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INTERVAL-VALUED INTUITIONISTIC GRADATION OF OPENNESS

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Abstract. In this paper, we introduce the concepts of interval-valued intuitionistic gradation of openness of fuzzy sets which is a generalization of intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mapping and then investigate their properties.

1. Introduction

After Zadeh [14] introduced the concept of fuzzy sets, there have been various generalizations of the concept of fuzzy sets. Chang [5] introduced the concept of fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X and Coker [7] introduced the concept of intuitionistic fuzzy topology on a set X by axiomatizing a collection T of intuitionistic fuzzy subsets of X. In their definitions of fuzzy topology and intuitionistic fuzzy topology, fuzzyness in the concept of openness of fuzzy subsets and intuitionistic fuzzy subsets was absent. Chattopadhyay, Hazra and Samanta [6,8] introduced the concept of gradation of openness of fuzzy subsets. Zadeh [15] introduced the concept of interval-valued fuzzy sets and Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Atanassov and Gargov [3] introduced the concept of interval-valued intuitionistic fuzzy sets which is a generalization of both interval-valued

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fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [9,13] introduced the concept of intuitionisti gradation of openness and defined an intuitionistic fuzzy topological space and investigated their properties.

In this paper, we introduce the concepts of interval-valued intuitionistic gradation of openness of fuzzy sets which is a generalization of intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mapping and then investigate some properties of interval-valued intuitionistic gradation of openness of fuzzy sets and interval-valued intuitionistic gradation preserving mappings.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. The family of all fuzzy sets of X will be denoted by I^X. By 0_X and 1_X we denote the characteristic functions of ϕ and X, respectively. For any $A \in I^X$, A^c denotes the complement of A, i.e., $A^c = 1_X - A$.

Definition 2.1. [4,6,12]. A *gradation of openness* (for short, GO) on X, which is also called a *smooth topology* on X, is a mapping $\tau : I^X \to I$ satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$,

(O2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for each $A, B \in I^X$,

(O3) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$, for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The pair (X, τ) is called a *smooth topological space* (for short, STS).

Definition 2.2. [9]. An *intuitionistic gradation of openness* (for short, IGO) on X, which is also called an *intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings from I^X to I satisfying the following conditions:

(IGO1) $\tau(A) + \tau^*(A) \leq 1$ for each $A \in I^X$,

(IGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$,

(IGO3) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$ for each $A, B \in I^X$,

(IGO4) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ and $\tau^*(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^*(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The triple (X, τ, τ^*) is called an *intuitionistic smooth topological space* (for short, ISTS). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.3. [9]. Let (X, τ, τ^*) and (Y, η, η^*) be two ISTSs and $f : X \to Y$ be a mapping. Then f is called a *gradation preserving mapping* (for short, a GP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$.

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$ and $M = [M^L, M^U]$, where M^L and M^U are respectively the lower and the upper end points. Especially, we denote $\mathbf{r} = [r, r]$ for each $r \in I$. The complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Note that $M = N$ iff $M^L = N^L$ and $M^U = N^U$ and that $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$.

Definition 2.4. [15]. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an *interval-valued fuzzy set* (for short, IVFS) on X, where $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$. $A^L(x)$ and $A^U(x)$ are called the *lower* and *upper end points* of $A(x)$, respectively.

Definition 2.5. [10]. Let A and B be IVFSs on X. Then
(a) $A = B$ iff $A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$ for all $x \in X$.
(b) $A \subset B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.
(c) The *complement* A^c of A is defined by $A^c(x) = [1 - A^U(x), 1 - A^L(x)]$ for all $x \in X$.
(d) For a family of IVFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\cup_{i\in\Gamma} A_i(x) = [\vee_{i\in\Gamma} A_i^L(x), \vee_{i\in\Gamma} A_i^U(x)],$$
$$\cap_{i\in\Gamma} A_i(x) = [\wedge_{i\in\Gamma} A_i^L(x), \wedge_{i\in\Gamma} A_i^U(x)]$$

for all $x \in X$.

Definition 2.6. [3]. A mapping $A = (\mu_A, \nu_A) : X \to D(I) \times D(I)$ is called an *interval-valued intuitionistic fuzzy set* (for short, IVIFS) on X, where $\mu_A : X \to D(I)$ and $\nu_A : X \to D(I)$ are interval-valued fuzzy sets on X with the condition $\sup_{x\in X} \mu_A^U(x) + \sup_{x\in X} \nu_A^U(x) \leq 1$. The intervals $\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\nu_A(x) = [\nu_A^L(x), \nu_A^U(x)]$ denote the degree of belongingness and the degree of nonbelongingness of the element x to the set A, respectively.

Definition 2.7. [11]. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IVIFSs on X. Then

(a) $A \subset B$ iff $\mu_A^L(x) \leq \mu_B^L(x)$, $\mu_A^U(x) \leq \mu_B^U(x)$ and $\nu_A^L(x) \geq \nu_B^L(x)$, $\nu_A^U(x) \geq \nu_B^U(x)$ for all $x \in X$.

(b) $A = B$ iff $A \subset B$ and $B \subset A$.

(c) The *complement* A^c of A is defined by $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$.

(d) For a family of IVIFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\mu_{\cup_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cup_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\nu_{A_i}(x),$$
$$\mu_{\cap_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cap_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\nu_{A_i}(x)$$

for all $x \in X$.

3. Interval-valued intuitionistic gradation of openness

Definition 3.1. An *interval-valued intuitionistic gradation of openness* (for short, IVIGO) on X, which is also called an *interval-valued intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGO2) $\tau(0_X) = \tau(1_X) = \mathbf{1}$ and $\tau^*(0_X) = \tau^*(1_X) = \mathbf{0}$,

(IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$,

(IVIGO4) $\tau^L(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^L(A_i)$, $\tau^U(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^U(A_i)$ and $\tau^{*L}(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^{*L}(A_i)$, $\tau^{*U}(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

The triple (X, τ, τ^*) is called an *interval-valued intuitionistic smooth topological space* (for short, IVISTS). τ and τ^* may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

Definition 3.2. An *interval-valued intuitionistic gradation of closedness* (for short, IVIGC) on X, which is also called an *interval-valued intuitionistic smooth cotopology* on X, is an ordered pair $(\mathcal{F}, \mathcal{F}^*)$ of mappings $\mathcal{F} = [\mathcal{F}^L, \mathcal{F}^U] : I^X \to D(I)$ and $\mathcal{F}^* = [\mathcal{F}^{*L}, \mathcal{F}^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGC1) $\mathcal{F}^L(A) \leq \mathcal{F}^U(A)$, $\mathcal{F}^{*L}(A) \leq \mathcal{F}^{*U}(A)$ and $\mathcal{F}^U(A)+\mathcal{F}^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = \mathbf{1}$ and $\mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = \mathbf{0}$,

(IVIGC3) $\mathcal{F}^L(A\cup B) \geq \mathcal{F}^L(A)\wedge\mathcal{F}^L(B)$, $\mathcal{F}^U(A\cup B) \geq \mathcal{F}^U(A)\wedge\mathcal{F}^U(B)$ and $\mathcal{F}^{*L}(A \cup B) \leq \mathcal{F}^{*L}(A) \vee \mathcal{F}^{*L}(B)$, $\mathcal{F}^{*U}(A \cup B) \leq \mathcal{F}^{*U}(A) \vee \mathcal{F}^{*U}(B)$ for each $A, B \in I^X$,

(IVIGC4) $\mathcal{F}^L(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^L(A_i)$, $\mathcal{F}^U(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^U(A_i)$ and $\mathcal{F}^{*L}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*L}(A_i)$, $\mathcal{F}^{*U}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subset I^X$.

THEOREM 3.3. *If* (τ, τ^*) *is an IVIGO on* X*, then* (τ^L, τ^{*L}) *and* (τ^U, τ^{*U}) *are IGOs on* X*.*

Proof. It follows immediately from Definition 2.2 and 3.1. □

For an IVIGO (τ, τ^*) and an IVIGC $(\mathcal{F}, \mathcal{F}^*)$ on X, we define

$$\tau_{\mathcal{F}}(A) = \mathcal{F}(A^c),\ \tau^*_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c),$$
$$\mathcal{F}_\tau(A) = \tau(A^c),\ \mathcal{F}^*_{\tau^*}(A) = \tau^*(A^c)$$

for each $A \in I^X$.

THEOREM 3.4. (a) (τ, τ^*) *is an IVIGO on* X *if and only if* $(\mathcal{F}_\tau, \mathcal{F}^*_{\tau^*})$ *is an IVIGC on* X*.*

(b) $(\mathcal{F}, \mathcal{F}^*)$ *is an IVIGC on* X *if and only if* $(\tau_{\mathcal{F}}, \tau^*_{\mathcal{F}^*})$ *is an IVIGO on* X*.*

(c) $\tau_{\mathcal{F}_\tau} = \tau$, $\tau^*_{\mathcal{F}^*{}_{\tau^*}} = \tau^*$, $\mathcal{F}_{\tau_{\mathcal{F}}} = \mathcal{F}$, $\mathcal{F}^*_{\tau^*{}_{\mathcal{F}^*}} = \mathcal{F}^*$.

Proof. (a) Since $\mathcal{F}^L_\tau(A) = \tau^L(A^c)$, $\mathcal{F}^U_\tau(A) = \tau^U(A^c)$, $\mathcal{F}^*_{\tau^*}{}^L(A) = \tau^{*L}(A^c)$, $\mathcal{F}^*_{\tau^*}{}^U(A) = \tau^{*U}(A^c)$, we have

$$\begin{aligned}\mathcal{F}^L_\tau(A) \leq \mathcal{F}^U_\tau(A),\ \forall A \in I^X &\Leftrightarrow \tau^L(A^c) \leq \tau^U(A^c),\ \forall A \in I^X\\ &\Leftrightarrow \tau^L(A) \leq \tau^U(A),\ \forall A \in I^X.\end{aligned}$$

Similarly,

$$\mathcal{F}^*_{\tau^*}{}^L(A) \leq \mathcal{F}^*_{\tau^*}{}^U(A),\ \forall A \in I^X \Leftrightarrow \tau^{*L}(A) \leq \tau^{*U}(A),\ \forall A \in I^X,$$

$$\mathcal{F}^U_\tau(A) + \mathcal{F}^*_{\tau^*}{}^U(A) \leq 1,\ \forall A \in I^X \Leftrightarrow \tau^U(A) + \tau^{*U}(A) \leq 1,\ \forall A \in I^X.$$

$$\begin{aligned}&\mathcal{F}_\tau(0_X) = \mathcal{F}_\tau(1_X) = \mathbf{1}, \mathcal{F}^*_{\tau^*}(0_X) = \mathcal{F}^*_{\tau^*}(1_X) = \mathbf{0}\\ &\qquad\Leftrightarrow \tau(1_X) = \tau(0_X) = \mathbf{1}, \tau^*(1_X) = \tau^*(0_X) = \mathbf{0}.\end{aligned}$$

$$\begin{aligned}\mathcal{F}_\tau^L(A\cup B)&\geq \mathcal{F}_\tau^L(A)\wedge \mathcal{F}_\tau^L(B),\ \forall A,B\in I^X\\ &\Leftrightarrow \tau^L(A^c\cap B^c)\geq \tau^L(A^c)\wedge\tau^L(B^c),\ \forall A,B\in I^X\\ &\Leftrightarrow \tau^L(A\cap B)\geq \tau^L(A)\wedge\tau^L(B),\ \forall A,B\in I^X.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}_\tau^U(A\cup B)&\geq \mathcal{F}_\tau^U(A)\wedge \mathcal{F}_\tau^U(B),\ \forall A,B\in I^X\\ &\Leftrightarrow \tau^U(A\cap B)\geq \tau^U(A)\wedge\tau^U(B),\ \forall A,B\in I^X,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\tau^*}^{*\,L}(A\cup B)&\leq \mathcal{F}_{\tau^*}^{*\,L}(A)\vee \mathcal{F}_{\tau^*}^{*\,L}(B),\ \forall A,B\in I^X\\ &\Leftrightarrow \tau^{*L}(A\cap B)\leq \tau^{*L}(A)\vee\tau^{*L}(B),\ \forall A,B\in I^X,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\tau^*}^{*\,U}(A\cup B)&\leq \mathcal{F}_{\tau^*}^{*\,U}(A)\vee \mathcal{F}_{\tau^*}^{*\,U}(B),\ \forall A,B\in I^X\\ &\Leftrightarrow \tau^{*U}(A\cap B)\leq \tau^{*U}(A)\vee\tau^{*U}(B),\ \forall A,B\in I^X.\end{aligned}$$

Let $\{A_i : i\in\Gamma\}\subset I^X$. Then

$$\begin{aligned}\mathcal{F}_\tau^L(\cap_{i\in\Gamma}A_i)&=\tau^L((\cap_{i\in\Gamma}A_i)^c)=\tau^L(\cup_{i\in\Gamma}A_i^c),\\ \mathcal{F}_\tau^U(\cap_{i\in\Gamma}A_i)&=\tau^U((\cap_{i\in\Gamma}A_i)^c)=\tau^U(\cup_{i\in\Gamma}A_i^c),\\ \mathcal{F}_{\tau^*}^{*\,L}(\cap_{i\in\Gamma}A_i)&=\tau^{*L}((\cap_{i\in\Gamma}A_i)^c)=\tau^{*L}(\cup_{i\in\Gamma}A_i^c),\\ \mathcal{F}_{\tau^*}^{*\,U}(\cap_{i\in\Gamma}A_i)&=\tau^{*U}((\cap_{i\in\Gamma}A_i)^c)=\tau^{*U}(\cup_{i\in\Gamma}A_i^c).\end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{F}_\tau^L(\cap_{i\in\Gamma}A_i)&\geq \wedge_{i\in\Gamma}\mathcal{F}_\tau^L(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X\\ &\Leftrightarrow \tau^L(\cup_{i\in\Gamma}A_i^c)\geq\wedge_{i\in\Gamma}\tau^L(A_i^c),\ \forall\{A_i:i\in\Gamma\}\subset I^X\\ &\Leftrightarrow \tau^L(\cup_{i\in\Gamma}A_i)\geq\wedge_{i\in\Gamma}\tau^L(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}_\tau^U(\cap_{i\in\Gamma}A_i)&\geq \wedge_{i\in\Gamma}\mathcal{F}_\tau^U(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X\\ &\Leftrightarrow \tau^U(\cup_{i\in\Gamma}A_i)\geq\wedge_{i\in\Gamma}\tau^U(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\tau^*}^{*\,L}(\cap_{i\in\Gamma}A_i)&\leq \vee_{i\in\Gamma}\mathcal{F}_{\tau^*}^{*\,L}(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X\\ &\Leftrightarrow \tau^{*L}(\cup_{i\in\Gamma}A_i)\leq\vee_{i\in\Gamma}\tau^{*L}(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X,\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\tau^*}^{*\,U}(\cap_{i\in\Gamma}A_i)&\leq \vee_{i\in\Gamma}\mathcal{F}_{\tau^*}^{*\,U}(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X\\ &\Leftrightarrow \tau^{*U}(\cup_{i\in\Gamma}A_i)\leq\vee_{i\in\Gamma}\tau^{*U}(A_i),\ \forall\{A_i:i\in\Gamma\}\subset I^X.\end{aligned}$$

Therefore (τ, τ^*) is an IVIGO on X if and only if $(\mathcal{F}_\tau, \mathcal{F}^*_{\tau^*})$ is an IVIGC on X.

(b) The proof is similar to (a).

(c) The proof is straightforward.

□

Let $\{(\tau_i, \tau_i^*)\}_{i\in\Gamma}$ be a family of IVIGOs on X. Then the intersection of $\{(\tau_i, \tau_i^*)\}_{i\in\Gamma}$ is defined by $\cap_{i\in\Gamma}(\tau_i, \tau_i^*) = (\wedge_{i\in\Gamma}\tau_i, \vee_{i\in\Gamma}\tau_i^*)$, where $(\wedge_{i\in\Gamma}\tau_i)(A) = [\wedge_{i\in\Gamma}\tau_i^L(A), \wedge_{i\in\Gamma}\tau_i^U(A)]$ and $(\vee_{i\in\Gamma}\tau_i^*)(A) = [\vee_{i\in\Gamma}{\tau_i^*}^L(A), \vee_{i\in\Gamma}{\tau_i^*}^U(A)]$ for each $A \in I^X$.

THEOREM 3.5. *If $\{(\tau_i, \tau_i^*)\}_{i\in\Gamma}$ is a family of IVIGOs on X, then $\cap_{i\in\Gamma}(\tau_i, \tau_i^*)$ is an IVIGO on X.*

Proof. The proof is straightforward.

□

Let (τ, τ^*) be an IVIGO on X. For $[r, s] \in D(I)$, we define

$\tau_{[r,s]} = \{A \in I^X : \tau(A) \geq [r, s]\}$,

${\tau^*}_{[r,s]} = \{A \in I^X : \tau^*(A) \leq [1 - s, 1 - r]\}$,

$(\tau, \tau^*)_{[r,s]} = \{A \in I^X : \tau(A) \geq [r, s] \text{ and } \tau^*(A) \leq [1 - s, 1 - r]\}$.

THEOREM 3.6. *Let (τ, τ^*) be an IVIGO on X and $[r, s] \in D(I)$. Then $\tau_{[r,s]}$, ${\tau^*}_{[r,s]}$ and $(\tau, \tau^*)_{[r,s]}$ are Chang's fuzzy topologies on X.*

Proof. Suppose that (τ, τ^*) is an IVIGO on X and $[r, s] \in D(I)$. We will prove that $(\tau, \tau^*)_{[r,s]}$ is a Chang's fuzzy topology on X. Since $\tau(0_X) = \tau(1_X) = \mathbf{1}$ and $\tau^*(0_X) = \tau^*(1_X) = \mathbf{0}$, $\tau^L(0_X) = 1 \geq r$, $\tau^U(0_X) = 1 \geq s$, $\tau^L(1_X) = 1 \geq r$, $\tau^U(1_X) = 1 \geq s$ and ${\tau^*}^L(0_X) = 0 \leq 1 - s$, ${\tau^*}^U(0_X) = 0 \leq 1 - r$, ${\tau^*}^L(1_X) = 0 \leq 1 - s$, ${\tau^*}^U(1_X) = 0 \leq 1 - r$. Thus $\tau(0_X) \geq [r, s]$, $\tau(1_X) \geq [r, s]$ and $\tau^*(0_X) \leq [1 - s, 1 - r]$, $\tau^*(1_X) \leq [1 - s, 1 - r]$. Hence $0_X, 1_X \in (\tau, \tau^*)_{[r,s]}$. Let $A, B \in (\tau, \tau^*)_{[r,s]}$. Then $\tau^L(A) \geq r$, $\tau^U(A) \geq s$, $\tau^L(B) \geq r$, $\tau^U(B) \geq s$ and ${\tau^*}^L(A) \leq 1 - s$, ${\tau^*}^U(A) \leq 1 - r$, ${\tau^*}^L(B) \leq 1 - s$, ${\tau^*}^U(B) \leq 1 - r$. So $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B) \geq r$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B) \geq s$ and ${\tau^*}^L(A \cap B) \leq {\tau^*}^L(A) \vee {\tau^*}^L(B) \leq 1 - s$, ${\tau^*}^U(A \cap B) \leq {\tau^*}^U(A) \vee {\tau^*}^U(B) \leq 1 - r$. Thus $\tau(A \cap B) \geq [r, s]$ and $\tau^*(A \cap B) \leq [1 - s, 1 - r]$. Hence $A \cap B \in (\tau, \tau^*)_{[r,s]}$. Let $\{A_i : i \in \Gamma\} \subset (\tau, \tau^*)_{[r,s]}$. Then $\tau^L(A_i) \geq r$, $\tau^U(A_i) \geq s$ and ${\tau^*}^L(A_i) \leq 1 - s$, ${\tau^*}^U(A_i) \leq 1 - r$ for each $i \in \Gamma$. So $\tau^L(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^L(A_i) \geq r$, $\tau^U(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau^U(A_i) \geq s$ and ${\tau^*}^L(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} {\tau^*}^L(A_i) \leq 1 - s$, ${\tau^*}^U(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} {\tau^*}^U(A_i) \leq$

$1-r$. Thus $\tau(\cup_{i\in\Gamma}\ A_i) \geq [r,s]$ and $\tau^*(\cup_{i\in\Gamma}\ A_i) \leq [1-s, 1-r]$. Hence $\cup_{i\in\Gamma} A_i \in (\tau,\tau^*)_{[r,s]}$. Therefore $(\tau,\tau^*)_{[r,s]}$ is a Chang's fuzzy topology on X.

Similarly, $\tau_{[r,s]}$ and ${\tau^*}_{[r,s]}$ are Chang's fuzzy topologies on X.

□

THEOREM 3.7. *Let (τ,τ^*) be an IVIGO on X. Then $\{\tau_{[r,s]}\}_{[r,s]\in D(I)}$ and $\{{\tau^*}_{[r,s]}\}_{[r,s]\in D(I)}$ are two descending families of Chang's fuzzy topologies on X such that $\tau_{[r,s]} = \cap_{[p,q]<[r,s]}\ \tau_{[p,q]}$ and ${\tau^*}_{[r,s]} = \cap_{[p,q]<[r,s]}\ {\tau^*}_{[p,q]}$ for each $[r,s] \in D(I_0)$.*

Proof. Let $[r,s],[t,u] \in D(I)$ with $[r,s] \leq [t,u]$. If $A \in \tau_{[t,u]}$, then $\tau^L(A) \geq t$ and $\tau^U(A) \geq u$. So $\tau^L(A) \geq r$ and $\tau^U(A) \geq s$. Thus $A \in \tau_{[r,s]}$. So $\tau_{[t,u]} \subset \tau_{[r,s]}$. Similarly, ${\tau^*}_{[t,u]} \subset {\tau^*}_{[r,s]}$. Therefore the families $\{\tau_{[r,s]}\}_{[r,s]\in D(I)}$ and $\{{\tau^*}_{[r,s]}\}_{[r,s]\in D(I)}$ are descending.

Let $[r,s] \in D(I_0]$. Since the family $\{\tau_{[r,s]}\}_{[r,s]\in D(I)}$ is descending, $\tau_{[r,s]} \subset \cap_{[p,q]<[r,s]}\ \tau_{[p,q]}$. If $A \notin \tau_{[r,s]}$, then $\tau^L(A) < r$ or $\tau^U(A) < s$. Hence there exists $[p,q] \in D(I_0)$ with $[p,q] < [r,s]$ such that $\tau^L(A) < p < r$ or $\tau^U(A) < q < s$. Hence $A \notin \cap_{[p,q]<[r,s]}\ \tau_{[p,q]}$. Thus $\cap_{[p,q]<[r,s]}\ \tau_{[p,q]} \subset \tau_{[r,s]}$. Therefore $\tau_{[r,s]} = \cap_{[p,q]<[r,s]}\ \tau_{[p,q]}$.

Similarly, ${\tau^*}_{[r,s]} = \cap_{[p,q]<[r,s]}\ {\tau^*}_{[p,q]}$.

□

Let $Y \subset X$. For each $A \in I^X$, a fuzzy set $A|_Y$, defined by $A|_Y(x) = A(x),\ x \in Y$, is the restriction of A on Y. For each $B \in I^Y$, a fuzzy set B_X, defined by $B_X(x) = \begin{cases} B(x), & x \in Y \\ 0, & x \in X-Y \end{cases}$, is the extension of B on X.

THEOREM 3.8. *Let (X,τ,τ^*) be an IVISTS and $Y \subset X$. Define two mappings $\tau_Y,\ \tau^*_Y : I^Y \to D(I)$ by $\tau_Y(A) = \vee\{\tau(B) : B \in I^X$ and $B|_Y = A\}$, $\tau^*_Y(A) = \wedge\{\tau^*(B) : B \in I^X$ and $B|_Y = A\}$ for each $A \in I^Y$. Then (τ_Y, τ^*_Y) is an IVIGO on Y and $\tau_Y(A) \geq \tau(A_X)$ and $\tau^*_Y(A) \leq \tau^*(A_X)$ for each $A \in I^Y$.*

Proof. For each $A \in I^Y$, let $B \in I^X$ with $B|_Y = A$. Since $\tau^L(B) \leq \tau^U(B)$ and ${\tau^*}^L(B) \leq {\tau^*}^U(B)$, ${\tau_Y}^L(A) = \vee\{\tau^L(B) : B \in I^X$ and $B|_Y = A\} \leq \vee\{\tau^U(B) : B \in I^X$ and $B|_Y = A\} = {\tau_Y}^U(A)$. Similarly, ${\tau^*_Y}^L(A) \leq {\tau^*_Y}^U(A)$. Since $0 \leq \tau^U(B) + {\tau^*}^U(B) \leq 1$, $\tau^U(B) \leq 1 - {\tau^*}^U(B)$. Hence we have

$$\begin{aligned}
{\tau_Y}^U(A) &= \vee\{\tau^U(B) : B \in I^X \text{ and } B|_Y = A\} \\
&\le \vee\{1 - {\tau^*}^U(B) : B \in I^X \text{ and } B|_Y = A\} \\
&= 1 - \wedge\{{\tau^*}^U(B) : B \in I^X \text{ and } B|_Y = A\} \\
&= 1 - {\tau_Y^*}^U(A).
\end{aligned}$$

Therefore ${\tau_Y}^U(A) + {\tau_Y^*}^U(A) \le 1$.

Clearly, $\tau_Y(0_Y) = \tau_Y(1_Y) = \mathbf{1}$ and $\tau_Y^*(0_Y) = \tau_Y^*(1_Y) = \mathbf{0}$.

Let $A_1, A_2 \in I^Y$. Then $\tau_Y^*(A_1 \cap A_2) = \wedge\{\tau^*(B) : B \in I^X$ and $B|_Y = A_1 \cap A_2\}$. If $\tau_Y^*(A_1) \vee \tau_Y^*(A_2) = \mathbf{1}$, then $\tau_Y^*(A_1 \cap A_2) \le \tau_Y^*(A_1) \vee \tau_Y^*(A_2) = \mathbf{1}$. If $\tau_Y^*(A_1) \vee \tau_Y^*(A_2) < \mathbf{1}$, take $[r, s]$ with $\tau_Y^*(A_1) \vee \tau_Y^*(A_2) < [r, s] < \mathbf{1}$. Then there exists $B_i \in I^X$ such that $B_i|_Y = A_i$ and $\tau^*(B_i) < [r, s]$ for $i = 1, 2$. Since $(B_1 \cap B_2)|_Y = (B_1|_Y) \cap (B_2|_Y) = A_1 \cap A_2$ and $\tau^*(B_1 \cap B_2) \le \tau^*(B_1) \vee \tau^*(B_2) < [r, s]$, $\tau_Y^*(A_1 \cap A_2) \le \tau^*(B_1 \cap B_2) < [r, s]$. Thus $\tau_Y^*(A_1) \vee \tau_Y^*(A_2) < [r, s]$ implies $\tau_Y^*(A_1 \cap A_2) < [r, s]$. Hence $\tau_Y^*(A_1 \cap A_2) \le \tau_Y^*(A_1) \vee \tau_Y^*(A_2)$. Therefore ${\tau_Y^*}^L(A_1 \cap A_2) \le {\tau_Y^*}^L(A_1) \vee {\tau_Y^*}^L(A_2)$ and ${\tau_Y^*}^U(A_1 \cap A_2) \le {\tau_Y^*}^U(A_1) \vee {\tau_Y^*}^U(A_2)$. Similarly, ${\tau_Y}^L(A_1 \cap A_2) \ge {\tau_Y}^L(A_1) \wedge {\tau_Y}^L(A_2)$ and ${\tau_Y}^U(A_1 \cap A_2) \ge {\tau_Y}^U(A_1) \wedge {\tau_Y}^U(A_2)$.

Let $\{A_i : i \in \Gamma\} \subset I^Y$. Then $\tau_Y^*(\cup_{i\in\Gamma} A_i) = \wedge\{\tau^*(B) : B \in I^X$ and $B|_Y = \cup_{i\in\Gamma} A_i\}$. If $\vee_{i\in\Gamma} \tau_Y^*(A_i) = \mathbf{1}$, then $\tau_Y^*(\cup_{i\in\Gamma} A_i) \le \vee_{i\in\Gamma} \tau_Y^*(A_i) = \mathbf{1}$. If $\vee_{i\in\Gamma} \tau_Y^*(A_i) < \mathbf{1}$, take $[r, s]$ with $\vee_{i\in\Gamma} \tau_Y^*(A_1) < [r, s] < \mathbf{1}$. Then $\tau_Y^*(A_1) < [r, s]$ for each $i \in \Gamma$. Hence there exists $B_i \in I^X$ such that $B_i|_Y = A_i$ and $\tau^*(B_i) < [r, s]$ for each $i \in \Gamma$. Since $(\cup_{i\in\Gamma} B_i)|_Y = \cup_{i\in\Gamma}(B_i|_Y) = \cup_{i\in\Gamma} A_i$ and $\tau^*(\cup_{i\in\Gamma} B_i) \le \vee_{i\in\Gamma} \tau^*(B_i) \le [r, s]$, $\tau_Y^*(\cup_{i\in\Gamma} A_i) \le \tau^*(\cup_{i\in\Gamma} B_i) \le [r, s]$. Thus $\vee_{i\in\Gamma} \tau_Y^*(A_i) < [r, s]$ implies $\tau_Y^*(\cup_{i\in\Gamma} A_i) \le [r, s]$. Hence $\tau_Y^*(\cup_{i\in\Gamma} A_i) \le \vee_{i\in\Gamma} \tau_Y^*(A_i)$. Therefore ${\tau_Y^*}^L(\cup_{i\in\Gamma} A_i) \le \vee_{i\in\Gamma} {\tau_Y^*}^L(A_i)$ and ${\tau_Y^*}^U(\cup_{i\in\Gamma} A_i) \le \vee_{i\in\Gamma} {\tau_Y^*}^U(A_i)$. Similarly, ${\tau_Y}^L(\cup_{i\in\Gamma} A_i) \ge \wedge_{i\in\Gamma} {\tau_Y}^L(A_i)$ and ${\tau_Y}^U(\cup_{i\in\Gamma} A_i) \ge \wedge_{i\in\Gamma} {\tau_Y}^U(A_i)$.

Therefore (τ_Y, τ_Y^*) is an IVIGO on Y.

Clearly, $\tau_Y(A) \ge \tau(A_X)$ and $\tau_Y^*(A) \le \tau^*(A_X)$ for each $A \in I^Y$.

□

THEOREM 3.9. *Let $(\mathcal{F}, \mathcal{F}^*)$ be an IVIGC on X and $Y \subset X$. Define two mappings $\mathcal{F}_Y,\ \mathcal{F}_Y^* : I^Y \to D(I)$ by $\mathcal{F}_Y(A) = \vee\{\mathcal{F}(B) : B \in I^X$ and $B|_Y = A\}$, $\mathcal{F}_Y^*(A) = \wedge\{\mathcal{F}^*(B) : B \in I^X$ and $B|_Y = A\}$ for each $A \in I^Y$. Then $(\mathcal{F}_Y, \mathcal{F}_Y^*)$ is an IVIGC on Y and $\mathcal{F}_Y(A) \ge \mathcal{F}(A_X)$ and $\mathcal{F}_Y^*(A) \le \mathcal{F}^*(A_X)$ for each $A \in I^Y$.*

Proof. The proof is similar to Theorem 3.8. □

When τ_Y and τ_Y^* are defined as in Theorem 3.8, (Y, τ_Y, τ_Y^*) is called an *interval-valued intuitionistic fuzzy subspace* of the IVISTS (X, τ, τ^*).

THEOREM 3.10. *Let (Y, τ_Y, τ_Y^*) be an interval-valued intuitionistic fuzzy subspace of the IVISTS (X, τ, τ^*). Then*
(a) $\mathcal{F}_{\tau_Y}(A) = \vee\{\mathcal{F}_\tau(B) : B \in I^X$ *and* $B|_Y = A\}$ *and*
$\mathcal{F}^*_{\tau_{Y^*}}(A) = \wedge\{\mathcal{F}^*_{\tau^*}(B) : B \in I^X$ *and* $B|_Y = A\}$
for each $A \in I^Y$.
(b) *If* $Z \subset Y \subset X$, *then* $\tau_Z = (\tau_Y)_Z$ *and* $\tau_Z^* = (\tau_Y^*)_Z$.

Proof. (a) For each $A \in I^Y$, we have

$$\begin{aligned}\mathcal{F}_{\tau_Y}(A) &= \tau_Y(A^c)\\ &= \vee\{\tau(B) : B \in I^X \text{ and } B|_Y = A^c\}\\ &= \vee\{\tau(B) : B^c \in I^X \text{ and } B^c|_Y = A\}\\ &= \vee\{\mathcal{F}_\tau(B^c) : B^c \in I^X \text{ and } B^c|_Y = A\}\\ &= \vee\{\mathcal{F}_\tau(B) : B \in I^X \text{ and } B|_Y = A\}.\end{aligned}$$

Similarly, $\mathcal{F}^*_{\tau_{Y^*}}(A) = \wedge\{\mathcal{F}^*_{\tau^*}(B) : B \in I^X \text{ and } B|_Y = A\}$
(b) For each $A \in I^Z$, we have

$$\begin{aligned}(\tau_Y)_Z(A) &= \vee\{\tau_Y(B) : B \in I^Y \text{ and } B|_Z = A\}\\ &= \vee\{\vee\{\tau(C) : C \in I^X \text{ and } C|_Y = B\} : B \in I^Y \text{ and } B|_Z = A\}\\ &= \vee\{\tau(C) : C \in I^X \text{ and } C|_Z = A\}\\ &= \tau_Z(A).\end{aligned}$$

Hence $\tau_Z = (\tau_Y)_Z$. Similarly, $\tau_Z^* = (\tau_Y^*)_Z$. □

4. Interval-valued intuitionistic gradation preserving mappings

DEFINITION 4.1. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $f : X \to Y$ be a mapping. Then f is called an *interval-valued intuitionistic gradation preserving mapping* (for short, an IVIGP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$.

THEOREM 4.2. *Let* (X, τ, τ^*) *and* (Y, η, η^*) *be two IVISTSs and* $f : X \to Y$ *be a mapping. Then* $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ *is an IVIGP-mapping if and only if* $f : (X, \tau^L, \tau^{*L}) \to (Y, \eta^L, \eta^{*L})$ *and* $f : (X, \tau^U, \tau^{*U}) \to (Y, \eta^U, \eta^{*U})$ *are GP-mappings.*

Proof. The proof is straightforward.

□

DEFINITION 4.3. [1]. Let (X, T, T^*) and (Y, S, S^*) be two bitopological spaces of fuzzy subsets. Then a mapping $f : (X, T, T^*) \to (Y, S, S^*)$ is said to be *continuous* if $f : (X, T) \to (Y, S)$ and $f : (X, T^*) \to (Y, S^*)$ are continuous.

THEOREM 4.4. *Let* (X, τ, τ^*) *and* (Y, η, η^*) *be two IVISTSs and* $f : X \to Y$ *be a mapping. Then* $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ *is an IVIGP-mapping if and only if* $f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})$ *is continuous for each* $[r, s] \in D(I_0)$.

Proof. Suppose that $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ is an IVIGP-mapping. Let $[r, s] \in D(I_0)$. If $A \in \eta_{[r,s]}$, then $\eta(A) \geq [r, s]$. By hypothesis, $\eta(A) \leq \tau(f^{-1}(A))$ and so $\tau(f^{-1}(A)) \geq [r, s]$, i.e., $f^{-1}(A) \in \tau_{[r,s]}$. Hence $f : (X, \tau_{[r,s]}) \to (Y, \eta_{[r,s]})$ is continuous. If $A \in \eta^*_{[r,s]}$, then $\eta^*(A) \leq [1 - s, 1 - r]$. By hypothesis, $\eta^*(A) \geq \tau^*(f^{-1}(A))$ and so $\tau^*(f^{-1}(A)) \leq [1 - s, 1 - r]$, i.e., $f^{-1}(A) \in \tau^*_{[r,s]}$. Hence $f : (X, \tau^*_{[r,s]}) \to (Y, \eta^*_{[r,s]})$ is continuous. Therefore $f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})$ is continuous.

Conversely, suppose that $f : (X, \tau_{[r,s]}, \tau^*_{[r,s]}) \to (Y, \eta_{[r,s]}, \eta^*_{[r,s]})$ is continuous for each $[r, s] \in D(I_0)$. Let $A \in I^Y$. If $\eta(A) = \mathbf{0}$, then $\eta(A) \leq \tau(f^{-1}(A))$. If $\eta(A) = [r, s] \in D(I_0)$, then $A \in \eta_{[r,s]}$. By hypothesis, $f^{-1}(A) \in \tau_{[r,s]}$, i.e., $\tau(f^{-1}(A)) \geq [r.s]$. Thus $\eta(A) \leq \tau(f^{-1}(A)$. If $\eta^*(A) = \mathbf{1}$, then $\eta^*(A) \geq \tau^*(f^{-1}(A))$. If $\eta^*(A) = [r, s] < \mathbf{1}$, then $[1 - s, 1 - r] \in D(I_0)$ and $\eta^*(A) = [r, s] = [1 - (1 - r), 1 - (1 - s)]$. Hence $A \in \eta^*_{[1-s,1-r]}$. By hypothesis, $f^{-1}(A) \in \tau^*_{[1-s,1-r]}$. Thus $\tau^*(f^{-1}(A)) \leq [1 - (1 - r), 1 - (1 - s)] = [r, s]$. Hence $\eta^*(A) \geq \tau^*(f^{-1}(A))$. Therefore $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ is an IVIGP-mapping.

□

DEFINITION 4.5. Let (X, τ, τ^*) be an IVISTS and $A \in I^X$. Then the $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy closure* and $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy interior* of A are defined by

$cl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subset K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\},$

$int_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subset A,\ \tau(G) \geq [r,s],\ \tau^*(G) \leq [t,u]\}$,
where $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

Note that $(cl_{[r,s],[t,u]}(A))^c = int_{[r,s],[t,u]}(A^c)$ and $(int_{[r,s],[t,u]}(A))^c = cl_{[r,s],[t,u]}(A^c)$ for each $A \in I^X$.

THEOREM 4.6. *Let* (X,τ,τ^*) *and* (Y,η,η^*) *be two IVISTSs and* $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ *with* $s+u \leq 1$. *If* $f : (X,\tau,\tau^*) \to (Y,\eta,\eta^*)$ *is an IVIGP-mapping, then*

(a) $f(cl_{[r,s],[t,u]}(A)) \subset cl_{[r,s],[t,u]}(f(A))$ *for each* $A \in I^X$.
(b) $cl_{[r,s],[t,u]}(f^{-1}(A)) \subset f^{-1}(cl_{[r,s],[t,u]}(A))$ *for each* $A \in I^Y$.
(c) $f^{-1}(int_{[r,s],[t,u]}(A)) \subset int_{[r,s],[t,u]}(f^{-1}(A))$ *for each* $A \in I^Y$.

Proof. (a) For each $A \in I^X$, we have

$$
\begin{aligned}
& f^{-1}(cl_{[r,s],[t,u]}(f(A))) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subset K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subset K,\ \eta(K^c) \geq [r,s],\ \eta^*(K^c) \leq [t,u]\}) \\
&\supset f^{-1}(\cap\{K \in I^Y : f(A) \subset K,\ \tau(f^{-1}(K^c)) \geq [r,s],\ \tau^*(f^{-1}(K^c)) \leq [t,u]\}) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subset K,\ \tau((f^{-1}(K))^c) \geq [r,s], \\
&\qquad \tau^*((f^{-1}(K))^c) \leq [t,u]\}) \\
&\supset f^{-1}(\cap\{K \in I^Y : A \subset f^{-1}(K),\ \mathcal{F}_\tau(f^{-1}(K)) \geq [r,s], \\
&\qquad \mathcal{F}^*_{\tau^*}(f^{-1}(K)) \leq [t,u]\}) \\
&= \cap\{f^{-1}(K) : K \in I^Y,\ A \subset f^{-1}(K),\ \mathcal{F}_\tau(f^{-1}(K)) \geq [r,s], \\
&\qquad \mathcal{F}^*_{\tau^*}(f^{-1}(K)) \leq [t,u]\} \\
&\supset \cap\{F \in I^X : A \subset F,\ \mathcal{F}_\tau(F) \geq [r,s],\ \mathcal{F}^*_{\tau^*}(F) \leq [t,u]\} \\
&= cl_{[r,s],[t,u]}(A).
\end{aligned}
$$

Hence $f(cl_{[r,s],[t,u]}(A)) \subset f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subset cl_{[r,s],[t,u]}(f(A))$.

(b) Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (a), we have

$$
\begin{aligned}
cl_{[r,s],[t,u]}(f^{-1}(A)) &\subset f^{-1}(f(cl_{[r,s],[t,u]}(f^{-1}(A)))) \\
&\subset f^{-1}(cl_{[r,s],[t,u]}(f(f^{-1}(A)))) \\
&\subset f^{-1}(cl_{[r,s],[t,u]}(A)).
\end{aligned}
$$

(c) Let $A \in I^Y$. By (b), $cl_{[r,s],[t,u]}(f^{-1}(A^c)) \subset f^{-1}(cl_{[r,s],[t,u]}(A^c))$ and so $(f^{-1}(cl_{[r,s],[t,u]}(A^c)))^c \subset (cl_{[r,s],[t,u]}(f^{-1}(A^c)))^c$. Hence

$$\begin{aligned} f^{-1}(int_{[r,s],[t,u]}(A)) &= (f^{-1}(cl_{[r,s],[t,u]}(A^c)))^c \\ &\subset (cl_{[r,s],[t,u]}(f^{-1}(A^c)))^c \\ &= int_{[r,s],[t,u]}(f^{-1}(A)). \end{aligned}$$

□

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