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MEAN VALUES OF THE HOMOGENEOUS DEDEKIND SUMS

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Abstract. Let a, b, q be integers with $q > 0$. The homogeneous Dedekind sum is defined by

$$S(a,b,q) = \sum_{r=1}^{q} \left(\left(\frac{ar}{q}\right)\right)\left(\left(\frac{br}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

In this paper we study the mean value of $S(a,b,q)$ by using mean value theorems of Dirichlet L-functions, and give some asymptotic formula.

1. Introduction

For integers a and $q > 0$, the classical Dedekind sum is defined by

$$S(a,q) = \sum_{r=1}^{q} \left(\left(\frac{r}{q}\right)\right)\left(\left(\frac{ar}{q}\right)\right),$$

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where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $S(a,q)$ plays an important role in the transformation theory of the Dedekind η function (see [6] and Chapter 3 of [1] for details).

Let a, b, q be integers with $q > 0$. According to [2], the homogeneous Dedekind sum is defined by

$$S(a,b,q) = \sum_{r=1}^{q} \left(\left(\frac{ar}{q}\right)\right)\left(\left(\frac{br}{q}\right)\right).$$

Z. Zheng [9] proved a generalized Knopp's identity for $S(a,b,q)$ as following:

$$\sum_{d|n}\sum_{r_1=1}^{d}\sum_{r_2=1}^{d} S\left(\frac{n}{d}a + r_1 q, \frac{n}{d}b + r_2 q, dq\right) = n\sigma(n)S(a,b,q),$$

where n is a positive integer, and $\sigma(n) = \sum_{d|n} d$.

H. Liu [4] studied the mean values of the homogeneous Dedekind sums in short intervals $\left[1, \frac{p}{3}\right]$ and $\left[1, \frac{p}{4}\right]$, and showed some asymptotic formulae.

Proposition 1.1. *Let $p \geq 5$ be a prime. We have*

$$\sum_{a\leq \frac{p}{3}}\sum_{b\leq \frac{p}{3}} S(a,b,p) = \frac{1}{18}p^2 + O\left(p^{1+\epsilon}\right),$$

$$\sum_{a\leq \frac{p}{4}}\sum_{b\leq \frac{p}{4}} S(a,b,p) = \frac{131}{2304}p^2 + O\left(p^{1+\epsilon}\right),$$

$$\sum_{a\leq \frac{p}{3}}\sum_{b\leq \frac{p}{4}} S(a,b,p) = \frac{3}{64}p^2 + O\left(p^{1+\epsilon}\right).$$

In this paper we further study the mean values of the homogeneous Dedekind sums, and give some formulae. Our main results are the following:

Theorem 1.2. *Let $p \geq 5$ be a prime. We have*

$$\sum_{a\leq \frac{p}{3}}\sum_{b\leq \frac{p}{3}} abS(a,b,p) = \left(\frac{1}{1296} + \frac{\sqrt{3}L^2(3,\chi_3)}{1296\pi L(5,\chi_3)} + \frac{135L^2(3,\chi_3)}{832\pi^6}\right)p^4 + \left(p^{3+\epsilon}\right),$$

$$\sum_{a\le\frac{p}{4}}\sum_{b\le\frac{p}{4}} abS(a,b,p) = \left(\frac{1}{4608}+\frac{L^2(3,\chi_4)}{2304\pi L(5,\chi_4)}+\frac{5L^2(3,\chi_4)}{24\pi^6}\right)p^4+\left(p^{3+\epsilon}\right),$$

$$\sum_{a\le\frac{p}{3}}\sum_{b\le\frac{p}{4}} abS(a,b,p) = \left(\frac{1}{13824}+\frac{13\sqrt{3}L^2(3,\chi_3)}{202752\pi L(5,\chi_3)}+\frac{23L^2(3,\chi_4)}{26352\pi L(5,\chi_4)}\right.$$
$$\left.+\frac{\sqrt{3}L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)}{96\pi^4 L(6,\chi_3\chi_4)}\right)p^4+O\left(p^{3+\epsilon}\right),$$

where χ_3 is the non-principal character modulo 3, χ_4 is the non-principal character modulo 4, and $L(s,\chi)=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^s}$ is the Dirichlet L-function.

2. Identities involving certain Dirichlet series

THEOREM 2.1. *Let k and l be fixed non-negative integers, and let $d(n)$ be the divisor function. Then we have*

$$\sum_{n=1}^{\infty}\frac{d(2^k n)d(3^l n)}{n^2}=\frac{(3k+5)(4l+5)\pi^4}{360}. \qquad (2.1)$$

Proof. By the property of the divisor function we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{d(2^k n)d(3^l n)}{n^2} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{n=1\\ 2^i|n,\ 2^{i+1}\nmid n\\ 3^j|n,\ 3^{j+1}\nmid n}}^{\infty}\frac{d(2^k n)d(3^l n)}{n^2}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{n=1\\ (n,6)=1}}^{\infty}\frac{d(2^{k+i}3^j n)d(2^i 3^{l+j}n)}{(2^i 3^j n)^2}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{d(2^{k+i})d(3^j)d(2^i)d(3^{l+j})}{2^{2i}3^{2j}}\sum_{\substack{n=1\\ (n,6)=1}}^{\infty}\frac{d^2(n)}{n^2}\\ &=\sum_{i=0}^{\infty}\frac{(k+i+1)(i+1)}{2^{2i}}\sum_{j=0}^{\infty}\frac{(j+1)(l+j+1)}{3^{2j}}\sum_{\substack{n=1\\ (n,6)=1}}^{\infty}\frac{d^2(n)}{n^2}.\end{aligned}$$

It is not hard to show that

$$\sum_{i=0}^{\infty} \frac{(k+i+1)(i+1)}{2^{2i}} = \frac{16(3k+5)}{27} \qquad \text{and}$$

$$\sum_{j=0}^{\infty} \frac{(j+1)(l+j+1)}{3^{2j}} = \frac{81(4l+5)}{256}.$$

On the other hand, from Euler product formula we have

$$\begin{aligned}
\sum_{\substack{n=1\\(n,6)=1}}^{\infty} \frac{d^2(n)}{n^2} &= \prod_{p\nmid 6}\left(1+\frac{d^2(p)}{p^2}+\frac{d^2(p^2)}{p^4}+\cdots\right)\\
&= \prod_{p\nmid 6}\left(1+\frac{2^2}{p^2}+\frac{3^2}{p^4}+\cdots\right)\\
&= \prod_{p\nmid 6}\frac{1+\frac{1}{p^2}}{\left(1-\frac{1}{p^2}\right)^3} = \prod_{p}\frac{1+\frac{1}{p^2}}{\left(1-\frac{1}{p^2}\right)^3}\prod_{p\mid 6}\frac{\left(1-\frac{1}{p^2}\right)^3}{1+\frac{1}{p^2}}\\
&= \frac{\zeta^4(2)}{\zeta(4)}\cdot\frac{16}{75} = \frac{2\pi^4}{135},
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function satisfying $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$. Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} &= \frac{16(3k+5)}{27}\cdot\frac{81(4l+5)}{256}\cdot\frac{2\pi^4}{135}\\
&= \frac{(3k+5)(4l+5)\pi^4}{360}.
\end{aligned}$$

□

THEOREM 2.2. *Let $p \geq 5$ be a prime, and let k be fixed non-negative integer. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Define $\tau(n) = \sum_{t\mid n} \frac{\chi_3(t)}{t}$ and*

$\rho(n) = \sum_{t|n} \frac{\chi_4(t)}{t}$. *Then we have*

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{45L^2(3,\chi_3)}{26}, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{\rho^2(n)}{n^2} = \frac{5L^2(3,\chi_4)}{3}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \frac{d(n)\tau(2^k n)}{n^2} = \left(1 + \frac{(-1)^k}{2^{k-1}\cdot 9}\right) \frac{\pi^4 L^2(3,\chi_3)}{44L(5,\chi_3)}, \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{d(n)\rho(3^k n)}{n^2} = \left(\frac{1}{4} + \frac{(-1)^k}{3^{k-1}\cdot 49}\right) \frac{49\pi^4 L^2(3,\chi_4)}{549L(5,\chi_4)}, \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)\rho(n)}{n^2} = \frac{\pi^2 L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)}{6L(6,\chi_3\chi_4)}. \tag{2.6}$$

Proof. We only prove (2.2) since similarly we can get others. Noting that $\tau(n)$ is a multiplicative function, we can write

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} = \sum_{j=0}^{\infty} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(3^j n)}{(3^j n)^2} = \sum_{j=0}^{\infty} \frac{1}{9^j} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{9}{8} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2}.$$

For the summation

$$\sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2},$$

by using the Euler product formula we have

$$\sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2} = \prod_{p\neq 3} \left(1 + \frac{\tau^2(p)}{p^2} + \frac{\tau^2(p^2)}{p^4} + \cdots\right)$$

$$= \prod_{p\neq 3} \left(1 + \left(\frac{1 + \frac{\chi_3(p)}{p}}{p}\right)^2 + \left(\frac{1 + \frac{\chi_3(p)}{p} + \left(\frac{\chi_3(p)}{p}\right)^2}{p^2}\right)^2 + \cdots\right)$$

$$
\begin{aligned}
&= \prod_{p\neq 3}\sum_{n=0}^{\infty}\left(\frac{1+\frac{\chi_3(p)}{p}+\left(\frac{\chi_3(p)}{p}\right)^2+\cdots+\left(\frac{\chi_3(p)}{p}\right)^n}{p^n}\right)^2\\
&= \prod_{p\neq 3}\sum_{n=0}^{\infty}\left(\frac{\frac{1-(\frac{\chi_3(p)}{p})^{n+1}}{1-\frac{\chi_3(p)}{p}}}{p^n}\right)^2\\
&= \prod_{p\neq 3}\frac{1}{(1-\frac{\chi_3(p)}{p})^2}\sum_{n=0}^{\infty}\left(\frac{1}{p^{2n}}-\frac{2(\chi_3(p))^{n+1}}{p^{3n+1}}+\frac{1}{p^{4n+2}}\right)\\
&= \prod_{p\neq 3}\frac{1}{(1-\frac{\chi_3(p)}{p})^2}\left(\frac{p^2}{p^2-1}-\frac{2\chi_3(p)p^2}{p^3-\chi_3(p)}+\frac{p^2}{p^4-1}\right)\\
&= \prod_{p\neq 3}\frac{p^2}{(1-\frac{\chi_3(p)}{p})^2}\times\frac{(p^2+1)(p^3-\chi_3(p))-2\chi_3(p)(p^4-1)+p^3-\chi_3(p)}{(p^4-1)(p^3-\chi_3(p))}\\
&= \prod_{p\neq 3}\frac{p^4}{(1-\frac{\chi_3(p)}{p})^2}\times\frac{p^3-2p^2\chi_3(p)+2p-\chi_3(p)}{(p^4-1)(p^3-\chi_3(p))}\\
&= \prod_{p\neq 3}\frac{p^6(p-\chi_3(p))(p+\chi_3(p))(p^2-\chi_3(p)p+(\chi_3(p))^2)}{(p-\chi_3(p))^2(p+\chi_3(p))(p^4-1)(p^3-\chi_3(p))}\\
&= \prod_{p\neq 3}\frac{p^6(p^3+\chi_3(p))}{(p^2-1)(p^4-1)(p^3-\chi_3(p))}=\prod_{p\neq 3}\frac{1-\frac{1}{p^6}}{\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^4}\right)\left(1-\frac{\chi_3(p)}{p^3}\right)^2}\\
&= \frac{80\zeta(2)\zeta(4)L^2(3,\chi_3)}{91\zeta(6)}=\frac{20L^2(3,\chi_3)}{13},
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta function satisfying $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$ and $\zeta(6)=\frac{\pi^6}{945}$. Therefore

$$
\sum_{n=1}^{\infty}\frac{\tau^2(n)}{n^2}=\frac{9}{8}\sum_{\substack{n=1\\(n,3)=1}}^{\infty}\frac{\tau^2(n)}{n^2}=\frac{45L^2(3,\chi_3)}{26}.
$$

This proves (2.2). □

3. Mean values of the Dirichlet L- functions

In this section, we shall prove some mean values of the Dirichlet L-functions, which will be used to prove Theorem 1.1.

THEOREM 3.1. *Let $p \geq 5$ be a prime, and let k and l be fixed non-negative integers. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\overline{\chi}(3^l)\,|L(1,\chi)|^4 = \frac{(3k+5)(4l+5)\pi^4}{2^k 3^l \cdot 720} p + O\left(p^{\epsilon}\right),$$

where $L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet L-function, and $\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}$ denotes the summation over all odd Dirichlet characters modulo p.

Proof. Let N be an integer with $p \leq N < p^4$, and let $d(n)$ be the divisor function. By Abel's identity we have

$$L^2(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)d(n)}{n} = \sum_{n=1}^{N} \frac{\chi(n)d(n)}{n} + \int_N^{\infty} \frac{\sum_{N<n\leq y} \chi(n)d(n)}{y^2} dy.$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\overline{\chi}(3^l)\,|L(1,\chi)|^4 \\
&= \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^{N} \frac{\chi(n_1)d(n_1)}{n_1} + \int_N^{\infty} \frac{\sum_{N<n_1\leq y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\
&\qquad \times \overline{\chi}(3^l) \left(\sum_{n_2=1}^{N} \frac{\overline{\chi}(n_2)d(n_2)}{n_2} + \int_N^{\infty} \frac{\sum_{N<n_2\leq y_2} \overline{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\
&= \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^{N} \frac{\chi(n_1)d(n_1)}{n_1} \right) \overline{\chi}(3^l) \left(\sum_{n_2=1}^{N} \frac{\overline{\chi}(n_2)d(n_2)}{n_2} \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^{N} \frac{\chi(n_1)d(n_1)}{n_1} \right) \\
&\qquad\qquad \overline{\chi}(3^l) \left(\int_N^\infty \frac{\sum\limits_{N<n_2\le y_2} \overline{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\
&+ \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\int_N^\infty \frac{\sum\limits_{N<n_1\le y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\
&\qquad\qquad \overline{\chi}(3^l) \left(\sum_{n_2=1}^{N} \frac{\overline{\chi}(n_2)d(n_2)}{n_2} \right) \\
&+ \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\int_N^\infty \frac{\sum\limits_{N<n_1\le y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\
&\qquad\qquad \overline{\chi}(3^l) \left(\int_N^\infty \frac{\sum\limits_{N<n_2\le y_2} \overline{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\
&:= M_1 + M_2 + M_3 + M_4. \qquad (3.1)
\end{aligned}
$$

From the Pólya-Vinogradov inequality we get

$$
\begin{aligned}
&\sum_{N<n\le y} \chi(n)d(n) = \sum_{N<nm\le y} \chi(nm) \\
&= 2 \sum_{n\le\sqrt{y}} \chi(n) \sum_{m\le\frac{y}{n}} \chi(m) - \left(\sum_{n\le\sqrt{y}} \chi(n) \right)^2
\end{aligned}
$$

$$-2\sum_{n\le\sqrt{N}}\chi(n)\sum_{m\le\frac{y}{n}}\chi(m)+\left(\sum_{n\le\sqrt{N}}\chi(n)\right)^2$$
$$\ll y^{\frac12}p^{\frac12}\log p+p(\log p)^2+N^{\frac12}p^{\frac12}\log p+p(\log p)^2$$
$$\ll y^{\frac12}p^{\frac12}(\log p)^2.$$

So we have

$$M_2\ll\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{n_1=1}^{N}\frac{d(n_1)}{n_1}\int_N^\infty\frac{y_2^{\frac12}p^{\frac12}(\log p)^2}{y_2^2}dy_2\ll p^{\frac32+\epsilon}N^{-\frac12}, \quad (3.2)$$

$$M_3\ll\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\int_N^\infty\frac{y_1^{\frac12}p^{\frac12}(\log p)^2}{y_1^2}dy_1\sum_{n_2=1}^{N}\frac{d(n_2)}{n_2}\ll p^{\frac32+\epsilon}N^{-\frac12}, \quad (3.3)$$

and

$$M_4\ll\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\int_N^\infty\frac{y_1^{\frac12}p^{\frac12}(\log p)^2}{y_1^2}dy_1\int_N^\infty\frac{y_2^{\frac12}p^{\frac12}(\log p)^2}{y_2^2}dy_2\ll p^{2+\epsilon}N^{-1}. \quad (3.4)$$

Now from (3.1)-(3.4) and the orthogonality relations for characters we get

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^k)\overline{\chi}(3^l)\left|L(1,\chi)\right|^4$$
$$=\sum_{n_1=1}^{N}\frac{d(n_1)}{n_1}\sum_{n_2=1}^{N}\frac{d(n_2)}{n_2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^kn_1)\overline{\chi}(3^ln_2)$$
$$+O\left(p^{\frac32+\epsilon}N^{-\frac12}\right)+O\left(p^{2+\epsilon}N^{-1}\right)$$
$$=\frac12\sum_{n_1=1}^{N}\frac{d(n_1)}{n_1}\sum_{n_2=1}^{N}\frac{d(n_2)}{n_2}\sum_{\chi \bmod p}\left(1-\chi(-1)\right)\chi(2^kn_1)\overline{\chi}(3^ln_2)$$
$$+O\left(p^{\frac32+\epsilon}N^{-\frac12}\right)+O\left(p^{2+\epsilon}N^{-1}\right)$$

$$= \frac{p-1}{2} \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}}^{N} \sum_{\substack{n_2=1 \\ (n_2,p)=1}}^{N} \frac{d(n_1)d(n_2)}{n_1 n_2} - \frac{p-1}{2} \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv -3^l n_2 (\bmod p)}}^{N} \sum_{\substack{n_2=1 \\ (n_2,p)=1}}^{N} \frac{d(n_1)d(n_2)}{n_1 n_2}$$

$$+O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right)$$

$$:= \frac{p-1}{2} M_{11} - \frac{p-1}{2} M_{12} + O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right). \quad (3.5)$$

First we consider M_{11}. We have

$$M_{11} = \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{1 \le n_2 \le \frac{p}{3^l}-1} \frac{d(n_1)d(n_2)}{n_1 n_2} + \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{\substack{\frac{p}{3^l} \le n_2 \le N \\ (n_2,p)=1}} \frac{d(n_1)d(n_2)}{n_1 n_2}$$

$$+ \sum_{\substack{\frac{p}{2^k} \le n_1 \le N \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{1 \le n_2 \le \frac{p}{3^l}-1} \frac{d(n_1)d(n_2)}{n_1 n_2} + \sum_{\substack{\frac{p}{2^k} \le n_1 \le N \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{\substack{\frac{p}{3^l} \le n_2 \le N \\ (n_2,p)=1}} \frac{d(n_1)d(n_2)}{n_1 n_2}$$

$$:= \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4. \quad (3.6)$$

It is not hard to show that

$$\Omega_1 = \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 2^k n_1 = 3^l n_2}} \sum_{1 \le n_2 \le \frac{p}{3^l}-1} \frac{d(n_1)d(n_2)}{n_1 n_2} = \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 3^l | n_1 \\ 2^k n_1 = 3^l n_2}} \sum_{\substack{1 \le n_2 \le \frac{p}{3^l}-1 \\ 2^k | n_2}} \frac{d(n_1)d(n_2)}{n_1 n_2}$$

$$= \sum_{\substack{1 \le n_1 \le \frac{p}{2^k 3^l}-1 \\ 2^k 3^l n_1 = 2^k 3^l n_2}} \sum_{1 \le n_2 \le \frac{p}{2^k 3^l}-1} \frac{d(3^l n_1) d(2^k n_2)}{3^l n_1 2^k n_2} = \sum_{1 \le n \le \frac{p}{2^k 3^l}-1} \frac{d(3^l n) d(2^k n)}{3^l 2^k n^2}$$

$$= \frac{1}{2^k 3^l} \sum_{n=1}^{\infty} \frac{d(2^k n) d(3^l n)}{n^2} + O\left(p^{\epsilon-1}\right), \quad (3.7)$$

$$\Omega_2 \ll p^{\epsilon} \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{\substack{\frac{p}{3^l} \le n_2 \le N \\ (n_2,p)=1}} \frac{1}{n_1 n_2} = p^{\epsilon} \sum_{\substack{1 \le n_1 \le \frac{p}{2^k}-1 \\ 2^k n_1 \equiv 3^l n_2 (\bmod p)}} \sum_{\substack{\frac{p}{3^l} \le n_2 \le N \\ (n_2,p)=1}} \frac{2^k 3^l}{2^k n_1 3^l n_2}$$

$$= \quad p^{\epsilon} \sum_{1\le r_1\le p-1} \sum_{\substack{1\le l_2\le \frac{3^l N}{p} \\ r_1\equiv l_2p+r_2(\bmod p)}} \sum_{1\le r_2\le p-1} \frac{2^k 3^l}{r_1(l_2p+r_2)}$$

$$\ll \quad p^{\epsilon} \sum_{1\le r\le p-1} \sum_{1\le l_2\le \frac{3^l N}{p}} \frac{1}{r(l_2p+r)}$$

$$\ll \quad p^{\epsilon} \sum_{1\le r\le p-1} \frac{1}{r} \sum_{1\le l_2\le \frac{3^l N}{p}} \frac{1}{l_2p} \ll p^{\epsilon-1}, \tag{3.8}$$

$$\Omega_3 \ll p^{\epsilon} \sum_{\substack{\frac{p}{2^k}\le n_1\le N \\ (n_1,p)=1 \\ 2^k n_1\equiv 3^l n_2(\bmod p)}} \sum_{1\le n_2\le \frac{p}{3^l}-1} \frac{1}{n_1n_2} = p^{\epsilon} \sum_{\substack{\frac{p}{2^k}\le n_1\le N \\ (n_1,p)=1 \\ 2^k n_1\equiv 3^l n_2(\bmod p)}} \sum_{1\le n_2\le \frac{p}{3^l}-1} \frac{2^k 3^l}{2^k n_1 3^l n_2}$$

$$= \quad p^{\epsilon} \sum_{1\le l_1\le \frac{2^k N}{p}} \sum_{\substack{1\le r_1\le p-1 \\ l_1p+r_1\equiv r_2(\bmod p)}} \sum_{1\le r_2\le p-1} \frac{2^k 3^l}{(l_1p+r_1)r_2}$$

$$\ll \quad p^{\epsilon} \sum_{1\le l_1\le \frac{2^k N}{p}} \sum_{1\le r\le p-1} \frac{1}{(l_1p+r)r}$$

$$\ll \quad p^{\epsilon} \sum_{1\le l_1\le \frac{2^k N}{p}} \frac{1}{l_1p} \sum_{1\le r\le p-1} \frac{1}{r} \ll p^{\epsilon-1}, \tag{3.9}$$

and

$$\Omega_4 \ll p^{\epsilon} \sum_{\substack{\frac{p}{2^k}\le n_1\le N \\ (n_1,p)=1 \\ 2^k n_1\equiv 3^l n_2(\bmod p)}} \sum_{\substack{\frac{p}{3^l}\le n_2\le N \\ (n_2,p)=1}} \frac{1}{n_1n_2} = p^{\epsilon} \sum_{\substack{\frac{p}{2^k}\le n_1\le N \\ (n_1,p)=1 \\ 2^k n_1\equiv 3^l n_2(\bmod p)}} \sum_{\substack{\frac{p}{3^l}\le n_2\le N \\ (n_2,p)=1}} \frac{2^k 3^l}{2^k n_1 3^l n_2}$$

$$= \quad p^{\epsilon} \sum_{1\le l_1\le \frac{2^k N}{p}} \sum_{1\le r_1\le p-1} \sum_{\substack{1\le l_2\le \frac{3^l N}{p} \\ l_1p+r_1\equiv l_2p+r_2(\bmod p)}} \sum_{1\le r_2\le p-1} \frac{2^k 3^l}{(l_1p+r_1)(l_2p+r_2)}$$

$$\ll \quad p^{\epsilon} \sum_{1\le l_1\le \frac{2^k N}{p}} \sum_{1\le r\le p-1} \sum_{1\le l_2\le \frac{3^l N}{p}} \frac{1}{(l_1p+r)(l_2p+r)}$$

$$
\begin{aligned}
&\ll p^{\epsilon} \sum_{1\le l_1\le \frac{2^kN}{p}} \sum_{1\le r\le p-1} \frac{1}{l_1p+r} \sum_{1\le l_2\le \frac{3^lN}{p}} \frac{1}{l_2p} \ll p^{\epsilon-1} \sum_{1\le n\le 2^kN} \frac{1}{n} \\
&\ll p^{\epsilon-1}. \qquad (3.10)
\end{aligned}
$$

Then from (3.6)-(3.10) we have

$$
M_{11} = \frac{1}{2^k3^l} \sum_{n=1}^{\infty} \frac{d(2^kn)d(3^ln)}{n^2} + O\left(p^{\epsilon-1}\right). \qquad (3.11)
$$

Similarly we can get

$$
\begin{aligned}
M_{12} &= \sum_{\substack{n_1=1\\(n_1,p)=1}}^{N} \sum_{\substack{n_2=1\\(n_2,p)=1\\ 2^kn_1\equiv -3^ln_2(\bmod p)}}^{N} \frac{d(n_1)d(n_2)}{n_1n_2} \ll p^{\epsilon} \sum_{\substack{n_1=1\\(n_1,p)=1}}^{N} \sum_{\substack{n_2=1\\(n_2,p)=1\\ 2^kn_1\equiv -3^ln_2(\bmod p)}}^{N} \frac{1}{n_1n_2} \\
&\ll p^{\epsilon} \sum_{0\le l_1\le \frac{2^kN}{p}} \sum_{1\le r_1\le p-1} \sum_{0\le l_2\le \frac{3^lN}{p}} \sum_{\substack{1\le r_2\le p-1\\ l_1p+r_1\equiv -(l_2p+r_2)(\bmod p)}} \frac{1}{(l_1p+r_1)(l_2p+r_2)} \\
&\ll p^{\epsilon} \sum_{0\le l_1\le \frac{2^kN}{p}} \sum_{1\le r\le p-1} \sum_{0\le l_2\le \frac{3^lN}{p}} \frac{1}{(l_1p+r)(l_2p+p-r)} \\
&= p^{\epsilon} \sum_{1\le r\le p-1} \frac{1}{r(p-r)} + p^{\epsilon} \sum_{1\le r\le p-1} \sum_{1\le l_2\le \frac{3^lN}{p}} \frac{1}{r(l_2p+p-r)} \\
&\quad + p^{\epsilon} \sum_{1\le l_1\le \frac{2^kN}{p}} \sum_{1\le r\le p-1} \frac{1}{(l_1p+r)(p-r)} \\
&\quad + p^{\epsilon} \sum_{1\le l_1\le \frac{2^kN}{p}} \sum_{1\le r\le p-1} \sum_{1\le l_2\le \frac{3^lN}{p}} \frac{1}{(l_1p+r)(l_2p+p-r)} \\
&\ll p^{\epsilon-1} \sum_{1\le r\le p-1} \left(\frac{1}{r} + \frac{1}{p-r}\right) + p^{\epsilon} \sum_{1\le r\le p-1} \frac{1}{r} \sum_{1\le l_2\le \frac{3^lN}{p}} \frac{1}{l_2p} \\
&+ p^{\epsilon} \sum_{1\le l_1\le \frac{2^kN}{p}} \frac{1}{l_1p} \sum_{1\le r\le p-1} \frac{1}{p-r} + p^{\epsilon} \sum_{1\le l_1\le \frac{2^kN}{p}} \sum_{1\le r\le p-1} \frac{1}{l_1p+r} \sum_{1\le l_2\le \frac{3^lN}{p}} \frac{1}{l_2p} \\
&\ll p^{\epsilon-1}. \qquad (3.12)
\end{aligned}
$$

Now taking $N = p^3$ and combining (3.5), (3.11), (3.12) we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\overline{\chi}(3^l)\left|L(1,\chi)\right|^4 = \frac{p}{2^{k+1}3^l}\sum_{n=1}^{\infty}\frac{d(2^k n)d(3^l n)}{n^2} + O\left(p^{\epsilon}\right) \quad (3.13)$$

Thus from Theorem 2.1 we immediately get Theorem 3.1. □

Corollary 3.1. *Let $p \geq 5$ be a prime, and let k be fixed non-negative integer. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\left|L(1,\chi)\right|^4 = \frac{(3k+5)\pi^4}{144\cdot 2^k}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k)\left|L(1,\chi)\right|^4 = \frac{(4k+5)\pi^4}{144\cdot 3^k}p + O\left(p^{\epsilon}\right).$$

Theorem 3.2. *Let $p \geq 5$ be a prime. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\left|L(2,\chi\chi_3)\right|^2 = \frac{45L^2(3,\chi_3)}{52}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\left|L(2,\chi\chi_4)\right|^2 = \frac{5L^2(3,\chi_4)}{6}p + O\left(p^{\epsilon}\right).$$

Proof. Define $\tau(n) = \sum_{t|n}\frac{\chi_3(t)}{t}$ and $\rho(n) = \sum_{t|n}\frac{\chi_4(t)}{t}$. By using Theorem 2.2, Theorem 2.3 and the methods of Theorem 3.1 we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\left|L(2,\chi\chi_3)\right|^2 = \frac{p}{2}\sum_{n=1}^{\infty}\frac{\tau^2(n)}{n^2} + O\left(p^{\epsilon}\right) = \frac{45L^2(3,\chi_3)}{52}p + O\left(p^{\epsilon}\right),$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\left|L(2,\chi\chi_4)\right|^2 = \frac{p}{2}\sum_{n=1}^{\infty}\frac{\rho^2(n)}{n^2} + O\left(p^{\epsilon}\right) = \frac{5L^2(3,\chi_4)}{6}p + O\left(p^{\epsilon}\right).$$

□

THEOREM 3.3. *Let $p \geq 5$ be a prime, and let k be a non-negative integer. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_3) = \left(1+\frac{(-1)^k}{2^{k-1}\cdot 9}\right)\frac{\pi^4 L^2(3,\chi_3)}{2^{k+3}\cdot 11L(5,\chi_3)}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) \left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_3) = \frac{\pi^4 L^2(3,\chi_3)}{3^{k+2}\cdot 8L(5,\chi_3)}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_4) = \frac{\pi^4 L^2(3,\chi_4)}{2^{k+3}\cdot 9L(5,\chi_4)}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) \left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_4) = \left(\frac{49}{4}+\frac{(-1)^k}{3^{k-1}}\right)\frac{\pi^4 L^2(3,\chi_4)}{3^{k}\cdot 1098L(5,\chi_4)}p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 L(2,\chi\chi_4)L(2,\overline{\chi}\chi_3) = \frac{\pi^2 L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)}{12L(6,\chi_3\chi_4)}p + O\left(p^{\epsilon}\right).$$

Proof. Define $\tau(n) = \sum_{t|n} \frac{\chi_3(t)}{t}$ and $\rho(n) = \sum_{t|n} \frac{\chi_4(t)}{t}$. It is easy to show that $\tau(3^k n) = \tau(n)$ and $\rho(2^k n) = \rho(n)$. By using Theorem 2.2, Theorem 2.3 and the methods of Theorem 3.1 we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_3) = \frac{p}{2^{k+1}}\sum_{n=1}^{\infty}\frac{d(n)\tau(2^k n)}{n^2} + O\left(p^{\epsilon}\right)$$

$$= \left(1 + \frac{(-1)^k}{2^{k-1} \cdot 9}\right) \frac{\pi^4 L^2(3,\chi_3)}{2^{k+3} \cdot 11 L(5,\chi_3)} p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) \left|L(1,\chi)\right|^2 L(1,\chi) L(2,\overline{\chi}\chi_3) = \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\tau(3^k n)}{n^2} + O\left(p^{\epsilon}\right)$$

$$= \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\tau(n)}{n^2} + O\left(p^{\epsilon}\right) = \frac{\pi^4 L^2(3,\chi_3)}{3^{k+2} \cdot 8 L(5,\chi_3)} p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left|L(1,\chi)\right|^2 L(1,\chi) L(2,\overline{\chi}\chi_4) = \frac{p}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n)\rho(2^k n)}{n^2} + O\left(p^{\epsilon}\right)$$

$$= \frac{p}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n)\rho(n)}{n^2} + O\left(p^{\epsilon}\right) = \frac{\pi^4 L^2(3,\chi_4)}{2^{k+3} \cdot 9 L(5,\chi_4)} p + O\left(p^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) \left|L(1,\chi)\right|^2 L(1,\chi) L(2,\overline{\chi}\chi_4) = \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\rho(3^k n)}{n^2} + O\left(p^{\epsilon}\right)$$

$$= \left(\frac{49}{4} + \frac{(-1)^k}{3^{k-1}}\right) \frac{\pi^4 L^2(3,\chi_4)}{3^k \cdot 1098 L(5,\chi_4)} p + O\left(p^{\epsilon}\right),$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 L(2,\chi\chi_4) L(2,\overline{\chi}\chi_3) = \frac{p}{2} \sum_{n=1}^{\infty} \frac{\tau(n)\rho(n)}{n^2} + O\left(p^{\epsilon}\right)$$

$$= \frac{\pi^2 L(3,\chi_3) L(3,\chi_4) L(4,\chi_3\chi_4)}{12 L(6,\chi_3\chi_4)} p + O\left(p^{\epsilon}\right).$$

□

4. Mean values of the homogeneous Dedekind sums

Let $q > 3$ be an integer, and let χ be a Dirichlet character modulo q. The generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$\sum_{a=1}^{q} \chi(a) \frac{t\mathrm{e}^{at}}{\mathrm{e}^{qt} - 1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n.$$

Let r be a positive integer prime to q, and let $k \geq 0$ be an integer. By (6) of [7] and (2.12) of [3] we have

$$\frac{1}{q^k} \sum_{1 \leq a \leq q/r} a^k \chi(a) = -\frac{1}{q^k(k+1)} B_{k+1,\chi} + \frac{\overline{\chi}(r)}{(rq)^k \phi(r)} \sum_{\psi \bmod r} \overline{\psi}(-q) \sum_{n=1}^{k+1} \frac{1}{k+1} \binom{k+1}{n} q^{k+1-n} B_{n,\chi\psi},$$

where $\sum_\psi$ is over all characters ψ modulo r.

By using the above formula H. Liu [5] showed the following lemma.

LEMMA 4.1. *Let χ be a primitive character modulo $q > 3$. Then*

$$\frac{1}{q} \sum_{1 \leq a \leq q/3} a\chi(a) = \begin{cases} \frac{1}{6\pi i} \left(1 - \overline{\chi}(3)\right) \tau(\chi) L(1, \overline{\chi}) + \frac{\sqrt{3} i}{(2\pi i)^2} \tau(\chi) L(2, \overline{\chi}\chi_3), & \text{if } \chi(-1) = -1, \\ \frac{\sqrt{3}}{6\pi} \tau(\chi) L(1, \overline{\chi}\chi_3) + \frac{1}{(2\pi i)^2} \left(3 - \frac{\overline{\chi}(3)}{3}\right) \tau(\chi) L(2, \overline{\chi}), & \text{if } \chi(-1) = 1, \end{cases}$$

$$\frac{1}{q} \sum_{1 \leq a \leq q/4} a\chi(a) = \begin{cases} \frac{1}{8\pi i} \left(\overline{\chi}(2) - \overline{\chi}(4)\right) \tau(\chi) L(1, \overline{\chi}) + \frac{2i}{(2\pi i)^2} \tau(\chi) L(2, \overline{\chi}\chi_4), \\ \qquad \text{if } \chi(-1) = -1, \\ \frac{1}{4\pi} \tau(\chi) L(1, \overline{\chi}\chi_4) + \frac{1}{(2\pi i)^2} \left(2 + \frac{\overline{\chi}(2)}{2} - \frac{\overline{\chi}(4)}{4}\right) \tau(\chi) L(2, \overline{\chi}), \\ \qquad \text{if } \chi(-1) = 1, \end{cases}$$

where $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$ denotes the Gauss sum, χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4.

On the other hand, by [8] we know that

$$S(a, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1) = -1}} \chi(a) \left|L(1, \chi)\right|^2 .$$

Then we have

$$S(a,b,p) = S(a\bar{b},p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)\overline{\chi}(b)\left|L(1,\chi)\right|^2.$$

Noting that $\tau(\chi)\tau(\overline{\chi}) = p\chi(-1)$. From Lemma 4.1 we have

$$\sum_{a\le \frac{p}{3}}\sum_{b\le \frac{p}{3}} abS(a,b,p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 \sum_{a\le \frac{p}{3}} a\chi(a) \sum_{b\le \frac{p}{3}} b\overline{\chi}(b)$$

$$= \frac{p^4}{18\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (1-\chi(3))\left|L(1,\chi)\right|^4$$

$$+\frac{\sqrt{3}p^4}{12\pi^5(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (1-\chi(3))\left|L(1,\chi)\right|^2 L(1,\chi)L(2,\overline{\chi}\chi_3)$$

$$+\frac{3p^4}{16\pi^6(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\left|L(2,\chi\chi_3)\right|^2$$

$$= \frac{p^4}{18\pi^4(p-1)}\left(\frac{5\pi^4 p}{144} - \frac{(4+5)\pi^4 p}{144\times 3}\right)$$

$$+\frac{\sqrt{3}p^4}{12\pi^5(p-1)}\left(\frac{\pi^4 L^2(3,\chi_3)p}{9\times 8L(5,\chi_3)} - \frac{\pi^4 L^2(3,\chi_3)p}{3^3\cdot 8L(5,\chi_3)}\right)$$

$$+\frac{3p^4}{16\pi^6(p-1)}\left(\frac{45L^2(3,\chi_3)p}{52}\right) + \left(p^{3+\epsilon}\right)$$

$$= \frac{p^4}{1296} + \frac{\sqrt{3}p^4}{12\pi^5}\left(\frac{1}{9\times 8} - \frac{1}{27\times 8}\right)\frac{\pi^4 L^2(3,\chi_3)}{L(5,\chi_3)} + \frac{135L^2(3,\chi_3)p^4}{832\pi^6} + \left(p^{3+\epsilon}\right)$$

$$= \left(\frac{1}{1296} + \frac{\sqrt{3}L^2(3,\chi_3)}{1296\pi L(5,\chi_3)} + \frac{135L^2(3,\chi_3)}{832\pi^6}\right)p^4 + \left(p^{3+\epsilon}\right).$$

Similarly we get

$$\sum_{a\le \frac{p}{4}}\sum_{b\le \frac{p}{4}} abS(a,b,p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 \sum_{a\le \frac{p}{4}} a\chi(a) \sum_{b\le \frac{p}{4}} b\overline{\chi}(b)$$

$$= \frac{p^4}{32\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (1-\chi(2)) |L(1,\chi)|^4$$
$$+\frac{p^4}{8\pi^5(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (\chi(2)-\chi(4)) |L(1,\chi)|^2 L(1,\chi)L(2,\overline{\chi}\chi_4)$$
$$+\frac{p^4}{4\pi^6(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 |L(2,\chi\chi_4)|^2$$
$$= \frac{p^4}{32\pi^4(p-1)} \left(\frac{5\pi^4 p}{144} - \frac{(3+5)\pi^4 p}{2\times 144}\right)$$
$$+\frac{p^4}{8\pi^5(p-1)} \left(\frac{\pi^4 L^2(3,\chi_4)p}{2^4\cdot 9L(5,\chi_4)} - \frac{\pi^4 L^2(3,\chi_4)p}{2^5\cdot 9L(5,\chi_4)}\right)$$
$$+\frac{p^4}{4\pi^6(p-1)} \times \frac{5L^2(3,\chi_4)p}{6} + \left(p^{3+\epsilon}\right)$$
$$= \frac{p^4}{32}\left(\frac{10-8}{2\times 144}\right) + \frac{p^4 L^2(3,\chi_4)}{8\pi L(5,\chi_4)}\left(\frac{2-1}{2^5\times 9}\right) + \frac{5L^2(3,\chi_4)p^4}{24\pi^6} + \left(p^{3+\epsilon}\right)$$
$$= \left(\frac{1}{4608} + \frac{L^2(3,\chi_4)}{2304\pi L(5,\chi_4)} + \frac{5L^2(3,\chi_4)}{24\pi^6}\right) p^4 + \left(p^{3+\epsilon}\right),$$

and

$$\sum_{a\le \frac{p}{3}}\sum_{b\le \frac{p}{4}} abS(a,b,p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 \sum_{a\le \frac{p}{3}} a\chi(a) \sum_{b\le \frac{p}{4}} b\overline{\chi}(b)$$
$$= \frac{p^4}{48\pi^4(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (\chi(2)-\chi(4)-\chi(2)\overline{\chi}(3)+\chi(4)\overline{\chi}(3)) |L(1,\chi)|^4$$
$$+\frac{\sqrt{3}p^4}{32\pi^5(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (\chi(2)-\chi(4)) |L(1,\chi)|^2 L(1,\chi)L(2,\overline{\chi}\chi_3)$$
$$+\frac{p^4}{12\pi^5(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (1-\chi(3)) |L(1,\chi)|^2 L(1,\chi)L(2,\overline{\chi}\chi_4)$$

$$+\frac{\sqrt{3}p^4}{8\pi^6(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2 L(2,\chi\chi_4)L(2,\overline{\chi}\chi_3)$$

$$\begin{aligned}
&= \frac{p^4}{48\pi^4(p-1)}\left(\frac{8\pi^4 p}{144\times 2}-\frac{11\pi^4 p}{144\times 4}-\frac{8\times 9\pi^4 p}{2\times 3\times 720}+\frac{11\times 9\pi^4 p}{4\times 3\times 720}\right)\\
&\quad+\frac{\sqrt{3}p^4}{32\pi^5(p-1)}\left(\left(1-\frac{1}{9}\right)\frac{\pi^4L^2(3,\chi_3)p}{2^4\cdot 11L(5,\chi_3)}-\left(1+\frac{1}{2\times 9}\right)\frac{\pi^4L^2(3,\chi_3)p}{2^5\cdot 11L(5,\chi_3)}\right)\\
&\quad+\frac{p^4}{12\pi^5(p-1)}\left(\left(\frac{49}{4}+3\right)\frac{\pi^4L^2(3,\chi_4)p}{1098L(5,\chi_4)}-\left(\frac{49}{4}-1\right)\frac{\pi^4L^2(3,\chi_4)p}{3\times 1098L(5,\chi_4)}\right)\\
&\quad+\frac{\sqrt{3}p^4}{8\pi^6(p-1)}\times\frac{\pi^2L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)p}{12L(6,\chi_3\chi_4)}+O\left(p^{3+\epsilon}\right)\\
&= \frac{p^4}{48\times 144}\left(4-\frac{11}{4}-\frac{12}{5}+\frac{33}{20}\right)+\frac{\sqrt{3}p^4L^2(3,\chi_3)}{32\times 11\times 16\times 9\pi L(5,\chi_3)}\left(8-\frac{19}{4}\right)\\
&\quad+\frac{p^4L^2(3,\chi_4)}{12\times 4\times 1098\pi L(5,\chi_4)}(61-15)+\frac{\sqrt{3}p^4L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)}{96\pi^4L(6,\chi_3\chi_4)}\\
&\quad+O\left(p^{3+\epsilon}\right)\\
&= \left(\frac{1}{13824}+\frac{13\sqrt{3}L^2(3,\chi_3)}{202752\pi L(5,\chi_3)}+\frac{23L^2(3,\chi_4)}{26352\pi L(5,\chi_4)}\right.\\
&\qquad\left.+\frac{\sqrt{3}L(3,\chi_3)L(3,\chi_4)L(4,\chi_3\chi_4)}{96\pi^4L(6,\chi_3\chi_4)}\right)p^4+O\left(p^{3+\epsilon}\right).
\end{aligned}$$

This proves Theorem 1.1.

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