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ESSENTIAL NORM OF THE PULL BACK OPERATOR

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Abstract. We obtain some estimations of the essential norm of a pull back operator induced by quasi-symmetric homeomorphisms. As a corollary, we deduce the compactness criterion of this operator.

1. Introduction

Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\Delta^* = \overline{\mathbb{C}} \backslash \overline{\Delta}$. A homeomorphism h is said to be quasis-ymmetric if there is some $M > 0$, called the quasis-ymmetric constant of h, such that

$$\frac{1}{M} \leq \left| \frac{h(e^{i(\theta+t)}) - h(e^{i\theta})}{h(e^{i\theta}) - h(e^{i(\theta-t)})} \right| \leq M$$

for all θ and $t > 0$. Denote by $QS(S^1)$ the group of quasi-symmetric homeomorphisms of the unit circle S^1. Beurling and Ahlfors [1] proved that a sense preserving self-homeomorphism h is quasi-symmetric if and only if there exists some quasi-conformal homeomorphism of Δ onto itself which has boundary value h. Later Douady and Earle [3] gave a quasi-conformal extension of h to the unit disk which is also conformally

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invariant. Let Möb(S^1) be the group of Möbius transformations mapping Δ onto itself. The universal Teichmüller space is the right coset space $T = QS(S^1)/\text{Möb}(S^1)$.

A quasis-ymmetric homeomorphism h is said to be symmetric if

$$\lim_{t\to 0^+} \frac{h(e^{i(\theta+t)}) - h(e^{i\theta})}{h(e^{i\theta}) - h(e^{i(\theta-t)})} = 1$$

for all θ and $t > 0$. Let $S(S^1)$ denote the set of all symmetric homeomorphisms of the unit circle. Then $S(S^1)$ is a subgroup of $QS(S^1)$. The little universal Teichmüller space is defined as $T_0 = S(S^1)/\text{Möb}(S^1)$. The class of symmetric homeomorphisms has several equivalent definitions and has been much investigated in classical complex analysis [6]. For any quasi-conformal homeomorphism f of the unit disk Δ onto itself with Beltrami coefficient $\mu(z)$, define $b^*(f)$ to be the infimum taken over all compact subset F contained in Δ of the essential supremum norm of $\mu(z)$ as z varies over $\Delta\backslash F$. We say a quasis-ymmetric homeomorphism f is asymptotically conformal if $b^*(f) = 0$. Define the boundary dilatation $b(h)$ of a quasis-ymmetric homeomorphism h to be the infimum of $b^*(f)$ taken over all quasi-conformal mapping f with the boundary value $f|_{S^1} = h$. The following results are well known.

Proposition 1.1. [4] *A quasis-ymmetric homeomorphism h is symmetric if and only if $b(h) = 0$.*

Actually, Gardiner and Sullivan [4] proved that for a symmetric homeomorphism, the Beurling-Ahlfors extension is asymptotically conformal. The Douady-Earle extension also has this property (see [2] and [5]).

Hu and Shen [5] introduced some pull-back operators and functions induced by quasis-ymmetric homeomorphism to study the universal Teichmüller space and some subspaces of the universal Teichmüller space. We recall some notations and definitions.

The Bergman space A^2 consists of all holomorphic functions ϕ in the unit disk Δ with finite norm

$$\|\phi\| = \left(\frac{1}{\pi}\iint_\Delta |\phi(z)|^2 dxdy\right)^{\frac{1}{2}} < \infty. \tag{1}$$

This is a Hilbert space with inner product defined as

$$\langle\phi,\psi\rangle = \frac{1}{\pi}\iint_\Delta \phi(z)\overline{\psi(z)}dxdy. \tag{2}$$

Let h be a quasis-ymmetric homeomorphism in the unit circle. The following two kernel functions induced by h were introduced in [5],

$$(3)\quad \phi_h(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))}dw, \qquad (\zeta, z) \in \Delta\times\Delta,$$

$$(4)\quad \psi_h(\zeta, z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(\zeta - w)^2(1-zh(w))}dw, \qquad (\zeta, z) \in \Delta\times\Delta.$$

Both ϕ_h and ψ_h are holomorphic functions. It is noted that the function ϕ_h was also appeared in Cui [2]. The two kernel functions induce the following two operators from Bergman space A^2 into itself respectively,

$$(5)\qquad T_h^-\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \phi_h(\zeta, z)\psi(\bar{z})dxdy, \qquad \zeta\in\Delta,$$

$$(6)\qquad T_h^+\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \psi_h(\zeta, z)\psi(\bar{z})dxdy, \qquad \zeta\in\Delta.$$

The two kernel functions also induce two functions,

$$(7)\qquad \phi_h(z) = \left(\frac{1}{\pi}\iint_\Delta |\phi_h(\zeta, z)|^2 d\xi d\eta\right)^{\frac{1}{2}}, \qquad z\in\Delta,$$

$$(8)\qquad \psi_h(z) = \left(\frac{1}{\pi}\iint_\Delta |\psi_h(\zeta, z)|^2 d\xi d\eta\right)^{\frac{1}{2}}, \qquad z\in\Delta.$$

These pull-back operators and functions play an important role in the Teichmüller theory (see [5], [8], [9]). They were used in [5] and [8] to characterize when a quasis-ymmetric homeomorphism is symmetric or belongs to the Weil-Petersson class. They were also used to study the BMO-Teichmüller theory in [9].

Proposition 1.2. [5] *$T_h^-: A^2\to A^2$ is bounded operator if and only if h is quasis-ymmetric and $\|T_h^-\| \le \frac{\|\mu\|_\infty}{\sqrt{1-\|\mu\|_\infty^2}}$, where μ is the Beltrami coefficient of a quasi-conformal extension of h.*

Proposition 1.3. [5] *Let h be a quasis-ymmetric homeomorphism. Then the following statements are equivalent:*
(1) $T_h^-: A^2\to A^2$ is a compact operator;
(2) h is symmetric;
(3) $\lim_{|z|\to 1}\phi_h(z)(1-|z|^2) = 0$.

A nature problem is how to estimate the essential norm of the pull-back operator T_h^-. In this note, we will give some estimations of the essential norm of the pull back operator T_h^- and then deduce the compactness criterion of this operator. We first recall the definition of essential norm of a operator in Banach space. Let X and Y be Banach spaces. For a bounded linear operator $T : X \to Y$, the essential norm $\|T\|_e$ is defined to be the distance from T to the set of the compact operators $K : X \to Y$, precisely

$$\|T\|_e = \inf \|T - K\| \tag{9}$$

where the infimum is taken over all compact operators K from X into Y and $\|\cdot\|$ denotes the usual operator norm. Note that T is compact if and only if $\|T\|_e = 0$.

We first give a representation formula for the essential norm of the pull-back operator by means of the degenerating sequence (see the definition in the next section). Then we obtain our main estimation of the essential norm of the pull back operator T_h^- as follows.

Theorem 1.4. *Let h be a quasis-ymmetric homeomorphism. Then there is a constant $C > 0$ which depends only on the quasis-ymmetric constant of h such that*

$$\overline{\lim_{|a|\to 1}}(1-|a|^2)\phi_h(a) \le \|T_h^-\|_e \le C\,\overline{\lim_{|a|\to 1}}(1-|a|^2)\phi_h(a). \tag{10}$$

By means of Proposition 1.1, Proposition 1.3 and Theorem 1.4, we have the following result.

Corollary 1.5. *Let h be a quasis-ymmetric homeomorphism. Then the following statements are equivalent:*
(1) $T_h^- : A^2 \to A^2$ *is a compact operator;*
(2) h *is symmetric;*
(3) $\lim_{|z|\to 1} \phi_h(z)(1-|z|^2) = 0$;
(4) $b(h) = 0$.

The formula of the essential norm of T_h^- will be given in the second section and the proofs of Theorems 3.1 and 1.4 will be presented in the next two sections.

2. A formula of the essential norm of T_h^-

Let ϕ be an analytic function in Δ with Taylor expansion

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k.$$

For any positive integer $n \geq 1$, define two operators as following

$$\mathbb{R}_n \phi(z) = \sum_{k=n}^{\infty} a_k z^k,$$

and

$$\mathbb{K}_n = I - \mathbb{R}_n,$$

where I is the identity operator. We need the following unified estimation for the operator $\mathbb{R}_n$.

Lemma 2.1. *For any $\epsilon > 0$ and $0 < r < 1$, there is a positive integer N_0 which depends only on r such that for any $n > N_0$, $|a| < r$ and $f \in A^2$,*

$$\sup_{\|f\| \leq 1} |\mathbb{R}_n f(a)| < \epsilon. \tag{11}$$

Proof. The reproducing kernel function in A^2 is

$$K_a(z) = \frac{1}{(1-\bar{a}z)^2}, \qquad a \in \Delta, z \in \Delta.$$

For fixed $a \in \Delta$, the function $K_a(z)$ is a bounded analytic function. It is easy to see that the operator $\mathbb{R}_n$ is a self-adjoint operator in A^2. Therefore, for any $f \in A^2$,

$$\langle \mathbb{R}_n f, K_a \rangle = \langle f, \mathbb{R}_n K_a \rangle.$$

This yields

$$|\mathbb{R}_n f(a)| = \frac{1}{\pi} |\langle \mathbb{R}_n f, K_a \rangle| = \frac{1}{\pi} |\langle f, \mathbb{R}_n K_a \rangle| \leq \frac{1}{\pi} \|f\| \|\mathbb{R}_n K_a\|_{\infty}.$$

It is noted that $K_a(z) = \sum_{n=0}^{\infty} (n+1) \bar{a}^n z^n$, therefore for any $|a| < r$,

$$|\mathbb{R}_n K_a(z)| = |\sum_{k=n}^{\infty} (k+1) \bar{a}^k z^k| \leq \sum_{k=n}^{\infty} (k+1) r^k.$$

For any $\epsilon > 0$, take N_0 such that $\frac{1}{\pi}\sum_{k=N_0}^{\infty}(k+1)r^k < \epsilon$, then when $n > N_0$ and $|a| < r$,

$$\sup_{\|f\|\leq 1} |\mathbb{R}_n f(a)| < \epsilon.$$

The proof of Lemma 2.1 is completed. □

We say a sequence $\{\varphi_n\} \in A^2$ is degenerating, if $\|\varphi_n\| \leq 1$ and $\{\varphi_n\}$ converges uniformly to zero uniformly on any compacted subset of Δ. The following result gives an expression of the essential norm of the operator T_h^-.

THEOREM 2.2. *Let h be a quasis-ymmetric homeomorphism. Then*

$$\|T_h^-\|_e = \sup_{\{\varphi_n\}} \left\{ \overline{\lim_{n\to\infty}} \|T_h^- \varphi_n\| \right\}, \tag{12}$$

where the supremum is taken over all degenerating sequence $\{\varphi_n\} \subset A^2$
.

Proof. Note that the degenerating sequence $\{\varphi_n\}$ in A^2 weakly converges to zero. Therefore, for any compact operator $K : A^2 \to A^2$, we have $\|K(\varphi_n)\| \to 0$ as $n \to \infty$. We deduce that

$$\begin{aligned} \|T_h^- - K\| &\geq \overline{\lim_{n\to\infty}} \|(T_h^- - K)(\varphi_n)\| \\ &\geq \overline{\lim_{n\to\infty}} \|T_h^-(\varphi_n)\| - \overline{\lim_{n\to\infty}} \|K(\varphi_n)\| \\ &= \overline{\lim_{n\to\infty}} \|T_h^-(\varphi_n)\|. \end{aligned}$$

Take the supremun over all degenerated sequence $\{\varphi_n\} \in A^2$, and then take the infimum over all compact operator $K : A^2 \to A^2$, we have

$$\|T_h^-\|_e \geq \sup_{\{\varphi_n\}} \left\{ \overline{\lim_{n\to\infty}} \|T_h^- \varphi_n\| \right\}.$$

Noting that h is a quasis-ymmetric homeomorphism, by Proposition 1.2, we know that T_h^- is a bounded operator in A^2. It is noted that for each n, $\mathbb{K}_n$ is a compact operator, which implies that $T_h^- \mathbb{K}_n$ is also a compact operator for all n. Therefore, we have

$$\|T_h^-\|_e = \|T_h^- \mathbb{R}_n + T_h^- \mathbb{K}_n\|_e \leq \|T_h^- \mathbb{R}_n\| \leq \overline{\lim_{n\to\infty}} \sup_{\|\phi\|\leq 1} \|T_h^- \mathbb{R}_n(\phi)\|.$$

For each n, there is a sequence $\{\phi_m^n\} \subset A^2$ with $\|\phi_m^n\| \leq 1$ such that

$$\|T_h^- \mathbb{R}_n\| = \overline{\lim_{m\to\infty}} \|T_h^- \mathbb{R}_n(\phi_m^n)\|.$$

We choose a sequence $\{\phi_n\} \subset A^2$ with $\|\phi_n\| \leq 1$ such that

$$\|T_h^- \mathbb{R}_n\| \leq \|T_h^- \mathbb{R}_n(\phi_n)\| + \frac{1}{n}.$$

Denote the sequence $\{\mathbb{R}_n(\phi_n)\}$ by $\{\varphi_n\}$. Note that for each n, $\mathbb{R}_n$ is a projection operator from A^2 to A^2, therefore $\|\mathbb{R}_n\| = 1$ and $\|\varphi_n\| \leq 1$. It follows from Lemma 2.1 that the sequence $\{\varphi_n\}$ converges uniformly to zero on any compact subset of Δ. Thus the sequence $\{\varphi_n\}$ is a degenerating sequence and

$$\|T_h^-\|_e \leq \overline{\lim_{n\to\infty}} \|T_h^- \mathbb{R}_n\| \leq \overline{\lim_{n\to\infty}} \|T_h^- \varphi_n\|.$$

Therefore we have

$$\|T_h^-\|_e \leq \sup_{\{\varphi_n\}} \left\{ \overline{\lim_{n\to\infty}} \|T_h^- \varphi_n\| \right\},$$

where the supremum is taken over all degenerating sequence $\{\varphi_n\} \subset A^2$. The proof of Theorem 2.2 is completed. □

3. Estimations in terms of boundary distortion

In this section, an estimation of the essential norm of the pull-back operator by means of the boundary distortion will be given. We prove the following result.

THEOREM 3.1. *Let h be a quasis-ymmetric homeomorphism. Then*

$$\overline{\lim_{|a|\to 1}} (1 - |a|^2)|\phi_h(a)| \leq \|T_h^-\|_e \leq \frac{b(h)}{\sqrt{1 - b(h)^2}}, \tag{13}$$

where $b(h)$ is the boundary dilatation of h.

To proof the theorem, we need the following results.

LEMMA 3.2. *[5] [9] Let h be a quasis-ymmetric homeomorphism.*
(1) For any $\psi \in A^2$, choosing ϕ such that $\phi' = \psi$, we have

$$T_h^- \psi(\zeta) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\phi(h(w))}{(1 - \zeta w)^2} dw.$$

(2) Let $a \in \Delta$ and $K_a(\zeta) = \frac{1-|a|^2}{(1-a\zeta)^2}$. Then we have

$$T_h^-(K_a(\zeta)) = (1 - |a|^2)\phi_h(\zeta, a).$$

We first estimate the lower bound. Let $a \in \Delta$. Consider the function

$$K_a(\zeta) = \frac{1-|a|^2}{(1-a\zeta)^2},$$

it is known that $K_a(\zeta) \in A^2$ and $\|K_a\| = 1$. Notice that the family K_a in A^2 converges uniformly to zero locally in Δ as $|a| \to 1$. Therefore, for any compact operator $K : A^2 \to A^2$, we have $\|K(K_a)\| \to 0$ as $|a| \to 1$. By Theorem 2.2, we deduce that

$$\|T_h^-\|_e \geq \overline{\lim_{|a|\to 1}} \|T_h^-(K_a)\|$$

By Lemma 3.2,

$$T_h^-(K_a(\zeta)) = (1-|a|^2)\phi_h(\zeta, a).$$

Therefore, we have

$$\|T_h^-\|_e \geq \overline{\lim_{|a|\to 1}} (1-|a|^2)|\phi_h(a)|.$$

We now estimate the upper bound. From the proof of Theorem 2.2, we have

$$\|T_h^-\|_e = \|T_h^-\mathbb{R}_n + T_h^-\mathbb{K}_n\|_e \leq \|T_h^-\mathbb{R}_n\|.$$

Thus, we will proceed to estimate the norm $\|T_h^-\mathbb{R}_n\|$ and obtain the upper bound estimation of $\|T_h^-\|_e$. Let f be a quasi-conformal extension of the quasis-ymmetric homeomorphism h into Δ. By Lemma 3.2, for any $\psi \in A^2$, choosing ϕ such that $\phi' = \psi$, we have

$$T_h^-\psi(\zeta) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{\phi(h(w))}{(1-\zeta w)^2} dw.$$

The Green formula yields

$$T_h^-\psi(\zeta) = \frac{1}{\pi}\iint_{\Delta} \frac{\psi(f(w))\overline{\partial} f(w)}{(1-\zeta w)^2} dudv.$$

Noting that the Hilbert transformation is isometry on $L^2(\mathbb{C})$, we deduce that

$$\begin{aligned}
\|T_h^-\mathbb{R}_n\psi\|^2 &= \frac{1}{\pi}\iint_\Delta \left|\frac{1}{\pi}\iint_\Delta \frac{\mathbb{R}_n\psi(f(w))\overline{\partial}f(w)}{(\frac{1}{\zeta}-w)^2}dudv\right|^2 \left|\frac{1}{\zeta^4}\right| d\xi d\eta \\
&= \frac{1}{\pi}\iint_{\Delta^*} \left|\frac{1}{\pi}\iint_\Delta \frac{\mathbb{R}_n\psi(f(w))\overline{\partial}f(w)}{(\zeta-w)^2}dudv\right|^2 d\xi d\eta \\
&\leq \frac{1}{\pi}\iint_\Delta \left|\mathbb{R}_n\psi(f(w))\overline{\partial}f(w)\right|^2 dudv \qquad (14)\\
&\leq \frac{1}{\pi}\iint_\Delta \frac{|\mu(w)|^2}{1-|\mu(w)|^2}\left|\mathbb{R}_n\psi(w)\right|^2 dudv,
\end{aligned}$$

where $\mu(w)$ is the Beltrami coefficient of f^{-1}.

Let $0 < r_0 < 1$, $A_{r_0} = \{z \in \Delta : |z| > r_0\}$ and $\Delta_{r_0} = \Delta\backslash A_{r_0}$. We divide the integral above into two parts,

$$\begin{aligned}
\|T_h^-\mathbb{R}_n\psi\|^2 &\leq \frac{1}{\pi}\iint_{\Delta_{r_0}} \frac{|\mu(w)|^2}{1-|\mu(w)|^2}\left|\mathbb{R}_n\psi(w)\right|^2 dudv \\
&+ \frac{1}{\pi}\iint_{A_{r_0}} \frac{|\mu(w)|^2}{1-|\mu(w)|^2}\left|\mathbb{R}_n\psi(w)\right|^2 dudv \\
&= J_1 + J_2.
\end{aligned}$$

We first estimate the term J_2. Let $M = \sup_{|w|>r_0}\frac{|\mu(w)|^2}{1-|\mu(w)|^2}$. Noting that $\sup\{\|\mathbb{R}_n\| : n \geq 1\} = 1$, we have

$$\begin{aligned}
J_2 &= \frac{1}{\pi}\iint_{A_{r_0}} \frac{|\mu(w)|^2}{1-|\mu(w)|^2}\left|\mathbb{R}_n\psi(w)\right|^2 dudv \\
&\leq M\|\mathbb{R}_n\psi\|^2 \\
&\leq M\sup\{\|\mathbb{R}_n\| : n \geq 1\}\|\psi\|^2 \\
&\leq M\|\psi\|^2.
\end{aligned}$$

Next, we estimate the term J_1. Let $k = \|\mu\|_\infty$, we have

$$\begin{aligned}
J_1 &= \frac{1}{\pi}\iint_{\Delta_{r_0}} \frac{|\mu(w)|^2}{1-|\mu(w)|^2}\left|\mathbb{R}_n\psi(w)\right|^2 dudv \\
&\leq \frac{k^2}{1-k^2}\frac{1}{\pi}\iint_{\Delta_{r_0}} \left|\mathbb{R}_n\psi(w)\right|^2 dudv.
\end{aligned}$$

Therefore we deduce that

$$\sup_{\|\psi\|\le 1}\|T_h^-\mathbb{R}_n\psi\| \le \left(M+\frac{k^2}{1-k^2}\frac{1}{\pi}\iint_{\Delta_{r_0}}\sup_{\|\psi\|\le 1}|\mathbb{R}_n\psi(w)|^2\,dudv\right)^{\frac{1}{2}}$$

Thus, it follows from Lemma 2.1 that

$$\lim_{n\to\infty}\sup_{\|\psi\|\le 1}\|T_h^-\mathbb{R}_n\psi\|\le M^{\frac{1}{2}}. \tag{15}$$

Let $r_0\to 1$ and then take the infimum over all Beltrami coefficient μ of quasi-conformal extension of the quasis-ymmetric homeomorphism h, we have

$$\|T_h^-\|_e\le \overline{\lim_{n\to\infty}}\|T_h^-\mathbb{R}_n\|\le \frac{b(h)}{\sqrt{1-b(h)^2}}.$$

The proof of Theorem 3.1 follows.

4. Proof of Theorem 1.4

From Theorem 3.1, we need only estimate the upper bound. Recall that the Douady-Earle extension $w=E(h)(z)$ of the quasis-ymmetric homeomorphism h is defined as the equation, for $z,w\in\Delta$,

$$F(z,w)=\frac{1}{2\pi}\int_{S^1}\frac{(h(t)-w)(1-\mid w\mid^2)}{(1-\overline{w}h(t))\mid z-t\mid^2}\mid dt\mid=0, \tag{16}$$

(see [3]). Let $\mu(w)$ be the Beltrami coefficient of the inverse mapping $E(h)^{-1}$ of the Douay-Earle extension $E(h)$ of quasis-ymmetric homeomorphism h. It follows from [2] and [5] that there is a constant $C>0$ which depends only on the quasis-ymmetric constant of h such that

$$\frac{|\mu(w)|^2}{1-|\mu(w)|^2}\le C(1-|w|^2)^2\phi_h^2(\bar{w}). \tag{17}$$

Therefore, from (15) and (17), we have

$$\overline{\lim_{n\to\infty}}\|T_h^-\mathbb{R}_n\|^2\le C\sup_{|w|>r_0}(1-|w|^2)^2\phi_h^2(\bar{w}).$$

Let $r_0\to 1$. We get

$$\|T_h^-\|_e\le C\overline{\lim_{|w|\to 1}}(1-|w|^2)\phi_h(w).$$

We complete the proof of Theorem 1.4.

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