times B)((p_1 \vee p_2, q_1 \vee q_2), (p_1, q_1)).
\end{aligned}$$

Similarly we may show that $(A \times B)((p_2, q_2), (p_1 \vee p_2, q_1 \vee q_2)) > 0$ and

$$(A \times B)((p_2, q_2), (p_1 \vee p_2, q_1 \vee q_2)) \geq (A \times B)((p_1 \vee p_2, q_1 \vee q_2), (p_2, q_2)).$$

Thus $(p_1 \vee p_2, q_1 \vee q_2)$ is an upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. Let (s, t) be an upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. Then $(A \times B)((p_1, q_1), (s, t)) > 0$, $(A \times B)((p_1, q_1), (s, t)) \geq (A \times B)((s, t), (p_1, q_1))$, $(A \times B)((p_2, q_2), (s, t)) > 0$, and $(A \times B)((p_2, q_2), (s, t)) \geq (A \times B)((s, t), (p_2, q_2))$.
Thus min $[A(p_1, s), B(q_1, t)] > 0$ and min $[A(p_2, s), B(q_2, t)] > 0$. Since $A(p_1, s) > 0$ and $A(p_2, s) > 0$, $A(p_1 \vee p_2, s) > 0$ by Proposition 2.5 (5). Since $B(q_1, t) > 0$ and $B(q_2, t) > 0$, $B(q_1 \vee q_2, t) > 0$ by Proposition 2.5 (5). Thus

$$(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) = \min\,[A(p_1 \vee p_2, s),\ B(q_1 \vee q_2, t)] > 0.$$

(i) Suppose $s = p_1 \vee p_2$ and $t = q_1 \vee q_2$. Then

$$(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) = (A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)) = 1.$$

(ii) Suppose $s \neq p_1 \vee p_2$. Then $A(s, p_1 \vee p_2) = 0$ or $A(p_1 \vee p_2, s) = 0$. Since $A(p_1 \vee p_2, s) > 0$, $A(s, p_1 \vee p_2) = 0$, and hence $(A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)) = 0$. Thus

$$(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) \geq (A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)).$$

(iii) Suppose $t \neq q_1 \vee q_2$. Then $B(t, q_1 \vee q_2) = 0$ or $B(q_1 \vee q_2, t) = 0$. Since $B(q_1 \vee q_2, t) > 0$, $B(t, q_1 \vee q_2) = 0$, and hence $(A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)) = 0$. Thus

$$(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) \geq (A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)).$$

From (i), (ii), and (iii),

$$(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) \geq (A \times B)((s, t), (p_1 \vee p_2, q_1 \vee q_2)).$$

Thus $(p_1 \vee p_2, q_1 \vee q_2)$ is the least upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. That is, for every $(p_1, q_1), (p_2, q_2) \in PQ$, there exists the least upper bound $(p_1 \vee p_2, q_1 \vee q_2)$ of $\{(p_1, q_1), (p_2, q_2)\}$.
Let $(p_1, q_1), (p_2, q_2) \in PQ$. Then

$$(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (p_1, q_1)) = \min[A(p_1 \wedge p_2, p_1),\ B(q_1 \wedge q_2, q_1)] > 0$$

and

$$\begin{aligned}(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (p_1, q_1)) &\geq \min\,[A(p_1, p_1 \wedge p_2),\ B(q_1, q_1 \wedge q_2)]\\ &= (A \times B)((p_1, q_1), (p_1 \wedge p_2, q_1 \wedge q_2)).\end{aligned}$$

Similarly we may show that $(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (p_2, q_2)) > 0$ and

$$(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (p_2, q_2)) \geq (A \times B)((p_2, q_2), (p_1 \wedge p_2, q_1 \wedge q_2)).$$

Thus $(p_1 \wedge p_2, q_1 \wedge q_2)$ is a lower bound of $\{(p_1, q_1), (p_2, q_2)\}$. Let (v, w) be a lower bound of $\{(p_1, q_1), (p_2, q_2)\}$. Then $(A\times B)((v, w), (p_1, q_1)) > 0$, $(A\times B)((v, w), (p_1, q_1)) \geq (A\times B)((p_1, q_1), (v, w))$, $(A\times B)((v, w), (p_2, q_2)) > 0$, and $(A \times B)((v, w), (p_2, q_2)) \geq (A \times B)((p_2, q_2), (v, w))$.
Thus min $[A(v, p_1), B(w, q_1)] > 0$ and $\min[A(v, p_2), B(w, q_2)] > 0$. Since $A(v, p_1) > 0$ and $A(v, p_2) > 0$, $A(v, p_1 \wedge p_2) > 0$ by Proposition 2.5 (6). Since $B(w, q_1) > 0$ and $B(w, q_2) > 0$, $B(w, q_1 \wedge q_2) > 0$ by Proposition 2.5 (6). Thus

$$(A \times B)((v, w), (p_1 \wedge p_2, q_1 \wedge q_2)) = \min\,[A(v, p_1 \wedge p_2),\ B(w, q_1 \wedge q_2)] > 0.$$

(i)$'$ Suppose $v = p_1 \wedge p_2$ and $w = q_1 \wedge q_2$. Then

$$(A \times B)((v, w), (p_1 \wedge p_2, q_1 \wedge q_2)) = (A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)) = 1.$$

(ii)$'$ Suppose $v \neq p_1 \wedge p_2$. Then $A(p_1 \wedge p_2, v) = 0$ or $A(v, p_1 \wedge p_2) = 0$. Since $A(v, p_1 \wedge p_2) > 0$, $A(p_1 \wedge p_2, v) = 0$, and hence $(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)) = 0$. Thus

$$(A \times B)((v, w), (p_1 \wedge p_2, q_1 \wedge q_2)) \geq (A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)).$$

(iii)$'$ Suppose $w \neq q_1 \wedge q_2$. Then $B(q_1 \wedge q_2, w) = 0$ or $B(w, q_1 \wedge q_2) = 0$. Since $B(w, q_1 \wedge q_2) > 0$, $B(q_1 \wedge q_2, w) = 0$, and hence $(A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)) = 0$. Thus

$$(A \times B)((v, w), (p_1 \wedge p_2, q_1 \wedge q_2)) \geq (A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)).$$

From (i)$'$, (ii)$'$, and (iii)$'$,

$$(A \times B)((v, w), (p_1 \wedge p_2, q_1 \wedge q_2)) \geq (A \times B)((p_1 \wedge p_2, q_1 \wedge q_2), (v, w)).$$

Thus $(p_1 \wedge p_2, q_1 \wedge q_2)$ is the greatest lower bound of $\{(p_1, q_1), (p_2, q_2)\}$. That is, for every $(p_1, q_1), (p_2, q_2) \in PQ$, there exists the greatest lower bound $(p_1 \wedge p_2, q_1 \wedge q_2)$ of $\{(p_1, q_1), (p_2, q_2)\}$. Hence $(PQ, A \times B)$ is a fuzzy lattice. $\square$

References

[1] N. Ajmal and K. V. Thomas, *Fuzzy lattices*, Inform. Sci. **79** (1994), 271–291.
[2] I. Chon, *Fuzzy partial order relations and fuzzy lattices*, Korean J. Math. **17** (2009), 361–374.
[3] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.
[4] E. Sanchez, *Resolution of composite fuzzy relation equation*, Inform. and Control **30** (1976), 38–48.
[5] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.

[6] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Inform. Sci. **3** (1971), 177–200.

Inheung Chon
Department of Mathematics
Seoul Women's University
Seoul 139-774, South Korea
E-mail: `ihchon@swu.ac.kr`